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The $:\phi^2:$ Field in the $P(\phi)_2$ Model

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Abstract. Euclidean Field Theory techniques are used to study the Schwinger functions and characteristic function of the $:\phi^2:$ field in even $P(\phi)_2$ models. The infinite volume limit is obtained for Half-Dirichlet boundary conditions by means of correlation inequalities. Analytic continuation yields Lorentz invariant Wightman functions. It is shown that, in the infinite volume limit, $\langle :\phi(x)^2: \rangle \geqq 0$ for both the Half and the Full-Dirichlet $(\lambda\phi^4)_2$ model. This result also holds for a finite volume with periodic boundary conditions.

1. Introduction

The path space approach to the self-interacting scalar Bose field in two space-time dimensions involves the introduction of the free Euclidean field ϕ which can be viewed (Nelson [1]) as the generalized Gaussian stochastic process $\phi(x)$ with mean zero and covariance

$$S(x-y) = \langle \phi(x)\,\phi(y) \rangle = \int \frac{d^2(p)}{(2\pi)^2} \frac{e^{ip\cdot x}}{p^2+m^2}\,. \tag{1}$$

The Schwinger functions associated with the $P(\phi)$ interaction in the open bounded region $\Lambda \subset R^2$ are given by

$$\langle \phi(f_1)\ldots\phi(f_n) \rangle_\Lambda = \frac{\int \phi(f_1)\ldots\phi(f_n)\, e^{-\int_\Lambda :P(\phi(x)):\, d^2(x)}\, d\mu_0}{\int e^{-\int_\Lambda :P(\phi(x)):\, d^2(x)}\, d\mu_0}, \tag{2}$$

where $\operatorname{supp} f_i \subset \Lambda$, $\phi(f) = \int d^2(x)\,\phi(x) f(x)$, and μ_0 is the free Gaussian measure, i.e. the Gaussian measure associated with the free Euclidean field ϕ. One is then interested in the Schwinger functions in the infinite volume limit $\Lambda \to R^2$. In the case of small coupling constant, the Glimm-Jaffe-Spencer [2] cluster expansion is a powerful tool for studying this infinite volume limit. If, however, it is desired to obtain results independent of the magnitude of the coupling constant, then correlation inequalities of Griffiths' type become the primary tool (Guerra, Rosen, and Simon [3], see also Simon [4]). This method is essentially restricted to even polynomial interactions and we assume this in the following.

Little is known about correlation inequalities for Wick powers (see [3]; for small coupling Wick powers have been studied by Schrader [12]) so it is of some interest to explore the properties of even the simplest Wick power, $:\phi^2:$. Indeed,

$:\phi^2:$ differs from ϕ^2 only by an "infinite constant" and thus it is reasonable to expect that methods useful for studying the field ϕ should also be applicable to $:\phi^2:$. Higher Wick powers involve polynomials in ϕ containing infinite negative coefficients and the application of correlation inequalities in these cases appears to be considerably more difficult.

We briefly summarize here the Dirichlet boundary conditions and the lattice approximation. For details we refer to Refs. [3] and [4]. The covariance $S(x-y)$ given by Eq. (1) satisfies

$$(-\Delta + m^2)\, S(x-y) = \delta(x-y)\,. \tag{3}$$

Let $S_{\Lambda,D}(x,y)$ be the solution to (3), for $x, y \in \Lambda$, that vanishes on the boundary of Λ. The Euclidean field with Dirichlet boundary conditions on Λ is the generalized Gaussian stochastic process with mean zero and covariance $S_{\Lambda,D}(x,y)$. Let $d\mu_{\Lambda,D}$ be the corresponding Gaussian measure, called the Dirichlet measure for the region Λ. The Dirichlet boundary condition decouples regions, hence its usefulness in the investigation of the infinite volume limit. The Half-Dirichlet Schwinger functions $\langle\phi(f_1)\ldots\phi(f_n)\rangle_\Lambda^{HD}$ are defined as in Eq. (2) with $d\mu_{\Lambda,D}$ in place of $d\mu_0$. For the Full-Dirichlet Schwinger functions $\langle\phi(f_1)\ldots\phi(f_n)\rangle_\Lambda^D$, the Wick ordering : : defined with respect to the free covariance $S(x-y)$ is replaced by the Wick ordering : $:_D$ defined with respect to the Dirichlet covariance $S_{\Lambda,D}(x,y)$ (see [3]). We shall always assume Λ to be sufficiently smooth (regular and log-normal [3]).

