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Spontaneous Breakdown in Two Dimensional Space-Time

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Abstract. We prove that in two-dimensional space-time, symmetry transformations which are generated by Poincaré covariant currents can not be spontaneously broken. This is also the case with the dilation current. We argue that other currents which involve explicit space-time dependence might lead to spontaneously broken symmetries accompanied by massless Goldstone bosons. We construct a trivial example where this phenomenon occurs.

Introduction and Statement of the Results

In Ref. [1] it was shown that in two dimensional space-time, the following result holds: Let $\phi(x)$ be a scalar field and $j^\mu(x)$ a conserved vector field, then

$$\int dx^1(\Omega, [j^0(x), \phi(y)]\Omega) = 0 \tag{1}$$

where Ω is the vacuum state. This is an indication that in two dimensional space-time, a symmetry transformation which is generated by $j^\mu(x)$ is always an exact one and cannot be spontaneously broken. There still exists the question what happens if one takes in Eq. (1) instead of $\phi(y)$ an arbitrary polynomial in tensorial and spinorial fields, or if one considers currents with different transformation properties under the Poincaré group. We discuss these questions here.

The framework is Wightman field theory [2] in two dimensional space-time. Let A denote any polynomial in the smeared fields where the smearing functions are infinitely differentiable and of compact support. Let $j^\mu(x)$ denote a set of four fields satisfying a conservation equation

$$\partial_\mu j^\mu(x) = 0 . \tag{2}$$

The Goldstone theorem states that [3]

$$\int dx^1(\Omega, [j^0(x), A]\Omega) = \int dx^1(\Omega, [j^0(x)E_0 A - AE_0 j^0(x)]\Omega) \tag{3}$$

where E_0 is the projection on the subspace of zero mass states. In Eq. (3), $j^\mu(x)$ does not have to be a vector field, actually, it can even contain an explicit space-time dependence through polynomials in x^μ (see Ref. [4]). For instance, one can take the dilation current

$$d^\mu(x) = x_\nu T^{\mu\nu}(x) \tag{4}$$

or the special conformal currents

$$c^{\mu\nu}(x) = (2x^\nu x_\lambda - \delta^\nu_\lambda x^2) T^{\mu\lambda}(x) \tag{5}$$

where

$$T^{\mu\nu}=T^{\nu\mu}\ ; \qquad \partial_\mu T^{\mu\nu}=0\ ; \qquad T^\mu_\mu=0\,. \tag{6}$$

Spontaneous breakdown is a situation in which the left hand side of Eq. (3) does not vanish for at least one A. Then, according to Eq. (3), $E_0 j^0(x)\Omega \neq 0$ and the spectrum contains massless particles - the Goldstone bosons.

We prove here the following results (Positive metric is assumed):

1. Let $t^{\mu m}(x)$ be a Hermitian and Poincaré covariant conserved tensor field

$$U(a,\Lambda)t^{\mu m}(x)U^{-1}(a,\Lambda)=(\Lambda^{-1})^\mu_\nu D^m_n(\Lambda^{-1})t^{\nu n}(\Lambda x+a)\ ; \qquad \partial_\mu t^{\mu m}(x)=0\,. \tag{7}$$

Here $U(a,\Lambda)$ is the unitary representation of the proper Poincaré group in the Hilbert space of states and $D(\Lambda)$ is a finite dimensional one valued matrix representation of the proper Lorentz group. In two dimensional space-time, one has for every A

$$\int dx^1(\Omega,[t^{0m}(x),A]\Omega)=0\,. \tag{8}$$

2. Let $d^\mu(x)$ satisfy Eq. (4) where $T^{\mu\nu}(x)$ is a Poincaré covariant Hermitian second rank tensor field, satisfying Eq. (6). Then, in two dimensional space-time, for every A one has

$$\int dx^1(\Omega,[d^0(x),A]\Omega)=0\,. \tag{9}$$

Results 1 and 2 exclude the possibility that, in two dimensional space-time, symmetry transformations which are generated by currents of the form $t^{\mu m}(x)$ and $d^\mu(x)$ might be spontaneously broken. This is in contrast to the situation in four dimensional space-time (see Ref. [5] for result 1 and Ref. [6] for result 2)

These results still do not exclude the possibility that symmetry transformation, which are generated by currents not covariant under translations, but have a different form than $d^\mu(x)$, might be spontaneously broken. Massless Goldstone bosons should then appear in the spectrum. In fact, a current of the form of $c^{\mu\nu}(x)$ of Eq. (5) might be spontaneously broken. To see this, let us consider a theory of a free Hermitian vector field $j^\mu(x)$ which is defined through the two point function

$$(\Omega,j^\mu(x)j^\nu(y)\Omega)=\int_{p_0\geqq 0}d^2p e^{-ip(x-y)}p^\mu p^\nu\delta(p^2)\,. \tag{10}$$

From Eq. (10) it follows that $j^\mu(x)$ satisfies (one uses theorem 4–3 of Ref. [2])

$$\partial^\mu j^\nu(x)=\partial^\nu j^\mu(x)\ ; \qquad \partial_\mu\partial^\mu j^\nu(x)=0\ ; \qquad \partial_\mu j^\mu(x)=0\,. \tag{11}$$

