

Collision Theory for Waves in Two Dimensions and a Characterization of Models with Trivial S -Matrix

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Abstract. Within the framework of local relativistic quantum theory in two space-time dimensions, we develop a collision theory for waves (the set of vectors corresponding to the eigenvalue zero of the mass operator). Since among these vectors there need not be one-particle states, the asymptotic Hilbert spaces do not in general have Fock structure. However, the definition and “physical interpretation” of an S -matrix is still possible. We show that this S -matrix is trivial if the correlations between localized operators vanish at large timelike distances.

1. Introduction

The aim of the present paper is to develop, within the framework of local relativistic quantum theory in two dimensions, a collision theory for the states corresponding to the eigenvalue zero of the mass operator. It is a peculiarity of the two-dimensional world that the set of these states corresponds in general to a highly reducible representation of the Poincaré group. Moreover, among these states there need not be one-particle states. As a trivial example take the even part of a free, massless theory (the set of even polynomials in the fields acting on the vectors with even particle number). Since the direct product of an arbitrary number of massless one-particle states with positive (or negative) momentum is again a state with mass zero, there are many massless states in such a theory. However, there are no one-particle states.

Therefore, instead of assuming the existence of one-particle states, we take as the basis of our construction the representation spaces \mathcal{H}_+ and \mathcal{H}_- of the Poincaré group corresponding to the right and left branch of the light-cone in momentum space¹. Since the vectors in these spaces are invariant under light-like translations in the $t_+ = (t, t)$ and $t_- = (t, -t)$ direction, respectively, they can be interpreted as excitations (*waves*) moving freely with the speed of light from the left to the right or in the opposite direction. Hence these states are suitable

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¹ Note that the branches of the light-cone in two dimensions are separately invariant under Lorentz transformations. Thus, there exist representations attached to either one of them.

quantities for the formulation of a collision theory in the sense of Brenig and Haag [1] although they need not be particles in the usual sense of the word.

Applying methods already used [2] we shall construct for each pair of vectors $\Psi_+ \in \mathcal{H}_+$ and $\Psi_- \in \mathcal{H}_-$ a collision state $\Psi_+ \times^{\text{out}} \Psi_-$ which behaves at large positive times like the waves Ψ_+ and Ψ_- going out to the right and left, respectively. The existence of an incoming configuration $\Psi_+ \times^{\text{in}} \Psi_-$ can be established by a similar procedure. As the multiplication symbol indicates, it turns out that the asymptotic spaces spanned by the collision states are a direct product of the representation spaces \mathcal{H}_+ and \mathcal{H}_- of the Poincaré group. The S -matrix for the waves can therefore be defined as an isometry mapping the vectors $\Psi_+ \times^{\text{out}} \Psi_-$ onto $\Psi_+ \times^{\text{in}} \Psi_-$ and it has the usual physical interpretation. It is a remarkable, yet not entirely surprising fact that this S -matrix is trivial if the correlations between localized operators vanish at large time-like distances.

Let us now give a brief list of our assumptions. Instead of working with the fundamental, but unbounded field operators, we prefer to express the basic structures in terms of a field algebra \mathcal{F} of bounded operators [3]. The field algebra \mathcal{F} is assumed to be a C^* -algebra which is generated by a net $\mathcal{O} \rightarrow \mathcal{F}(\mathcal{O})$ of local algebras attached to the open, bounded regions $\mathcal{O} \subset \mathbb{R}^2$ and to act irreducibly on a Hilbert space \mathcal{H} . We suppose that the local operators commute at space-like distances,

$$\mathcal{F}(\mathcal{O}_1) \subseteq \mathcal{F}(\mathcal{O}_2)' \quad \text{for} \quad \mathcal{O}_1 \subseteq \mathcal{O}_2'. \quad (1)$$

However, our results could easily be extended to a situation where also Fermi operators are present. Furthermore, we assume that there is a continuous, unitary representation $L \rightarrow U(L)$ of the Poincaré group \mathcal{P} in \mathcal{H} which induces automorphisms of the local net,

$$\alpha_L(\mathcal{F}(\mathcal{O})) = U(L)\mathcal{F}(\mathcal{O})U(L)^{-1} = \mathcal{F}(L\mathcal{O}), \quad L \in \mathcal{P}. \quad (2)$$