In deriving and applying the Griffiths inequalities the lattice approximation is used. For each $\delta > 0$ one considers the lattice $\{n\delta\}$ in R^2, where $n = (n_1, n_2)$, $n\delta = (n_1\delta, n_2\delta)$ and $n_i = 0, \pm 1, \pm 2, \ldots$; let $|n| = |n_1| + |n_2|$. For each lattice point there is a lattice field $\phi^\delta(n\delta)$. The lattice fields are Gaussian random variables with mean zero and covariance $S^\delta(n,n') = \langle\phi^\delta(n\delta)\,\phi^\delta(n'\delta)\rangle$ in the free boundary condition case, and $S^\delta_{\Lambda,D}(n,n')$ for Dirichlet boundary conditions on Λ, where

$$[S^\delta_{\Lambda,D}]^{-1}(n,n') = A^\delta(n,n') = \begin{cases} 4 + m^2\delta^2 & \text{if } n = n' \\ -1 & \text{if } |n-n'| = 1 \\ 0 & \text{otherwise}\,. \end{cases}$$

The restriction of S^δ to lattice points in the region Λ satisfies $[S^\delta \restriction \Lambda]^{-1}(n,n') = A^\delta(n,n') - B^\delta(n,n')$ where B is a positive definite matrix with non-negative elements (see [3]). In the lattice approximation $\phi(h)$ is replaced by $\sum\limits_n \delta^2 \phi^\delta(n\delta)\, h(n\delta)$, and $\int\limits_\Lambda :P(\phi(x)): d^2(x)$ by $\sum\limits_{n\delta\in\Lambda} \delta^2 :P(\phi^\delta(n\delta)):$, the Wick ordering here being with respect to the lattice covariance S^δ_Λ. The lattice fields suitably approximate the corresponding continuum fields as $\delta \to 0$ in the sense that the lattice Schwinger functions converge to the continuum Schwinger functions. Given an open bounded region Λ, let $q_1, \ldots, q_N$ be the lattice fields $\phi^\delta(n\delta)$; $n\delta \in \Lambda$. With the $P(\phi)$ interaction, the joint distribution of the lattice fields has the form

$$dP^\delta_\Lambda = \eta\, e^{\sum\limits_{i\,j=1}^{N} (B_{ij} - A_{ij})\, q_i q_j} \prod_{i=1}^{N} F_i(q_i)\, dq_1 \ldots dq_N\,,$$

where $F_i = e^{-\delta^2 :P(q_i):}$ and η is a normalization constant. The Half-Dirichlet lattice measure $dP^\delta_{\Lambda,HD}$ is obtained from dP^δ_Λ by setting the matrix $B = 0$, and in addition for the Full-Dirichlet measure $dP^\delta_{\Lambda,D}$ the Wick ordering in $:P(q_i):$ is with respect to the Dirichlet covariance $S^\delta_{\Lambda,D}$. The measures dP^δ_Λ, $dP^\delta_{\Lambda,HD}$, and $dP^\delta_{\Lambda,D}$ are even ferromagnetic measures and expectations of $q^\alpha = q_1^{\alpha_1}\dots q_N^{\alpha_N}$ with respect to these measures satisfy the Griffiths inequalities [3]

$$\text{(G I)}\ \langle q^\alpha\rangle \geqq 0\,,$$

$$\text{(G II)}\ \langle q^\alpha q^\beta\rangle \geqq \langle q^\alpha\rangle\langle q^\beta\rangle\,.$$

It follows by an argument of Nelson [5] that

$$\langle\phi(x_1)\dots\phi(x_n)\rangle^{HD}_\Lambda \leqq \langle\phi(x_1)\dots\phi(x_n)\rangle_\Lambda$$

and $\langle\phi(x_1)\dots\phi(x_n)\rangle^{HD}_\Lambda$ is monotone increasing in Λ. The Half-Dirichlet Schwinger functions are uniformly bounded (Frohlich [6]) independently of Λ and thus the limit $\Lambda\to R^2$ can be taken. The resulting Schwinger functions satisfy the Osterwalder-Schrader axioms and thus yield Wightman functions by analytic continuation (see [6, 4]). This also holds for the Full-Dirichlet Schwinger functions, in the case of a ϕ^4 interaction ([3, 4]). In the following sections we obtain similar results for the Wick square $:\phi(x)^2:$, and in the last section we prove that, in the infinite volume limit, $\langle:\phi(x)^2:\rangle \geqq 0$ for both the Half and the Full-Dirichlet $(\lambda\phi^4)_2$ model. This result also holds for a finite volume with periodic boundary conditions.

