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Finite Mass Renormalizations in the Euclidean Yukawa$_2$ Field Theory*

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Abstract. We show that arbitrary finite boson mass renormalizations are possible in the Euclidean Yukawa$_2$ theory. We work in the Matthews-Salam representation with the fermions "integrated out".

I. Introduction and Results

We study the Yukawa$_2$ quantum field theory in a finite volume as a Euclidean boson field theory with the fermions "integrated out". The possibility of integrating out the fermions in the Yukawa theory was first demonstrated, in the external boson field case, by Matthews and Salam [1] and in the two-dimensional finite volume interacting theory, by Seiler [2] who showed that the resulting Fredholm determinants are L_p functions of the boson field. He thus obtained estimates on Schwinger functions of the form:

$$\begin{aligned} &|S(f_1, \ldots, f_n; g_1, \ldots, g_m; h_1, \ldots, h_m)| \\ &\quad = |\langle \textstyle\prod_{i=1}^n \phi(f_i) \prod_{j=1}^m \Psi^{(1)}(g_j) \prod_{k=1}^m \Psi^{(2)}(h_k) e^{-V} \rangle| \\ &\quad \leqq c_1 c_2^{n+2m} (n!)^{\frac{1}{2}} \textstyle\prod_{i=1}^n \|f_i\|_{-1} \prod_{j=1}^m \|g_j\|_{-\frac{1}{2}} \|h_j\|_{-\frac{1}{2}}, \end{aligned} \tag{1.1}$$

with c_1, c_2 independent of ultraviolet cutoffs. The norms are those for the boson and fermion test-function spaces $\mathscr{H}_{-1}^{(\mu_0)}$, $\mathscr{H}_{-\frac{1}{2}}^{(m_0)} \otimes C^2$, where $\mathscr{H}_s^{(m)} = L_2(R^2, (k^2+m^2)^s d^2k)$.

In this paper we give a derivation of (1.1) even in the presence of an arbitrarily large negative boson mass renormalization. That such a renormalization is possible has already been demonstrated in the Hamiltonian formalism by Glimm [3, 4]. The basic idea is to decompose V into parts with high and low Fermi momenta. The high momentum part requires a "smaller" infinite ultraviolet mass renormalization than does the whole interaction, and the difference can then be used to dominate the finite boson mass renormalization. The low momentum part of e^{-V} may be expanded in a power series, since both fermions are bounded operators. Thus the Schwinger functions of the theory can be expressed as a power series in the Schwinger functions of the high momentum part. We prove a bound of the form (1.1) for the high momentum interaction and, applied to each term of the power series, this yields a bound of the same form for the full interaction. Our principal result is:

Theorem 1.1. *In the presence of a finite boson mass renormalization $-M$, a bound of the form* (1.1) *still applies, uniformly in the ultraviolet cutoffs, and the Schwinger functions converge as the cutoffs are removed.*

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In section II we prove Theorem 1.1, assuming the result for the high momentum interaction. In Section III we prove the result for the high momentum interaction, where the lower bound ϱ on at least one of the fermi momenta is determined by the size of the coupling constant λ and the negative mass renormalization $-M$.

II. The Low-momentum Expansion

The ultraviolet cutoff interaction and un-normalized Schwinger functions are given by:

$$V_{\kappa,\sigma}^{(\lambda,M)} = \lambda{:}V_{I,\kappa,\sigma}{:} - \frac{\lambda^2}{2}\langle :V_{I,\kappa,\sigma}:^2\rangle - \tfrac{1}{2}(\lambda^2\delta m_\sigma^2 + M^2){:}\phi_\kappa^2(g^2){:}\,,$$

$$\begin{aligned} V_{I,\kappa,\sigma} &= \int d^2x g(x)\phi_\kappa(x)\Psi_\sigma^{(2)}(x)\Psi_\sigma^{(1)}(x) \\ &= 2\pi\int d^2p d^2q (g\phi_\kappa)\tilde{}(-p-q)\tilde{\chi}_\sigma(p)\tilde{\chi}_\sigma(q)\Psi^{(2)}(p)\Psi^{(1)}(q)\,, \end{aligned}$$