The spectrum of the generators of the translations $x = (x_0, x_1) \rightarrow U(x)$ is contained in the closed forward light-cone, and there exists (up to a phase) just one unit vector Ω , the vacuum, in \mathcal{H} which is invariant under all $U(L)$, $L \in \mathcal{P}$. Finally we assume, as argued above, that there are two representation spaces \mathcal{H}_+ and \mathcal{H}_- of the Poincaré group corresponding to the right and left branch of the light-cone in momentum space. It will be convenient to include Ω in the definition of both spaces. Then $\mathcal{H}_+ \cap \mathcal{H}_- = [c \cdot \Omega]$, and the quotient spaces $\mathcal{H}_+/[c \cdot \Omega]$ and $\mathcal{H}_-/[c \cdot \Omega]$ are orthogonal to each other.

2. Asymptotic Fields

We now turn to our objective, the formulation of a collision theory for the waves. In analogy to the case of massless Fermions in four dimensions considered in Ref. [2], we start our investigations with the construction of asymptotic fields corresponding to the states in \mathcal{H}_+ and \mathcal{H}_- . Owing to the special geometry of the two-dimensional Minkowski space, these fields have some unusual properties which we shall briefly discuss. Having obtained these results it will then be possible

to build up the collision states and to verify that they have all the properties needed to define a physically sensible S -matrix.

As in Ref. [2] we construct the asymptotic fields as “adiabatic limits” of local operators. For this procedure we use smooth, non-negative functions h with compact support which are normalized according to $\int dt h(t) = 1$. Together with h , the functions h_T , $T \neq 0$ defined by

$$h_T(t) = |T|^{-\varepsilon} \cdot h(|T|^{-\varepsilon} \cdot (t - T)), \quad 0 < \varepsilon < 1 \quad (3)$$

have also these properties. Therefore if F is any local operator from some $\mathcal{F}(\mathcal{O})$, the integrals

$$F_{\pm}(h_T) = \int dt h_T(t) F(t_{\pm}) \quad \text{with} \quad t_{\pm} = (t, \pm t) \quad (4)$$

exist as Riemann integrals in the strong topology.

In the subsequent lemma we shall prove that the sequences $F_{\pm}(h_T)$ converge in the limit of large T . To abbreviate the argument, it will be convenient to label certain regions in configuration space; if \mathcal{O} is any simply connected bounded region of \mathbb{R}^2 , its space-like complement is the union of two disjoint, wedge-shaped regions. We call the wedge which extends to $-\infty$ the *left tangent* of \mathcal{O} and the wedge which extends to $+\infty$ the *right tangent* of \mathcal{O} . One easily checks that the space-like complements of the regions $\mathcal{O} + t_{\pm}$ still contain either one of the tangents of \mathcal{O} . So owing to locality and the support properties of h_T , the elements of the sequence $F_{\pm}(h_T)$, for example, commute with all operators F' localized in the left tangent of \mathcal{O} for large enough $T > 0$. In the proofs that follow we shall make full use of this fact.

Lemma 1. *Let F be any operator which is localized in some simply connected, bounded region \mathcal{O} , $F \in \mathcal{F}(\mathcal{O})$, and let h_T be any sequence of functions with the properties specified above.*

a) *Then the limits $\lim_{T \rightarrow \pm\infty} F_{\pm}(h_T)$ exist in the strong topology and $\|\lim_{T \rightarrow \pm\infty} F_{\pm}(h_T)\| \leq \|F\|$.*

b) *The operators $\lim_{T \rightarrow \pm\infty} F_{\pm}(h_T)$ are uniquely determined by the states which they create from the vacuum.*

Proof. Consider for example $F_{+}(h_T)$. Since $\|F_{+}(h_T)\| \leq \|F\| \cdot \int dt |h_T(t)| = \|F\|$ it suffices to show that the sequence $F_{+}(h_T)$, $T > 0$ converges strongly on a dense set of vectors in \mathcal{H} . Now because of the invariance of the vacuum under translations, one gets

$$F_{+}(h_T)\Omega = \int dt h_T(t) U(t_{+}) \cdot F\Omega$$

and it is then a consequence of the mean ergodic theorem applied to the unitary group $t \rightarrow U(t_{+})$ that [4]

$$\text{s-lim}_{T \rightarrow \infty} F_{+}(h_T)\Omega = P_{+} \cdot F\Omega.$$

Here P_{+} is the projection onto \mathcal{H}_{+} , the space of vectors which are invariant under $t \rightarrow U(t_{+})$. Therefore if F' is any operator which is localized in the left tangent of \mathcal{O} one has

$$\text{s-lim}_{T \rightarrow \infty} F_{+}(h_T) \cdot F'\Omega = \text{s-lim}_{T \rightarrow \infty} F' \cdot F_{+}(h_T)\Omega = F' \cdot P_{+} F\Omega.$$