2. Basic Inequalities

Let $:\phi^2:(g)$ denote $\int :\phi(x)^2: g(x)\,d^2(x)$ where $:\phi(x)^2:$ is Wick ordered with respect to the free covariance $S(x-y)$. Let $\mathscr{D}$ denote the set of infinitely differentiable functions with compact support, $\mathscr{D}(\Lambda)$ the functions in $\mathscr{D}$ with support in the open set Λ, and let $\mathscr{S}$ denote the set of infinitely differentiable functions which decrease faster than any inverse power at infinity.

The Griffiths' inequalities yield the following theorems:

Theorem 1[1]. *Let* $g\in\mathscr{D}(\Lambda)$. *Then*

$$\langle e^{:\phi^2:(g)}\rangle^{HD}_\Lambda \leqq \langle e^{:\phi^2:(g)}\rangle_\Lambda \quad \textit{if} \quad g\geqq 0$$

$$\langle e^{:\phi^2:(g)}\rangle^{HD}_\Lambda \geqq \langle e^{:\phi^2:(g)}\rangle_\Lambda \quad \textit{if} \quad g\leqq 0\,.$$

Proof. Take $g\geqq 0$. In the lattice approximation $\langle e^{\sigma:\phi^2:(g)}\rangle^{HD}_\Lambda$ is given by

$$e^{-\sigma S^\delta\Sigma g_i}\int e^{\sigma\Sigma g_i q_i^2}\,dP^\delta_{\Lambda,HD}\,,$$

where $S^\delta = S^\delta(n,n)$. It suffices to show that

$$\frac{\int e^{\sigma\Sigma q_i^2 g_i}\,e^{\gamma\Sigma B_{ij}q_iq_j}\,dP^\delta_{\Lambda,HD}}{\int e^{\gamma\Sigma B_{ij}q_iq_j}\,dP^\delta_{\Lambda,HD}}$$

is increasing in γ if $\sigma>0$ and decreasing in γ if $\sigma<0$.

[1] Here and throughout this paper we assume P has degree $\geqq 4$. The results are still true in case P is of second degree but one must then be careful about the integrability of $e^{:\phi^2:(g)}$, it suffices to notice $\langle e^{\lambda:\phi^2:(g)}\rangle_\Lambda<\infty$ for $|\lambda|$ sufficiently small.

The derivative of this expression with respect to γ is

$$\frac{\int e^{\sigma\Sigma q_i^2 g_i}\Sigma B_{ij}q_iq_j e^{\gamma\Sigma B_{ij}q_iq_j} dP^\delta_{\Lambda,HD}}{\int e^{\gamma\Sigma B_{ij}q_iq_j}\, dP^\delta_{\Lambda,HD}} - \frac{\int e^{\sigma\Sigma q_i^2 g_i} e^{\gamma\Sigma B_{ij}q_iq_j} dP^\delta_{\Lambda,HD} \int \Sigma B_{ij}q_iq_j e^{\gamma\Sigma B_{ij}q_iq_j} dP^\delta_{\Lambda,HD}}{(\int e^{\gamma\Sigma B_{ij}q_iq_j} dP^\delta_{\Lambda,HD})^2}$$

and this is positive or negative according to whether the following expression is positive or negative:

$$\langle \Sigma B_{ij}q_iq_j\rangle_\sigma - \langle \Sigma B_{ij}q_iq_j\rangle_\gamma \qquad (*)$$

where $\langle\ \rangle_\gamma$ is the expectation with respect to the measure

$$dP_\gamma = \frac{e^{\gamma\Sigma B_{ij}q_iq_j} dP^\delta_{\Lambda,HD}}{\int e^{\gamma\Sigma B_{ij}q_iq_j} dP^\delta_{\Lambda,HD}}$$

and $\langle\ \rangle_\sigma$ is the expectation with respect to the measure

$$dP_\sigma = \frac{e^{\sigma\Sigma q_i^2 g_i} dP_\gamma}{\int e^{\sigma\Sigma q_i^2 g_i} dP_\gamma}.$$