$$\begin{aligned} &(ZS)_{\kappa,\sigma}^{(\lambda,M)}(f_1,\ldots,f_n;g_1,\ldots,g_m;h_1,\ldots,h_m) \\ &\quad\equiv \langle \textstyle\prod_{i=1}^n \phi(f_i)\prod_{j=1}^m \Psi^{(1)}(g_j)\prod_{k=1}^m \Psi^{(2)}(h_k)e^{-V_{\kappa,\sigma}^{(\lambda,M)}}\rangle\,, \end{aligned}$$

where $\phi_\kappa(x) = (\chi_\kappa * \phi)(x)$, $\Psi_\sigma^{(i)}(x) = (\chi_\sigma * \Psi^{(i)})(x)$ and the ultraviolet cutoff function χ_σ satisfies $\chi_\sigma(\cdot) \to \delta(\cdot)$ as $\sigma \to \infty$, $|\tilde{\chi}_\sigma(k)| \leqq \frac{1}{2\pi}$. In particular we will choose either $\tilde{\chi}_\sigma(k) \equiv \frac{1}{2\pi}\theta(\sigma - |k|)$ or $\tilde{\chi}_\sigma(k) = \frac{1}{2\pi}\theta(\sigma - |k_1|)$, with $\theta(s) = 1$, $s \geqq 0$, $\theta(s) = 0$ otherwise, the latter cutoff being necessary in order to give a connection with the Hamiltonian formalism via the Feyman-Kac formula of Osterwalder and Schrader [5]. The space-time cutoff $g(\cdot)$, $0 \leqq g \leqq 1$, is either in $C_0^\infty(R^2)$ or else is the characteristic function of a compact region. In either case we denote by $\Lambda = [-t, t] \times [-a, a]$ a rectangular region containing suppt. g. We define the high and low momentum parts of the interaction as the operators $V_{I,\kappa,\varrho,\sigma}$, $\delta V_{I,\kappa,\varrho}$ obtained by the decomposition, for $\varrho \leqq \sigma$,

$$\tilde{\chi}_\sigma(p)\tilde{\chi}_\sigma(q) = \tilde{\chi}_\sigma(p)\tilde{\chi}_\sigma(q)\{(1 - \theta_\varrho(p)\theta_\varrho(q)) + \theta_\varrho(p)\theta_\varrho(q)\}, \qquad \theta_\varrho(p) \equiv \theta(\varrho - |p|)\,,$$

and the full renormalized high momentum action is then defined as:

$$V_{\kappa,\varrho,\sigma}^{(\lambda,M)} = V_{\kappa,\sigma}^{(\lambda,M)} - \lambda\delta V_{I,\kappa,\varrho}\,. \tag{2.1}$$

Expanding the low momentum action in a power series, we have:

$$\begin{aligned} &(ZS)_{\kappa,\sigma}^{(\lambda,M)}(f_1,\ldots,f_n;g_1,\ldots,g_m;h_1,\ldots,h_m) \\ &\quad= \textstyle\sum_{r=0}^\infty \frac{(-\lambda)^r}{r!}\langle \prod_{i=1}^n \phi(f_i)\prod_{j=1}^m \Psi^{(1)}(g_j)\prod_{k=1}^m \Psi^{(2)}(h_k)(\delta V_{I,\kappa,\varrho})^r e^{-V_{\kappa,\varrho,\sigma}^{(\lambda,M)}}\rangle\,. \end{aligned}$$

In order to obtain estimates uniform in κ, we integrate by parts in $\delta V_{I,\kappa,\varrho}$:

$$\begin{aligned} \delta V_{I,\kappa,\varrho} &= \int_{-t}^t dx_0 \int_{-a}^a dx_1 \left(\frac{\partial}{\partial x_1}\int_{-a}^{x_1} dx_1' g(x_0, x_1')\phi_\kappa(x_0, x_1')\right)\Psi_\varrho^{(2)}(x)\Psi_\varrho^{(1)}(x) \\ &= \textstyle\sum_{i=1}^3 \int_\Lambda d^2x \phi(F_{\kappa,x}^{(i)})\Psi^{(2)}(G_{\varrho,x}^{(i)})\Psi^{(1)}(H_{\varrho,x}^{(i)})\,, \end{aligned}$$

where

$$F^{(1)}_{\kappa,x}(y)=\frac{1}{2a}(\chi_\kappa(y_0-x_0,\cdot)*g(x_0,\cdot))(y_1),$$