But the set of vectors $F'\Omega$ with F' localized in the left tangent of \mathcal{O} is dense in \mathcal{H} and so the first part of the statement follows. It is also obvious from the last equation that the vacuum is separating for the linear span generated by the operators $\lim_{T \rightarrow \infty} F_+(h_T)$. This proves the second part of the lemma. (Compare also the more detailed proofs given in Ref. [2].) \square

Corresponding to the four light-like directions in \mathbb{R}^2 we can now define four mappings $\Phi_{\pm}^{\text{in}}, \Phi_{\pm}^{\text{out}}$ of the set of local operators F into the set of bounded operators on \mathcal{H} :

$$\Phi_{\pm}^{\text{in}}(F) = \text{s-lim}_{T \rightarrow -\infty} F_{\pm}(h_T) \quad \text{and} \quad \Phi_{\pm}^{\text{out}}(F) = \text{s-lim}_{T \rightarrow \infty} F_{\pm}(h_T). \quad (5)$$

These mappings are linear, positive-semidefinite (since the functions h_T are non-negative) and they can be extended to the whole field-algebra \mathcal{F} by continuity. In the next lemma we give a list of further properties of Φ_{\pm}^{out} . (An analogous lemma holds for Φ_{\pm}^{in} .)

Lemma 2. *Let Φ_{\pm}^{out} be the mappings of \mathcal{F} into $\mathcal{B}(\mathcal{H})$ defined above. Then:*

a) $\Phi_{\pm}^{\text{out}}(F) \cdot \Psi_{\pm} = P_{\pm} F \cdot \Psi_{\pm}$, where $\Psi_{\pm} \in \mathcal{H}_{\pm}$ and P_{\pm} are the projections onto \mathcal{H}_{\pm} .

b) $\alpha_L \cdot \Phi_{\pm}^{\text{out}} = \Phi_{\pm}^{\text{out}} \cdot \alpha_L$ for all Poincaré transformations $L \in \mathcal{P}$. In particular, $\alpha_{t_{\pm}} \cdot \Phi_{\pm}^{\text{out}} = \Phi_{\pm}^{\text{out}}$, where $t_{\pm} = (t, \pm t)$, $t \in \mathbb{R}$.

c) If \mathcal{O} is a simply connected, bounded region, $F \in \mathcal{F}(\mathcal{O})$, and if $F'_l(t_-)$ and $F'_r(t_+)$ are – after suitable light-like translations t_{\pm} – localized in the right and left tangent of \mathcal{O} , respectively, then $[\Phi_{+}^{\text{out}}(F), F'_l] = [\Phi_{-}^{\text{out}}(F), F'_r] = 0$.

d) If $\mathcal{O}_1, \mathcal{O}_2$ are two simply connected, bounded regions which cannot be connected by a light-like vector t_+ , $\mathcal{O}_1 \cap (\mathcal{O}_2 + t_+) = \emptyset$ for all t_+ , and if $F_1 \in \mathcal{F}(\mathcal{O}_1)$, $F_2 \in \mathcal{F}(\mathcal{O}_2)$ then $[\Phi_{\pm}^{\text{out}}(F_1), \Phi_{\pm}^{\text{out}}(F_2)] = 0$.

e) For arbitrary $F, G \in \mathcal{F}$ the relation $[\Phi_{+}^{\text{out}}(F), \Phi_{-}^{\text{out}}(G)] = 0$ holds.

Proof. (The proofs are completely analogous for Φ_{+}^{out} and Φ_{-}^{out} and we therefore sketch the arguments only for Φ_{+}^{out} .)

a) As in the preceding lemma, the statement follows from the invariance of $\Psi_{+} \in \mathcal{H}_{+}$ under the translations $t \rightarrow U(t_+)$ and the mean ergodic theorem.