The derivative of the quantity $(*)$ with respect to σ is

$$\langle(\Sigma g_iq_i^2)(\Sigma B_{ij}q_iq_j)\rangle_\sigma - \langle\Sigma g_iq_i^2\rangle_\sigma \langle\Sigma B_{ij}q_iq_j\rangle_\sigma$$

which is non-negative by (G II) for the measure dP_σ. Taking the limit $\delta\to 0$ completes the proof of the theorem. □

Precisely the same sort of argument $\left(\text{with } \Sigma B_{ij}q_iq_j \text{ replaced by } -\sum_{\substack{i\in\Lambda \\ j\in\Lambda'\setminus\Lambda}} A_{ij}q_iq_j\right)$ leads to the following result:

Theorem 2. *Let $g\in\mathscr{D}(\Lambda)$. Then if $\Lambda\subset\Lambda'$*

$$\langle e^{:\phi^2:(g)}\rangle^{HD}_\Lambda \leqq \langle e^{:\phi^2:(g)}\rangle^{HD}_{\Lambda'} \quad \textit{if} \quad g\geqq 0$$
$$\langle e^{:\phi^2:(g)}\rangle^{HD}_\Lambda \geqq \langle e^{:\phi^2:(g)}\rangle^{HD}_{\Lambda'} \quad \textit{if} \quad g\leqq 0.$$

Thus the Half-Dirichlet expectation is monotone increasing if $g\geqq 0$ and *monotone decreasing* if $g\leqq 0$.

We consider now the Full-Dirichlet expectation $\langle e^{:\phi^2:_D(g)}\rangle^D_\Lambda$ in the case $P(\phi)=\lambda\phi^4$, where $:\phi^2:_D$ denotes Wick ordering with respect to the Dirichlet covariance $S_{\Lambda,D}$. As

$$:\phi^2:_D(g) = :\phi^2:(g) + \int_\Lambda [S(x-x) - S_{\Lambda,D}(x,x)]\, g(x)\, d^2(x),$$

$$S(x-x) - S_{\Lambda,D}(x,x) \in L_p(\Lambda, d^2(x)) \quad \text{for all} \quad p<\infty,$$

and

$$S(x-x) - S_{\Lambda,D}(x,x) \searrow 0 \quad \text{as} \quad \Lambda\nearrow R^2 \text{ [3]},$$

it follows that

$$\lim_{\Lambda\nearrow R^2} \langle e^{:\phi^2:_D(g)}\rangle^D_\Lambda = \lim_{\Lambda\nearrow R^2} \langle e^{:\phi^2:(g)}\rangle^D_\Lambda$$

if the limits exist (and if one exists, so does the other). It suffices therefore to consider $\langle e^{:\phi^2:(g)}\rangle^D_\Lambda$.

Theorem 1′. *Let $P(\phi)=\lambda\phi^4$, and suppose $g\in\mathscr{D}(\Lambda)$. Then*

$$\langle e^{:\phi^2:(g)}\rangle_\Lambda^D \leqq \langle e^{:\phi^2:(g)}\rangle_\Lambda^{HD} \leqq \langle e^{:\phi^2:(g)}\rangle_\Lambda \quad \text{if} \quad g\geqq 0$$

$$\langle e^{:\phi^2:(g)}\rangle_\Lambda^D \geqq \langle e^{:\phi^2:(g)}\rangle_\Lambda^{HD} \geqq \langle e^{:\phi^2:(g)}\rangle_\Lambda \quad \textit{if} \quad g\leqq 0 .$$

Proof. To prove the inequalities $\langle\ \rangle_{\Lambda}^{D} \lesseqqgtr \langle\ \rangle_\Lambda^{HD}$ use

$$dP_{\Lambda,HD}^\delta=\alpha e^{6\lambda\sum\delta^2(S^\delta-S_{\Lambda,D}^\delta)(i,i)q_i^2}dP_{\Lambda,D}^\delta .$$

where $S^\delta-S_{\Lambda,D}^\delta>0$ and α is a normalization constant. The proof then proceeds exactly as in Theorem 1. The remaining inequalities are a restatement of Theorem 1. □

Theorem 2′. *Let $P(\phi)=\lambda\phi^4$, and suppose $g\in\mathscr{D}(\Lambda)$. Then if $\Lambda\subset\Lambda'$*

$$\langle e^{:\phi^2:(g)}\rangle_\Lambda^D \leqq \langle e^{:\phi^2:(g)}\rangle_{\Lambda'}^D \quad \textit{if} \quad g\geqq 0$$

$$\langle e^{:\phi^2:(g)}\rangle_\Lambda^D \geqq \langle e^{:\phi^2:(g)}\rangle_{\Lambda'}^D \quad \textit{if} \quad g\leqq 0 .$$