$$F^{(i)}_{\kappa,x}(y)=-(\chi_\kappa(y_0-x_0,\cdot)*g_{x_1}(x_0,\cdot))(y_1),\qquad i=2,3,$$

$$G^{(1)}_{\varrho,x}(y)=H^{(1)}_{\varrho,x}(y)=\frac{1}{2\pi}\tilde\theta_\varrho((x_0,a)-y),$$

$$G^{(2)}_{\varrho,x}(y)=H^{(3)}_{\varrho,x}(y)=\frac{1}{2\pi}\frac{\partial}{\partial x_1}\tilde\theta_\varrho(x-y),$$

$$G^{(3)}_{\varrho,x}(y)=H^{(2)}_{\varrho,x}(y)=\frac{1}{2\pi}\tilde\theta_\varrho(x-y)$$

– here $g_{x_1}(y)\equiv\theta(x_1-y_1)g(y)$ and $*$ denotes convolution in the space variable only. It is easily seen that:

Lemma 2.1. *$F\equiv\sup_{\kappa,x,i}\|F^{(i)}_{\kappa,x}\|_{-1}$, $G_\varrho\equiv\sup_{x,i}\|G^{(i)}_{\varrho,x}\|_{-\frac{1}{2}}$ are finite and $F^{(i)}_{\kappa,x}$ converges in $\mathscr{H}_{-1}$ as $\kappa\to\infty$, uniformly in i and x.*

Inserting the expression for $\delta V_{I,\kappa,\varrho}$ into $(ZS)^{(\lambda,M)}_{\kappa,\sigma}$ we obtain:

$$\begin{aligned}&(ZS)^{(\lambda,M)}_{\kappa,\sigma}(f_1,\dots,f_n;g_1,\dots,g_m;h_1,\dots,h_m)\\&=\sum_{r=0}^{\infty}\frac{((-)^{m+\frac{r-1}{2}}\lambda)^r}{r!}\int_{\Lambda^r}d^2x_1\dots d^2x_r\sum_{i_1,\dots,i_r=1}^{3}\\&(ZS)^{(\lambda,M)}_{\kappa,\varrho,\sigma}(f_1,\dots,f_n,F^{(i_1)}_{\kappa,x_1},\dots,F^{(i_r)}_{\kappa,x_r};g_1,\dots,g_m,G^{(i_1)}_{\varrho,x_1},\dots,G^{(i_r)}_{\varrho,x_r};\\&h_1,\dots,h_m,H^{(i_1)}_{\varrho,x_1},\dots,H^{(i_r)}_{\varrho,x_r})\end{aligned}\tag{2.2}$$

where $(ZS)^{(\lambda,M)}_{\kappa,\varrho,\sigma}$ denote the Schwinger functions for $V^{(\lambda,M)}_{\kappa,\varrho,\sigma}$. In Theorem 3.1 we prove that a bound of the form (1.1) holds for $(ZS)^{(\lambda,M)}_{\kappa,\varrho,\sigma}$ if $\varrho\geqq\varrho_0=\varrho_0(\lambda,M)$ and $\sigma\geqq\sigma_0(\varrho,\lambda,M)$, with constants $c_i(\varrho,\lambda,M)$ independent of κ,σ and furthermore that these Schwinger functions converge as $\sigma\to\infty,\kappa\to\infty$. Applying this bound to (2.2) term by term we obtain, for $\varrho\geqq\varrho_0(\lambda,M)$, $\sigma\geqq\sigma_0(\varrho,\lambda,M)$:

$$\begin{aligned}&|(ZS)^{\lambda,M}_{\kappa,\sigma}(f_1,\dots,f_n;g_1,\dots,g_m;h_1,\dots,h_m)\\&\leqq\sum_{r=0}^{\infty}\frac{|\lambda|^r}{r!}\int_{\Lambda^r}d^2x_1\dots d^2x_r\sum_{i_1,\dots,i_r=1}^{3}c_1(\varrho,\lambda,M)c_2(\varrho,\lambda,M)^{n+2m+3r}(n+r)!^{\frac{1}{2}}\\&\prod_{i=1}^{n}\|f_i\|_{-1}\prod_{j=1}^{m}\|g_j\|_{-\frac{1}{2}}\|h_j\|_{-\frac{1}{2}}\prod_{l=1}^{r}\|F^{(i_l)}_{\kappa,x_l}\|_{-1}\|G^{(i_l)}_{\varrho,x_l}\|_{-\frac{1}{2}}\|H^{(i_l)}_{\varrho,x_l}\|_{-\frac{1}{2}}\\&\leqq c_3(\varrho,\lambda\ M)c_4(\varrho,\lambda,M)^{n+2m}(n!)^{\frac{1}{2}}\prod_{i=1}^{n}\|f_i\|_{-1}\prod_{j=1}^{m}\|g_j\|_{-\frac{1}{2}}\|h_j\|_{-\frac{1}{2}},\end{aligned}$$