Therefore $\partial^\mu j^\nu(x)$ can be used to construct a conserved current of the form of Eq. (5)

$$c^{\mu\nu}(x)=(2x^\nu x_\lambda-\delta^\nu_\lambda x^2)\partial^\mu j^\lambda(x)\,. \tag{12}$$

Explicit calculation gives

$$\int dx^1(\Omega,[c^{0\nu}(x),j^\lambda(f)]\Omega)=4\pi i(\delta^\nu_0\delta^\lambda_0-\delta^\nu_1\delta^\lambda_1)\int d^2x f(x)\neq 0\,. \tag{13}$$

Proof of the Results

We follow the technique of Maison and Reeh [5]. Let us define the smearing functions $\theta_r(x^1)$ and $\eta(x^0)$ to be infinitely differentiable real functions of compact support satisfying

$$\theta_r(x^1) = \theta\left(\frac{x^1}{r}\right) = \begin{cases} 1, \left|\frac{x^1}{r}\right| < 1 \\ 0, \left|\frac{x^1}{r}\right| > 2 \end{cases} ; \quad \int dx^0 \eta(x^0) = 1 \tag{14}$$

Let us prove the following lemma, which is a generalization of lemma 2 of Ref. [5]

Lemma. *Let $j^\mu(x)$ be a conserved Hermitian current which may involve explicit space-time dependence through polynomials in x. $j^\mu(x)$ need not be a vector field or a Lorentz covariant field. Then the condition*

$$\lim_{r\to\infty} (\Omega, j^0(\theta_r\eta) E_0 j^0(\theta_r\eta)\Omega) = b < \infty \tag{15}$$

yields

$$\int dx^1 (\Omega, [j^0(x), A]\Omega) = 0 . \tag{16}$$

Positivity of the Hilbert space metric is assumed.

Proof. Under the assumptions of the lemma, it follows that $E_0 j^0(\theta_r\eta)\Omega$ converges weakly to a vector χ as $r\to\infty$, i.e., for every vector ψ

$$\lim_{r\to\infty} (\psi, E_0 j^0(\theta_r\eta)\Omega) = (\psi, \chi) . \tag{17}$$

Let us first show that $P^\mu\chi = 0$. Current conservation gives

$$\partial_\mu j^\mu(x) = \delta_\mu j^\mu(x) + i[P_\mu, j^\mu(x)] = 0 \tag{18}$$

where δ_μ denotes derivation with respect to a possible explicit x dependence, which is assumed to be of polynomial type. Equation (18) and locality give that for r large enough

$$[\delta_0 j^0(\theta_r\eta), A] = -i[[P_0, j^0(\theta_r\eta)], A] . \tag{19}$$

Repeated application of Eq. (19) yields

$$[\delta_0^{2n+1} j^0(\theta_r\eta), A] = (-i)^{2n+1}[[P_0, \ldots [P_0, j^0(\theta_r\eta)]\ldots], A] \tag{20}$$

For n large enough, Eq. 20 vanishes. This is due to the polynomial dependence on x^0. Taking Eq. 20 between two Ω yields then

$$(\Omega, [j^0(\theta_r\eta) P_0^{2n+1} A + A P_0^{2n+1} j^0(\theta_r\eta)]\Omega) = 0$$

or

$$(\Omega, [j^0(\theta_r\eta), [P_0, \ldots [P_0, A]\ldots]]\Omega) = 0 .$$

Using the Goldstone theorem [Eq. (3)] this yields

$$(\Omega, [j^0(\theta_r\eta) E_0 P_0^{2n+1} A + A P_0^{2n+1} E_0 j^0(\theta_r\eta)]\Omega) = 0 .$$

Using $E_0 = E_0^2$ and taking $r\to\infty$ this gives

$$(\chi, E_0 P_0^{2n+1} A\Omega) + (\Omega, A P_0^{2n+1} E_0 \chi) = 0 .$$

Let us take $A=j^0(\theta_r\eta)$ and take $r\to\infty$. Using $E_0P_0=P_0E_0$ one gets

$$(P_0^n\chi, P_0P_0^n\chi)=0\,.$$

As zero is the minimum value for P_0 this yields

$$P_0P_0^n\chi=0$$

by multiplying with χ from the left and using similar arguments, one finally gets

$$P_0\chi=0\,.$$

Therefore

$$0\leqq(\chi, P_\mu P^\mu\chi)=-(P^1\chi, P^1\chi)\leqq 0$$

and also

$$P^1\chi=0\,. \tag{21}$$

The general form of $j^0(x)$ is

$$j^0(x)=\sum_{mn}(x^0)^m(x^1)^n\phi_{mn}(x)$$

where $\phi_{mn}(x)$ are local fields. Using translation invariance of χ and Ω one gets

$$\begin{aligned}(\chi,\chi)&=\lim_{r\to\infty}(\chi, E_0j^0(\theta_r\eta)\Omega)\\ &=\lim_{r\to\infty}\sum_{mn}(\chi, E_0\phi_{mn}(0)\Omega)\int dx^0(x^0)^m\eta(x^0)\cdot r^{n+1}\int dx^1(x^1)^n\theta(x^1)\,.\end{aligned} \tag{22}$$