Proof. As in the discussion of the Schwinger functions for the ϕ field given in Ref. [3], one proceeds in two stages. In the first stage one does not change the region Λ but the Wick ordering is changed to that appropriate to Λ':

$$dP_{\Lambda,\Lambda',D}^\delta=\alpha e^{6\lambda\sum(S_{\Lambda',D}^\delta-S_{\Lambda,D}^\delta)(i,i)q_i^2}dP_{\Lambda,D}^\delta$$

and notice $S_{\Lambda',D}^\delta>S_{\Lambda,D}^\delta$. In the second stage one turns on the couplings to the region $\Lambda'\setminus\Lambda$. In each of these stages the required inequalities follow as in Theorems 1 and 2. □

Remark. By a limiting argument Theorem 1 implies that if $g\in L_1\cap L_2$

$$\langle e^{:\phi^2:(g)}\rangle_\Lambda^{HD} \underset{(\geqq)}{\leqq} \langle e^{:\phi^2:(g_\Lambda)}\rangle_\Lambda \quad \text{if} \quad \underset{(g\leqq 0)}{g\geqq 0},$$

where g_Λ is the restriction of g to the set Λ. ($\langle e^{:\phi^2:(g)}\rangle_\Lambda^{HD}=\langle e^{:\phi^2:(g_\Lambda)}\rangle_\Lambda^{HD}$ since the Dirichlet field vanishes outside Λ.)

Similarly, in Theorem 2, if $g\in L_1\cap L_2$

$$\langle e^{:\phi^2:(g)}\rangle_\Lambda^{HD} \underset{(\geqq)}{\leqq} \langle e^{:\phi^2:(g_\Lambda)}\rangle_{\Lambda'}^{HD} \quad \text{if} \quad \underset{(g\leqq 0)}{g\geqq 0}$$

and likewise for the other inequalities.

3. Uniform Bounds

According to Theorem 2, $\langle e^{:\phi^2:(g)}\rangle_\Lambda^{HD}$ (and $\langle e^{:\phi^2:(g)}\rangle_\Lambda^D$ for a ϕ^4 interaction) are monotone increasing (decreasing) in Λ for $g\geqq 0 (g\leqq 0)$ if Λ contains the support of g. A study of the infinite volume limit is completed by obtaining bounds uniform in Λ.

Theorem 3. *Let $g\in\mathscr{D}$ and* supp $g\subset\Lambda_0$ *where Λ_0 is a finite union of unit squares, and let $\Lambda\supset\Lambda_0$. There exists a norm $\|\ \|_s$ defined on $\mathscr{S}$ such that $\|g\|_s\leqq c<\infty$ implies $|\langle e^{:\phi^2:(g)}\rangle_\Lambda^{HD}|\leqq c'<\infty$ where c' depends only on c and is independent of Λ and Λ_0.*

Proof.

$$|\langle e^{:\phi^2:(g)}\rangle_\Lambda^{HD}| \leqq (\langle e^{:\phi^2:(2g_+)}\rangle_\Lambda^{HD})^{\frac{1}{2}} (\langle e^{-:\phi^2:(2g_-)}\rangle_\Lambda^{HD})^{\frac{1}{2}}$$

where g_+, g_- are the positive and negative parts of Reg, the real part of g. The first term can be estimated by an argument of Frohlich [6] (see also Simon [7]). Let $f = 2g_+$. By Theorems 1 and 2,

$$\langle e^{:\phi^2:(f)}\rangle_\Lambda^{HD} \leqq \langle e^{:\phi^2:(f)}\rangle_{lxt}^{HD} \leqq \langle e^{:\phi^2:(f)}\rangle_{lxt} \quad \text{if} \quad \Lambda \subset lxt\,,$$

where lxt denotes the rectangle with sides of length l and t centered at the origin. Thus

$$\langle e^{:\phi^2:(f)}\rangle_\Lambda^{HD} \leqq \lim_{t\to\infty} \langle e^{:\phi^2:(f)}\rangle_{lxt} \leqq e^{-\int E[\hat{H}_l - :\phi^2:(f_t)]dt}\,,$$

where $E[A]$ denotes the lowest eigenvalue of the operator A,

$$\hat{H}_l = H_l - E[H_l]\,, \qquad H_l = H_0 + \int_{-l/2}^{l/2} :P(\phi(x,0)):\,dx\,,$$

and $f_t(x) = f(x,t)$.