where

$$c_3(\varrho,\lambda,M)=c_1(\varrho,\lambda,M)\sum_{r=0}^{\infty}(6|\Lambda||\lambda|c_2(\varrho,\lambda,M)^3FG_\varrho^2)^r\frac{1}{r!^{\frac{1}{2}}},$$

$c_4(\varrho,\lambda,M)=2c_2(\varrho,\lambda,M)$, and we have used the estimates of Lemma 2.1. Convergence of $(ZS)^{(\lambda,M)}_{\kappa,\sigma}$ as $\sigma\to\infty$, $\kappa\to\infty$, also follows immediately from (2.2), using the convergence of $(ZS)^{(\lambda,M)}_{\kappa,\varrho,\sigma}$, and of $F^{(i_l)}_{\kappa,x_l}$ in $\mathscr{H}_{-1}$. This completes the proof of Theorem

1.1, since $Z_{\kappa,\sigma}^{(\lambda,M)}>0$, uniformly in κ, σ, for sufficiently large σ, follows from $Z_\kappa^{(\lambda,0)})>0$, which has been shown in [2].

III. The High-momentum Interaction

We now study the Schwinger functions for the high-momentum interaction $V_{\kappa,\varrho,\sigma}^{(\lambda,M)}$, (2.1). Since λ and M are fixed throughout, we suppress them in our notation. Defining the compact operator on $\mathscr{H}=\mathscr{H}_{\frac{1}{2}}^{(m_0)}\otimes C^2$ with kernel

$$K_{\kappa,\varrho,\sigma}(p,q)=2\pi\frac{\not p+m_0}{p^2+m_0^2}(\phi_\kappa g)\tilde{}(p-q)(q^2+m^2)^{-\frac{1}{2}}\tilde{\chi}_\sigma(p)\tilde{\chi}_\sigma(q)(1-\theta_\varrho(p)\theta_\varrho(q)),$$

one finds that the Schwinger functions $(ZS)_{\kappa,\varrho,\sigma}$ are expressed in terms of solutions of a Fredholm equation by [1, 2],[2]

$$\begin{aligned}(ZS)_{\kappa,\varrho,\sigma}&(f_1,\dots,f_n;g_1,\dots,g_m;h_1,\dots,h_m)\\ &\equiv\langle\textstyle\prod_{i=1}^n\phi(f_i)\prod_{j=1}^m\Psi^{(1)}(g_j)\prod_{k=1}^m\Psi^{(2)}(h_k)e^{-V_{\kappa,\varrho,\sigma}}\rangle\\ &=\textstyle\int d\mu_0\prod_{i=1}^n\phi(f_i)\cdot(-)^{\left[\frac{m}{2}\right]}\det_{jk}S'_{\varrho,\sigma}(g_j,h_k;\phi_\kappa)D_{\varrho,\sigma}(\phi_\kappa),\end{aligned}$$

where $S'_{\varrho,\sigma}(g,h;\phi_\kappa)\equiv(D_0g,(1+\lambda K_{\kappa,\varrho,\sigma})^{-1}S_0h)_{\frac{1}{2}}$, D_0 and S_0 are multiplication by $(p^2+m_0^2)^{-\frac{1}{2}}$ and $\dfrac{\not p+m_0}{p^2+m_0^2}$ respectively, and