b) It is obvious from the definition of Φ_{+}^{out} that $\alpha_x \cdot \Phi_{+}^{\text{out}} = \Phi_{+}^{\text{out}} \cdot \alpha_x$ for arbitrary translations. To prove $\alpha_{\Lambda} \cdot \Phi_{+}^{\text{out}} = \Phi_{+}^{\text{out}} \cdot \alpha_{\Lambda}$ for Lorentz boosts Λ , note that the vectors t_+ are eigenvectors of Λ , $\Lambda t_+ = \lambda \cdot t_+$ for some $\lambda > 0$. Hence

$$\alpha_{\Lambda}(\Phi_{+}^{\text{out}}(F)) = \text{s-lim}_{T \rightarrow \infty} \int dt h_T(t) \cdot F_{\Lambda}(\lambda \cdot t_+) = \text{s-lim}_{T \rightarrow \infty} \int dt \hat{h}_{\lambda T}(t) \cdot F_{\Lambda}(t_+),$$

where $F_{\Lambda} = \alpha_{\Lambda}(F)$ and $\hat{h}(t) = \lambda^{\varepsilon-1} \cdot h(\lambda^{\varepsilon-1} \cdot t)$. Clearly \hat{h} belongs to the set of functions h introduced above. Then the statement follows from the fact that Φ_{+}^{out} does not depend on the special choice of h within this set.

The relation $\alpha_{t_+} \cdot \Phi_{+}^{\text{out}} = \Phi_{+}^{\text{out}}$ is a consequence of the estimate

$$\|\alpha_{t_+}(\Phi_{+}^{\text{out}}(F)) - \Phi_{+}^{\text{out}}(F)\| \leq \lim_{T \rightarrow \infty} \int ds |h(s - |T|^{-\varepsilon} \cdot t) - h(s)| \cdot \|F\| = 0.$$

c) For F'_l which are localized in the left tangent of \mathcal{O} this is trivial since the approximations $F_+(h_T)$ of $\Phi_{+}^{\text{out}}(F)$ commute with F'_l for large T . However because of $\alpha_{t_+} \cdot \Phi_{+}^{\text{out}} = \Phi_{+}^{\text{out}}$ one has furthermore

$$[\alpha_{t_+}(F'_l), \Phi_{+}^{\text{out}}(F)] = \alpha_{t_+}([\Phi'_l, \Phi_{+}^{\text{out}}(F)]) = 0.$$

d) Since the regions \mathcal{O}_1 and \mathcal{O}_2 are simply connected there exists a translation t_+ such that $(\mathcal{O}_1 + t_+)$ and \mathcal{O}_2 are space-like separated. If $(\mathcal{O}_1 + t_+)$, for example, is localized in the left tangent of \mathcal{O}_2 , then according to (c),

$$[\Phi_+^{\text{out}}(F_1), \Phi_+^{\text{out}}(F_2)] = \text{s-lim}_{T \rightarrow \infty} [F_{1+}(h_T), \Phi_+^{\text{out}}(F_2)] = 0.$$

e) If F and G are local operators one gets from (c),

$$[\Phi_+^{\text{out}}(F), \Phi_-^{\text{out}}(G)] = \text{s-lim}_{T \rightarrow \infty} [\Phi_+^{\text{out}}(F), G_-(h_T)] = 0.$$

But every $F, G \in \mathcal{F}$ can be uniformly approximated by local operators and thus the statement follows from the continuity of Φ_{\pm}^{out} and Φ_{\pm}^{in} . \square

Since the operators $\Phi_{\pm}^{\text{in}}(F)$ and $\Phi_{\pm}^{\text{out}}(F)$ are the restrictions of the local fields F to the light-cone (the “mass-shell”) in momentum space, we shall call them *asymptotic fields* in analogy with ordinary collision theory. It is an amusing although not unexpected fact that the asymptotic fields are bounded. So even in a model with underlying particle structure one does not get, by our construction, the fundamental free fields themselves but only bounded functions of them. In the general case the wave states should also be decomposable into elementary subsystems. Therefore one expects that the asymptotic fields $\Phi_{\pm}^{\text{in}}(F)$ and $\Phi_{\pm}^{\text{out}}(F)$ are always built up out of certain basic field operators. However, we shall not discuss this question here.

The unusual localization properties of the asymptotic fields in configuration space are due to the peculiar geometry of the two-dimensional Minkowski space which does not allow the spreading of waves with time. So the waves behave almost classically, and the commutation relations for the asymptotic fields given in the preceding lemma may be interpreted as the Huyghens principle translated into field theory. The non-dispersive character of the motion of waves can also be read off the following lemma, which is the analog of the well-known LSZ asymptotic condition [5].