By a result of Spencer [8] (see Simon [7])

$$-E[\hat{H}_l - :\phi^2:(f_t)] \leqq d - c^{-1} E\left[H_0 + c\left\{\int_a^b :P(\phi(x,0)):\,dx - :\phi^2:(f_t)\right\}\right]$$

independent of l, if supp $f_t \subset [a,b]$. The constants c, d depend on a, b. As shown in Lemma 1 below, $(\int |f(x,t)|^2\,dx)^{\frac{1}{2}} \leqq c_1' < \infty$ implies

$$-E\left[H_0 + c\left\{\int_a^b :P(\phi(x,0)):\,dx - :\phi^2:(f_t)\right\}\right] \leqq c_2'$$

where c_2' depends on c_1' and on a and b.

Defining the norm $|||g|||^2 = \sup_t \int |g(x,t)|^2\,dx$ we have shown

$$\langle e^{:\phi^2:(2g_+)}\rangle_\Lambda^{HD} \leqq c_2 \quad \text{if} \quad |||g||| \leqq c_1\,,$$

where c_2 depends on c_1 and on Λ_0.

We must now estimate $\langle e^{-:\phi^2:(2g_-)}\rangle_\Lambda^{HD}$. As the first Griffiths' inequality does not apply to $:\phi^2:$ we cannot estimate this in terms of $\langle e^{:\phi^2:(2g_-)}\rangle_\Lambda^{HD}$. We proceed instead using the monotone decrease of $\langle e^{-:\phi^2:(2g_-)}\rangle_\Lambda^{HD}$ with Λ for $\Lambda \supset \Lambda_0$ (Theorem 2). Let $f = 2g_-$

$$\langle e^{-:\phi^2:(f)}\rangle_\Lambda^{HD} \leqq \langle e^{-:\phi^2:(f)}\rangle_{\Lambda_0}^{HD} = \frac{\int e^{-:\phi^2:(f)} e^{-V_{\Lambda_0}} d\mu_{\Lambda_0,D}}{\int e^{-V_{\Lambda_0}} d\mu_{\Lambda_0,D}}$$

$$\leqq c\left(\int e^{-:\phi^2:(2f)} d\mu_{\Lambda_0,D}\right)^{\frac{1}{2}}\,,$$

where V_{Λ_0} denotes the Half-Dirichlet interaction associated with Λ_0, and c depends on Λ_0.

But

$$\int e^{-:\phi^2:(2f)} d\mu_{\Lambda_0,D} = \int e^{-:\phi^2:_D(2f)} d\mu_{\Lambda_0,D} e^{\int [S(x,x) - S_{\Lambda_0,D}(x,x)]2f(x)d^2(x)}\,.$$

As $S(x,x) - S_{\Lambda_0,D}(x,x) \in L_p(\Lambda_0, d^2(x))$ for all $p < \infty$ [3] we have

$$\int [S(x,x) - S_{\Lambda_0,D}(x,x)]\, 2f(x)\, d^2(x) \leqq c\|f\|_2\,. \tag{4}$$

By an application of Jensen's inequality for conditional expectations [3]

$$\int e^{-:\phi^2:_D(2f)}d\mu_{\Lambda_0,D} \leqq \int e^{-:\phi^2:(2f)}d\mu_0 \leqq e^{\alpha\|f\|_2^2}$$

for some constant $\alpha > 0$. Since $\|f\|_2 \leqq b|||g|||$, b depending on the support of g, we have shown

$$\langle e^{-:\phi^2:(2g_-)}\rangle_\Lambda^{HD} \leqq c_2 \quad \text{if} \quad |||g||| \leqq c_1\,,$$

where c_2 depends on c_1 and on Λ_0. Thus, for $\Lambda \supset \Lambda_0$

$$|\langle e^{:\phi^2:(g)}\rangle_\Lambda^{HD}| \leqq c' < \infty \quad \text{if} \quad |||g||| \leqq c < \infty\,,$$