$$\begin{aligned}D_{\varrho,\sigma}(\phi_\kappa)=&\,e^{-\lambda\mathrm{Tr}(K_{\kappa,0,\sigma}-K_{\kappa,\varrho,\sigma})+\frac{\lambda^2}{2}\langle\mathrm{Tr}(K_{\kappa,0,\sigma}^2-K_{\kappa,\varrho,\sigma}^2)\rangle}\\ &\cdot e^{\frac{1}{2}\lambda^2:\left(\mathrm{Tr}K_{\kappa,\varrho,\sigma}^*K_{\kappa,\varrho,\sigma}+\left(\delta m_\sigma^2+\frac{M^2}{\lambda^2}\right)\phi_\kappa^2(g^2)\right):}\\ &e^{-\frac{\lambda^2}{4}:\mathrm{Tr}(K_{\kappa,\varrho,\sigma}^*+K_{\kappa,\varrho,\sigma})^2:}\ \det_3(1+\lambda K_{\kappa,\varrho,\sigma}).\end{aligned}$$

We choose the ultraviolet boson mass renormalization $\delta m_\sigma^2=-2\int d^2p\dfrac{\chi_\sigma(p)^2}{p^2+m_0^2}$.

Theorem 3.1. *$(ZS)_{\kappa,\varrho,\sigma}(f_1,\dots,f_n;g_1,\dots,g_m;h_1,\dots,h_m)$ satisfy a bound of the form* (1.1), *for $\varrho\geqq\varrho_0=\varrho_0\left(\lambda,\dfrac{M}{\lambda}\right)$, with constants $c_i(\varrho,\lambda,M)$ uniform in κ and $\sigma\geqq\sigma_0(\varrho,\lambda,M)$, and converge as $\sigma,\kappa\to\infty$.*

Proof. Since $\|\prod_{i=1}^n\phi(f_i)\|_2\leqq c_1c_2^n(n!)^{\frac{1}{2}}\prod_{i=1}^n\|f_i\|_{-1}$, it is sufficient to show that $\det_{jk}S'_{\varrho,\sigma}(g_j,h_k;\phi_\kappa)D_{\varrho,\sigma}(\phi_\kappa)$ converges in $L_2(d\mu_0)$ and that for some $p\geqq2$,

$$\|\det_{jk}S'_{\varrho,\sigma}(g_j,h_k;\phi_\kappa)D_{\varrho,\sigma}(\phi_\kappa)\|_p\leqq c_1c_2^{2m}\textstyle\prod_{j=1}^m\|g_j\|_{-1}\|h_j\|_{-1}$$

for some constants c_i uniform in κ, σ and for ϱ, σ sufficiently large.

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[1] The kernel $B(p,q)$ of an integral operator B on the weight space $\mathscr{H}_s^{(m)}$ is defined by $(B\theta)(p)=\int d^2q(q^2+m^2)^sB(p,q)\theta(q)$, $\theta\in\mathscr{H}_s^{(m)}$.

[2] We use the notation $\det_p(1+B)\equiv\det((1+B)\exp(\sum_{n=1}^{p-1}(-)^nn^{-1}B^n))$ most commonly used in the literature, [6]. This differs from the notation of [2] which would use $\det_2$ instead of $\det_3$.

The first term in $D_{\varrho,\sigma}(\phi_\kappa)$ clearly converges in $L_p(d\mu_0)$ as $\sigma, \kappa \to \infty$ as is clear since $\mathrm{Tr}(K_{\kappa,0,\sigma} - K_{\kappa,\varrho,\sigma}) = \int dx g(x)\phi_\kappa(x)\langle \Psi_\varrho^{(2)}(x)\Psi_\varrho^{(1)}(x)\rangle$, while the second part of its exponent is a constant in q space. We also easily see the second term in $D_{\varrho,\sigma}(\phi_\kappa)$ converges in L_p, $p \geqq 1$, for $\varrho \geqq \varrho_1\left(\frac{M}{\lambda}\right)$ and $\sigma \geqq \sigma_0(\varrho, \lambda, M, p)$, as $\sigma, \kappa \to \infty$. To prove this, note that its exponent may be written as $\frac{\lambda^2}{2}:(\phi, G_{\kappa,\varrho,\sigma}\phi)_1:$. Here $G_{\kappa,\varrho,\sigma}$ is the integral operator on $\mathscr{H}_1$ with kernel:

$$G_{\kappa,\varrho,\sigma}(p,q) \equiv (2\pi)^{-2}\chi_\kappa(p)\chi_\kappa(q)\mu(p)^{-2}\mu(q)^{-2}\int d^2k G_{\varrho,\sigma}(k)\tilde{g}(p-k)\tilde{g}(k-q),$$