Lemma 3. *If $F \in \mathcal{F}$ and $t_{\pm} = (t, \pm t)$ then*

$$\text{w-lim}_{t \rightarrow -\infty} F(t_{\pm}) = \Phi_{\pm}^{\text{in}}(F) \quad \text{and} \quad \text{w-lim}_{t \rightarrow \infty} F(t_{\pm}) = \Phi_{\pm}^{\text{out}}(F).$$

Proof. Applying methods such as those of Borchers [6] or Maison [7], one can show that

$$\text{w-lim}_{t \rightarrow \infty} U(t_{\pm}) = P_{\pm}.$$

Therefore if $F \in \mathcal{F}(\mathcal{O})$ and F' is any operator which is localized in the left tangent of \mathcal{O} , one gets

$$\text{w-lim}_{t \rightarrow \infty} F(t_+) \cdot F' \Omega = \text{w-lim}_{t \rightarrow \infty} F' \cdot U(t_+) F \Omega = F' \cdot P_+ F \Omega = \Phi_+^{\text{out}}(F) \cdot F' \Omega.$$

Here Lemma 2 has been used for the last equality. Since every operator $F \in \mathcal{F}$ can be uniformly approximated by local ones and since the set of vectors $F' \Omega$ is dense in \mathcal{H} the statement follows. \square

To conclude this section let us briefly sketch how to construct the asymptotic field algebras $\mathcal{F}_{\pm}^{\text{out}}$ and $\mathcal{F}_{\pm}^{\text{in}}$. If \mathcal{O}_1 is a simply connected, bounded region

in configuration space, we define the local algebra $\mathcal{F}_+^{\text{out}}(\mathcal{O}_1)$ as the von Neumann algebra which is generated by the operators $\Phi_+^{\text{out}}(F)$, $F \in \mathcal{F}(\mathcal{O}_1)$. For arbitrary bounded regions \mathcal{O} we define $\mathcal{F}_+^{\text{out}}(\mathcal{O})$ as the smallest von Neumann algebra containing all $\mathcal{F}_+^{\text{out}}(\mathcal{O}_1)$ with $\mathcal{O}_1 \subseteq \mathcal{O}$. The asymptotic field algebra $\mathcal{F}_+^{\text{out}}$ is then the global C^* -algebra of the local algebras $\mathcal{F}_+^{\text{out}}(\mathcal{O})$. With this definition the net $\mathcal{O} \rightarrow \mathcal{F}_+^{\text{out}}(\mathcal{O})$ is local, covariant, and enjoys all the properties usually required in relativistic quantum theory.

3. Colliding Waves and the S-Matrix

From our discussion of the asymptotic fields it is now almost evident how to construct states corresponding to a given asymptotic configuration of waves. Take any pair of vectors $\Psi_{\pm} \in \mathcal{H}_{\pm}$. Since the C^* -algebra \mathcal{F} is irreducibly represented in \mathcal{H} , there exists a pair of operators $F_{\pm} \in \mathcal{F}$ such that $\Psi_+ = F_+ \Omega$ and $\Psi_- = F_- \Omega$ [8, Theorem 1.21.17]. We then define the incoming and outgoing collision states of the waves Ψ_+ , Ψ_- by ²

$$\Psi_+ \overset{\text{in}}{\times} \Psi_- = \Phi_+^{\text{in}}(F_+) \Phi_-^{\text{in}}(F_-) \Omega \quad \text{and} \quad \Psi_+ \overset{\text{out}}{\times} \Psi_- = \Phi_+^{\text{out}}(F_+) \Phi_-^{\text{out}}(F_-) \Omega. \quad (6)$$

This definition does not depend on the special choice of F_+ and F_- . In fact, the states $\Psi_+ \overset{\text{in}}{\times} \Psi_-$ and $\Psi_+ \overset{\text{out}}{\times} \Psi_-$ are a direct product of the vectors Ψ_+ and Ψ_- , as the next lemma shows.

Lemma 4. *For the collision states $\Psi_+ \overset{\text{out}}{\times} \Psi_-$ one has the relations*

$$\text{a) } (\Psi_+ \overset{\text{out}}{\times} \Psi_-, \Psi'_+ \overset{\text{out}}{\times} \Psi'_-) = (\Psi_+, \Psi'_+) \cdot (\Psi_-, \Psi'_-).$$

$$\text{b) } U(L) \cdot (\Psi_+ \overset{\text{out}}{\times} \Psi_-) = (U_+(L) \Psi_+) \overset{\text{out}}{\times} (U_-(L) \Psi_-), \quad L \in \mathcal{P}.$$

$U_{\pm}(L)$ are the unitary representations of the Poincaré group \mathcal{P} in \mathcal{H}_+ and \mathcal{H}_- , respectively. (These relations hold also for $\Psi_+ \overset{\text{in}}{\times} \Psi_-$ if “out” is replaced by “in”.)