Expression of the form of Eq. (22) is either zero or divergent. As (χ,χ) is finite, it should vanish and one concludes that

$$\chi=0\,.$$

Therefore

$$\begin{aligned}\int dx^1(\Omega, [j^0(x), A]\Omega)&=\lim_{r\to\infty}(\Omega, [j^0(\theta_r\eta)E_0A-AE_0j^0(\theta_r\eta)]\Omega)\\ &=(\chi, A\Omega)-(\Omega, A\chi)=0\,.\end{aligned}$$

In order to prove result 1, one has to show that Eq. (15) is satisfied for $j^0(x)=t^{0m}(x)$. Poincaré covariance implies that

$$(\Omega, t^{\mu m}(x)E_0t^{\nu n}(y)\Omega)=\int_{p_0\geqq 0}d^2pe^{-ip(x-y)}\delta(p^2)P^{\mu m\nu n}(p)$$

where $P^{\mu m\nu n}$ is an even polynomial in p. Since $\delta(p^2)$ is not a permissable distribution in two dimensional space-time (see also the example in Ref. [1]), $P^{\mu m\nu n}$ is at least of second degree in p. Using Eq. (14) one gets

$$\begin{aligned}(\Omega, t^{0m}(\theta_r\eta)E_0t^{0m}(\theta_r\eta)\Omega)&=\int dp^1|\tilde\theta_r(p^1)|^2|\tilde\eta(|p^1|)|^2\frac{1}{2|p^1|}P^{0m0m}(|p^1|, p^1)\\ &=\int dp^1|\tilde\theta(p^1)|^2\left|\tilde\eta\left(\left|\frac{p^1}{r}\right|\right)\right|^2\frac{r^2}{2|p^1|}P^{0m0m}\left(\left|\frac{p^1}{r}\right|, \frac{p^1}{r}\right)\xrightarrow[r\to\infty]{} b<\infty\end{aligned}$$

where, in the last step, use is made of the fact that the degree of the polynomial is at least 2, and therefore there is $1/r^2$ factor which cancels the r^2.

Let us now consider $d^{\mu}(x)$

$$d^0(x)=x^0T^{00}(x)-x^1T^{01}(x)$$

therefore

$$(\Omega, d^0(x)E_0d^0(y)\Omega)=x^0y^0(\Omega, T^{00}(x)E_0T^{00}(y)\Omega)-x^0y^1(\Omega, T^{00}(x)E_0T^{01}(y)\Omega)$$
$$-x^1y^0(\Omega, T^{01}(x)E_0T^{00}(y)\Omega)+x^1y^1(\Omega, T^{01}(x)E_0T^{01}(y)\Omega).$$

For $T^{\mu\nu}(x)$ which satisfies Eq. (6) one has

$$(\Omega, T^{\mu\nu}(x)E_0T^{\lambda\sigma}(y)\Omega)=\int_{p_0\geqq 0}d^2pe^{-ip(x-y)}\delta(p^2)P^{\mu\nu\lambda\sigma}(p)$$

where

$$P^{\mu\nu\lambda\sigma}(p)=a_1(p^{\mu}p^{\lambda}g^{\nu\sigma}+p^{\mu}p^{\sigma}g^{\nu\lambda}+p^{\nu}p^{\sigma}g^{\mu\lambda}+p^{\nu}p^{\lambda}p^{\mu\sigma}-2p^{\lambda}p^{\sigma}g^{\mu\nu}-2p^{\mu}p^{\nu}g^{\lambda\sigma})$$
$$+a_2p^{\mu}p^{\nu}p^{\lambda}p^{\sigma}.$$

Calculation of the four terms of $(\Omega, d^0(\theta_r\eta)E_0d^0(\theta_r\eta)\Omega)$ yields that the first three terms vanish as $r\to\infty$, whereas the last one equals

$$\int dp^1|\partial\theta_r(p^1)|^2|\tilde{\eta}(|p^1|)|^2\frac{1}{2|p^1|}a_2|p^1|^4=\int dp^1|\partial\tilde{\theta}(p^1)|^2\left|\tilde{\eta}\left(\left|\frac{p^1}{r}\right|\right)\right|^2\frac{a_2}{2}|p_1|^3\xrightarrow[r\to\infty]{}b<\infty$$

Equation (15) is therefore satisfied for $j^0(x)=d^0(x)$ and result 2 follows.

Note: Part of the results were independently proven by Prof. H. Reeh (private communication).

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Note Added in Proof. In the proof of the lemma, use is made of the fact that P^{μ} is defined on χ. This can be justified by noting that if Eq. 15 holds, then also $\lim_{r\to\infty}(P^{\mu}E_0j^0(\theta_r\eta)\Omega, P^{\mu}E_0j^0(\theta_r\eta)\Omega)=c<\infty$. This is due to the fact that multiplication by P^{μ} produces p^{μ} factors which improve the convergence. Actually $c=\infty$ and this fact can be used to show directly that $P^{\mu}\chi=0$.