$$G_{\varrho,\sigma}(k) \equiv 2\int d^2l\left\{\frac{\chi_\sigma(l+\frac{1}{2}k)\chi_\sigma(l-\frac{1}{2}k)(1-\theta_\varrho(l+\frac{1}{2}k)\theta_\varrho(l-\frac{1}{2}k))}{\omega(l+\frac{1}{2}k)\omega(l-\frac{1}{2}k)} - \frac{\chi_\sigma(l)^2}{\omega(l)^2}\right\} + \frac{M^2}{\lambda^2}$$

where $\mu(p) = (p^2+\mu_0^2)^{\frac{1}{2}}$, $\omega(p) = (p^2+m_0^2)^{\frac{1}{2}}$.

It is easily shown that the function $G_{\varrho,\sigma}(k)$ satisfies:

Lemma 3.2. (i) $G_{\varrho,\sigma}(k) \to G_\varrho(k) \equiv G_{\varrho,\infty}(k)$ *for each fixed* k, ϱ;

(ii) $|G_{\varrho,\sigma}(k)| \leqq \mathrm{const}\ \ln\left(1+\frac{k^2}{m_0^2}\right) + \mathrm{const}$ *uniformly in* σ *and*

(iii) $G_\varrho(k) < 0$ *if* $\varrho \geqq \varrho_1\left(\frac{M}{\lambda}\right)$.

These conditions are sufficient to ensure that $G_{\kappa,\varrho,\sigma}$ are Hilbert-Schmidt operators converging to $G_{\kappa,\varrho}$ in the Hilbert-Schmidt norm, uniformly in κ, as $\sigma \to \infty$, and to $G_\varrho \equiv G_{\infty,\varrho}$ as $\kappa \to \infty$. Also $G_{\kappa,\varrho} < 0$ if $\varrho \geqq \varrho_1\left(\frac{M}{\lambda}\right)$ and thus $G_{\kappa,\varrho,\sigma} < \frac{1}{2\lambda^2 p}$ if $\varrho \geqq \varrho_1\left(\frac{M}{\lambda}\right)$ and $\sigma \geqq \sigma_1 = \sigma_1(\varrho, \lambda, M, p)$. If follows that $e^{\lambda^2/2:(\phi, G_{\kappa,\varrho,\sigma}\phi)_1:}$ converges a.e. in q-space and that (since $p\lambda^2 G_{\kappa,\varrho,\sigma} < \frac{1}{2}$)

$$\|e^{\frac{\lambda^2}{2}:(\phi, G_{\kappa,\varrho,\sigma}\phi)_1:}\|_p^p = \frac{1}{\sqrt{\det_2(1-p\lambda^2 G_{\kappa,\varrho,\sigma})}}$$
$$\leqq e^{\|p\lambda^2 G_{\kappa,\varrho,\sigma}\|_2^2} \leqq \mathrm{const},$$

if $\varrho \geqq \varrho_1$ and $\sigma \geqq \sigma_1$. L_p convergence follows immediately by uniform integrability.

Finally we come to

$$X_{\kappa,\varrho,\sigma}(\phi) \equiv \det_{jk} S'_{\varrho,\sigma}(g_j, h_k; \phi_\kappa) e^{\frac{\lambda^2}{4}:\mathrm{Tr}(K^*_{\kappa,\varrho,\sigma}+K_{\kappa,\varrho,\sigma})^2:} \det_3(1+\lambda K_{\kappa,\varrho,\sigma}),$$
$$= (\Psi, \bigotimes^m(1+\lambda K_{\kappa,\varrho,\sigma})^{-1}\Phi)_{\Lambda^m\mathscr{H}}$$
$$e^{-\frac{\lambda^2}{4}:\mathrm{Tr}(K^*_{\kappa,\varrho,\sigma}+K_{\kappa,\varrho,\sigma})^2:} \det_3(1+\lambda K_{\kappa,\varrho,\sigma}) \tag{3.1}$$

where $\Psi = D_0 g_1 \Lambda \ldots \Lambda D_0 g_m$, $\Phi = S_0 g_1 \Lambda \ldots \Lambda S_0 g_m$ are vectors in the m-fold antisymmetric tensor product $\Lambda^m\mathscr{H}$ of $\mathscr{H} \equiv \mathscr{H}_{\frac{1}{2}} \otimes C^2$. Here both D_0 and S_0 are isometries from $\mathscr{H}^*$ to $\mathscr{H}$. It is easily seen that $K_{\kappa,\varrho,\sigma} \in C_3$ and converges in the C_3 norm as $\kappa, \sigma \to \infty$, for almost all ϕ; Seiler shows that this is true in the C_4 norm and a simple interpolation argument gives the result for the C_3 norm also, see the appendix.