Proof. Both statements are a simple consequence of Lemma 2.

a) Bearing in mind that $\Psi_{\pm} = F_{\pm} \Omega$ and $\Psi'_{\pm} = F'_{\pm} \Omega$ for certain operators $F_{\pm}, F'_{\pm} \in \mathcal{F}$, one gets the string of equalities

$$\begin{aligned} (\Psi_+ \overset{\text{out}}{\times} \Psi_-, \Psi'_+ \overset{\text{out}}{\times} \Psi'_-) &= (\Phi_+^{\text{out}}(F_+) \Phi_-^{\text{out}}(F_-) \Omega, \Phi_+^{\text{out}}(F'_+) \Phi_-^{\text{out}}(F'_-) \Omega) \\ &= (\Phi_+^{\text{out}}(F'_+) * \Phi_+^{\text{out}}(F_+) \Omega, \Phi_-^{\text{out}}(F_-) * \Phi_-^{\text{out}}(F'_-) \Omega) \\ &= (\Phi_+^{\text{out}}(F'_+) * \Phi_+^{\text{out}}(F_+) \Omega, \Omega) (\Omega, \Phi_-^{\text{out}}(F_-) * \Phi_-^{\text{out}}(F'_-) \Omega) \\ &= (F_+ \Omega, F'_+ \Omega) (F_- \Omega, F'_- \Omega) = (\Psi_+, \Psi'_+) (\Psi_-, \Psi'_-), \end{aligned}$$

where use has been made of the fact that the quotient spaces $\mathcal{H}_+ / [c \cdot \Omega]$ and $\mathcal{H}_- / [c \cdot \Omega]$ are orthogonal.

b) The second half of the lemma follows from the commutativity of the mappings Φ_{\pm}^{out} with the Poincaré transformations, $\alpha_L \cdot \Phi_{\pm}^{\text{out}} = \Phi_{\pm}^{\text{out}} \cdot \alpha_L$. \square

As a consequence of the lemma, the spaces \mathcal{H}^{in} and \mathcal{H}^{out} which are generated by the vectors $\Psi_+ \overset{\text{in}}{\times} \Psi_-$ and $\Psi_+ \overset{\text{out}}{\times} \Psi_-$ are a direct product of \mathcal{H}_+ and \mathcal{H}_- ,

$$\mathcal{H}^{\text{in}} = \mathcal{H}_+ \overset{\text{in}}{\times} \mathcal{H}_- \quad \text{and} \quad \mathcal{H}^{\text{out}} = \mathcal{H}_+ \overset{\text{out}}{\times} \mathcal{H}_-, \quad (7)$$

² Note that with this definition $\Omega = \Omega \overset{\text{in}}{\times} \Omega = \Omega \overset{\text{out}}{\times} \Omega$, $\Psi_+ = \Psi_+ \overset{\text{in}}{\times} \Omega = \Psi_+ \overset{\text{out}}{\times} \Omega$ and $\Psi_- = \Omega \overset{\text{in}}{\times} \Psi_- = \Omega \overset{\text{out}}{\times} \Psi_-$.

and the Poincaré transformations $L \rightarrow U(L)$ act like a tensor product on these spaces. This is the mathematical expression of the physical fact that there are no correlations between the constituents Ψ_+ and Ψ_- of the collision states. However in order to make sure that the vectors $\Psi_+ \overset{\text{out}}{\times} \Psi_-$, for example, really correspond to an outgoing configuration of waves Ψ_+ and Ψ_- , one has to calculate the expectation values of local observables within these states at large positive times. Using the results of the preceding section it is an easy exercise to verify that for arbitrary $A, B \in \mathcal{F}$

$$\lim_{t \rightarrow \infty} (\Psi_+ \overset{\text{out}}{\times} \Psi_-, A(t_+)B(t_-)\Psi_+ \overset{\text{out}}{\times} \Psi_-) = (\Psi_+, A\Psi_+)(\Psi_-, B\Psi_-) \quad (8)$$

and in space-like or positive time-like directions n

$$\lim_{t \rightarrow \infty} (\Psi_+ \overset{\text{out}}{\times} \Psi_-, A(tn)\Psi_+ \overset{\text{out}}{\times} \Psi_-) = (\Omega, A\Omega) \cdot \|\Psi_+ \overset{\text{out}}{\times} \Psi_-\|^2. \quad (9)$$

So the vectors $\Psi_+ \overset{\text{out}}{\times} \Psi_-$ behave at asymptotic times like the waves Ψ_+ and Ψ_- going out to the right and left, respectively. This justifies our construction.