It follows immediately that $\bigotimes^m(1+\lambda K_{\kappa,\varrho,\sigma})^{-1}\det_3(1+\lambda K_{\kappa,\varrho,\sigma})$ converges in norm in $\Lambda^m\mathscr{H}$ as $\sigma,\kappa\to\infty$, for almost all ϕ. Similarly $K^*_{\kappa,\varrho,\sigma}+K_{\kappa,\varrho,\sigma}$ converges in Hilbert Schmidt norm as $\sigma\to\infty$ and by arguments similar to those for $:(\phi, G_{\kappa,\varrho,\sigma}\phi)_1:$ one shows that $:\mathrm{Tr}(K^*_{\kappa,\varrho,\sigma}+K_{\kappa,\varrho,\sigma})^2:$ converges a.e. as $\sigma,\kappa\to\infty$. Thus we have shown that $X_{\kappa,\varrho,\sigma}(\phi)$ converges a.e. as $\sigma,\kappa\to\infty$.

To obtain a uniform L_p bound, we note that by (3.1)

$$|X_{\kappa,\varrho,\sigma}(\phi)| \leqq \prod_{j=1}^m \|g_j\|_{-\frac{1}{2}}\|h_j\|_{-\frac{1}{2}} e^{-\frac{\lambda^2}{4}:\mathrm{Tr}(K^*_{\kappa,\varrho,\sigma}+K_{\kappa,\varrho,\sigma})^2:} \|\bigotimes^m(1+\lambda K_{\kappa,\varrho,\sigma})^{-1}\det_3(1+\lambda K_{\kappa,\varrho,\sigma})\|\,.$$

By an elementary generalization of the Carleman inequalities [6] one has

Lemma 3.3. $\|\bigotimes^m(1+\lambda K_{\kappa,\varrho,\sigma})^{-1}\det_3(1+\lambda K_{\kappa,\varrho,\sigma})\|_{\Lambda^m\mathscr{H}} \leqq e^{\frac{m}{2}+\frac{1}{4}\mathrm{Tr}(K^*_{\kappa,\varrho,\sigma}+K_{\kappa,\varrho,\sigma})^2}$,

with immediately gives an upper bound uniform in σ:

$$\begin{aligned}|X_{\kappa,\varrho,\sigma}(\phi)| &\leqq \prod_{j=1}^m \|g_j\|_{-\frac{1}{2}}\|h_j\|_{-\frac{1}{2}} e^{\frac{m}{2}+\frac{\lambda^2}{4}\langle\mathrm{Tr}(K^*_{\kappa,\varrho,\sigma}+K_{\kappa,\varrho,\sigma})^2\rangle} \\ &\leqq \prod_{j=1}^m \|g_j\|_{-\frac{1}{2}}\|h_j\|_{-\frac{1}{2}} e^{\frac{m}{2}} e^{\mathrm{const}\,\ln\left(1+\frac{\kappa}{\mu_0}\right)} \in L_p(d\mu_0)\,.\end{aligned}$$

Thus $X_{\kappa,\varrho,\sigma}(\phi)$ converges in $L_p(d\mu_0)$ to $X_{\kappa,\varrho}(\phi)$ and we need only demonstrate a bound uniform in κ. However this is essentially what was done by Seiler, [2], for the case $\varrho=0$, and his method of proof extends immediately to $\varrho\neq 0$. This completes the proof of the $L_p(d\mu_0)$ convergence of $X_{\kappa,\varrho,\sigma}(\phi)$ and thus also of Theorem 3.1.

We remark that we could equally well have considered a pseudoscalar Yukawa_2 theory or a more general mass counterterm such as $M\int dx f(x):\phi^2(x):$, $|f|\leqq g$.

Note: After submitting this paper for publication, I received a manuscript from E. Seiler and B. Simon proving essentially the same result, [7].

Appendix

We will show that the operators $K=K(\phi_\kappa)\in C_p$ for any $p>2$. We introduce operators $K^{(\alpha)}$ on $\mathscr{H}$ with kernels obtained by replacing $(p^2+m_0^2)^{-1}$ by $(p^2+m_0^2)^{-\alpha}$ in the definition of $K(p,q)$, and the operators $I_B^{(\alpha)}$, $0\leqq B<\infty$, of multiplication by $(p^2+m_0^2)^\alpha\theta((B-|p|)\operatorname{sgn}\alpha)$. Thus

$$K=K^{(1)}=I_B^{(-\alpha)}K^{(1-\alpha)}+I_B^{(\beta)}K^{(1+\beta)}\,,\qquad -\tfrac{1}{4}<\alpha,\beta<\tfrac{1}{4}\,.$$

Since the characteristic numbers $\mu_n(C)$ (the eigenvalues of the operator $|C|$ in decreasing order) of a compact operator satisfy $\mu_n(AC)\leqq\|A\|\mu_n(C)$, $\mu_{2n+1}(C_1+C_2)\leqq\mu_n(C_1)+\mu_n(C_2)$, we obtain:

$$\mu_{2n+1}(K)\leqq(B^2+m_0^2)^{-\alpha}\|K^{(1-\alpha)}\|_4 n^{-\frac{1}{4}}+(B^2+m_0^2)^\beta\|K^{(1+\beta)}\|_2 n^{-\frac{1}{2}}\,, \tag{A.1}$$

where we have used $\|I_B^{(\alpha)}\|=(B^2+m_0^2)^\alpha$ and $\mu_n(C)\leqq\|C\|_p n^{-\frac{1}{p}}, C\in C_p$. We now choose B, depending on n, to maximize (A.1). Thus, for n sufficiently large, we define $B^2+m^2=(n^{\frac{1}{4}}\|K^{(1-\alpha)}\|_4/\|K^{(1+\beta)}\|_2)^{1/(\alpha+\beta)}$ and thus by (A.1):

$$\mu_{2n+1}(K)\leqq\mathrm{const}\,\|K^{(1-\alpha)}\|_4^{\beta/(\alpha+\beta)}\|K^{(1+\beta)}\|_2^{\alpha/(\alpha+\beta)}n^{-\frac{1}{2}+\frac{1}{4}\beta/(\alpha+\beta)}\,. \tag{A.2}$$

By studying the corresponding Feyman diagrams, it is clear that $K^{(1-\alpha)} \in C_4$, $K^{(1+\alpha)} \in C_2$ for $\kappa > \infty$, $0 < \alpha < \frac{1}{4}$. It follows from (A.2) that $K(\phi_\kappa) \in C_p$, $p > 2$, $\kappa < \infty$, for we need only choose $\alpha/\beta > (4-p)/(2p-4)$, and then

$$\|K\|_p \leqq \text{const} \, \|K^{(1-\alpha)}\|_4^{\beta/(\alpha+\beta)} \|K^{(1+\beta)}\|_2^{\alpha/(\alpha+\beta)} .$$

To discuss the limit $\kappa \to \infty$ we first consider the inequality corresponding to (A.1) for $(\int d\mu_0 \mu_{2n+1}(K)^p)^{1/p}$ and again choose B as a function of n to maximize this inequality. We conclude that $K(\phi_\kappa)$ converges in C_p, $p > 2$, to an operator $K(\phi)$ for almost all ϕ.

References

1. Matthews, P. T., Salam, A.: Nuovo Cim. **12**, 563—565 (1954)
2. Seiler, E.: Schwinger functions for the Yukawa model in two dimensions with space-time cutoff, I.A.S. preprint.
3. Glimm, J.: Commun. math. Phys. **5**, 343—386 (1967)
4. Glimm, J.: Commun. math. Phys. **6**, 120—127 (1967)
5. Osterwalder, K., Schrader, R.: Helv. Phys. Acta **46**, 277—302 (1973)
6. Dunford, N., Schwartz, J.: Linear operators. New York: Interscience Publishers
7. Seiler, E., Simon, B.: On finite mass renormalizations in the two-dimensional Yukawa model

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