Knowing how to interpret the collision states it makes physical sense to define an S -matrix for the waves. We put

$$S \cdot \Psi_+ \overset{\text{out}}{\times} \Psi_- = \Psi_+ \overset{\text{in}}{\times} \Psi_- \quad (10)$$

and extend the domain of S to arbitrary vectors in \mathcal{H}^{out} by linearity. As a consequence of Lemma 4, S is an isometry which maps \mathcal{H}^{out} onto \mathcal{H}^{in} and it commutes with the Poincaré transformations $U(L)$. So S has the usual physical interpretation [1] and it is therefore the appropriate quantity to describe collision processes of waves.

4. Models with Trivial S -Matrix

Stimulated by the hope that one may gain a better understanding of the structural differences between interacting and non-interacting theories, there has always been some interest in conceptually simple criteria which characterize models with a trivial S -matrix. Several results in this direction have been obtained in the Wightman formalism [9–11]. However, almost nothing is known in the algebraic framework of field theory. We therefore believe that it is worth while to place the following criterion on record, in which it is shown that waves do not scatter if the correlations between localized operators vanish at large time-like distances. Again this result is due the non-dispersive motion of the waves in two dimensions and it fits completely with intuitive expectations.

Lemma 5. *If for arbitrary $F, G \in \mathcal{F}$ the relation*

$$s\text{-}\lim_{t \rightarrow \infty} [F(t), G] = 0 \quad \text{with} \quad F(t) = U(t)FU(t)^{-1}$$

holds then the S -matrix defined in (10) is trivial.

Proof. There exists a pair of operators $F_{\pm} \in \mathcal{F}$ which create the states $\Psi_{\pm} \in \mathcal{H}_{\pm}$ from the vacuum, $F_{\pm}\Omega = \Psi_{\pm}$. Thus, as an immediate consequence of Lemma 2

and Lemma 3, one gets

$$\begin{aligned}
\|\Psi_+ \overset{\text{out}}{\times} \Psi_- - \Psi_+ \overset{\text{in}}{\times} \Psi_-\| &= \|\Phi_+^{\text{out}}(F_+) \Phi_-^{\text{out}}(F_-) \Omega - \Phi_+^{\text{in}}(F_+) \Phi_-^{\text{in}}(F_-) \Omega\| \\
&= \|\Phi_+^{\text{out}}(F_+) F_- \Omega - \Phi_-^{\text{in}}(F_-) F_+ \Omega\| \\
&= \|\text{w-lim}_{t \rightarrow \infty} (F_+(t_+) F_- \Omega - F_-(-t_- t_-) F_+ \Omega)\| \\
&= \|\text{w-lim}_{t \rightarrow \infty} (F_+(t_+) F_-(-t_- t_-) \Omega - F_-(-t_- t_-) F_+(t_+) \Omega)\| \\
&\leq \lim_{t \rightarrow \infty} \|[F_+(2t), F_-] \Omega\| = 0
\end{aligned}$$

and therefore $\Psi_+ \overset{\text{in}}{\times} \Psi_- = \Psi_+ \overset{\text{out}}{\times} \Psi_-$. \square

It is obvious from the proof of the lemma that the assumptions could be considerably weakened. So it would suffice for the argument that all operators $F_{\pm} \in \mathcal{F}$ which create vectors $\Psi_{\pm} \in \mathcal{H}_{\pm}$ from the vacuum have asymptotically vanishing time-like correlations, $s\text{-lim}_{t \rightarrow \infty} [F_+(t), F_-] \Omega = 0$. However, the strong convergence of this sequence is crucial for the proof and cannot be replaced by weak convergence, for example. In fact it follows from the basic assumptions that $\text{w-lim}_{t \rightarrow \infty} [F_+(t), F_-] \Omega = 0$ regardless of whether the S -matrix is trivial or not.

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