where c' depends on c and the support of g but is independent of Λ. To complete the proof of the theorem the dependence of the constants on the support of g must be eliminated. This is accomplished using a technique due to Glimm and Jaffe. Decompose R^2 into (closed) unit squares Λ_{ij}, $ij \in Z$, such that $(x, y) \in \Lambda_{ij}$ implies $|x| \geqq |i|$, $|y| \geqq |j|$. Let $p_{ij} = A(i^2+1)(j^2+1)$ where A is chosen so $\Sigma\, 1/p_{ij} = 1$. Let g_{ij} denote the restriction of g to Λ_{ij} and let $\|g\|_s = \Sigma_{ij}\, p_{ij}|||g_{ij}|||$. By Holders' inequality,

$$|\langle e^{:\phi^2:(g)}\rangle_\Lambda^{HD}| \leqq \prod_{ij} (\langle e^{:\phi^2:(p_{ij}\mathrm{Re}\, g_{ij}):}\rangle_\Lambda^{HD})^{\frac{1}{p_{ij}}}.$$

As $\mathrm{supp}\, g_{ij} \subset \Lambda_{ij} \subset \Lambda$, we can use our uniform estimate for $\Lambda \supset \Lambda_0$, and replace Λ_0 by Λ_{ij}. If $\|g\|_s \leqq c_1$ then $|||p_{ij}g_{ij}||| \leqq c_1$ for all i, j and thus

$$\langle e^{:\phi^2:(p_{ij}\mathrm{Re}\, g_{ij})}\rangle_{\Lambda_{ij}}^{HD} \leqq c_2\,,$$

where c_2 depends only on c_1. Thus

$$|\langle e^{:\phi^2:(g)}\rangle_\Lambda^{HD}| \leqq c_2^{\Sigma \frac{1}{p_{ij}}} = c_2$$

and the theorem is proved. □

We now prove Lemma 1.

Lemma 1. $H_0 + \int_a^b :P(\phi(x,0)):\, dx - \int :\phi(x,0)^2:\, g(x)\, dx \geqq c_1$ *if* $\|g\|_2 \leqq c_2$, *where* $\mathrm{supp}\, g \subset [a, b]$ *and* c_1 *depends on* c_2, a, *and* b.

Proof.

$$e^{-E\left[H_0 + \int_a^b :P(\phi(x,0)):dx - \int :\phi(x,0)^2:\, g(x)dx\right]} \leqq \|e^{-\int_a^b :P(\phi(x,0)):\, dx + \int :\phi(x,0)^2:\, g(x)\, dx}\|_{2/m}\,,$$

where m is the mass associated with the ϕ field (see Segal [9], Guerra-Rosen-Simon [10], Klein-Landau [11]). This L_p norm is estimated by Nelson's method (e.g. [4]). Undoing the Wick ordering (with a momentum cutoff k, and dropping the x variable temporarily),

$$:P(\phi_k): - :\phi_k^2:\, g = a_{2n}\phi_k^{2n} + a'_{2n-2}\phi_k^{2n-2} + \cdots + a'_0 - \phi_k^2 g + S_k g\,,$$

where $P(z) = \sum_{i=0}^{n} a_{2i} z^{2i}$ and the a'_i include the change in the coefficients due to the Wick ordering. This is equal to

$$[\tfrac{1}{2}a_{2n}\phi_k^{2n} + a'_{2n-2}\phi_k^{2n-2} + \cdots + a'_0] + [\tfrac{1}{2}a_{2n}\phi_k^{2n} - \phi_k^2 g + S_k g]\,.$$

The first term is estimated as usual to be $\geqq -c_1(\ln k)^n$ and the second $\geqq -c_2 g^{n/n-1}$. Thus

$$\int_a^b :P(\phi_k(x,0)):\, dx - \int :\phi_k(x,0)^2:\, g(x)\, dx \geqq -c_1(\ln k)^n - c_2 \int g(x)^{n/n-1}\, dx\,.$$

Also

$$\left\|\left[\int_a^b :P(\phi_k(x,0)):\, dx - \int :\phi_k(x,0)^2:\, g(x)\, dx\right] - \left[\int_a^b :P(\phi(x,0)):\, dx - \int :\phi(x,0)^2:\, g(x)\, dx\right]\right\|_2$$
$$\leqq b(1+\|g\|_2)\, k^{-\alpha}$$

for some constants b, α as usual. It follows that if $\|g\|_2 \leqq c'$ Nelson's estimate shows that the $L_{2/m}$ norm is bounded uniformly in g and thus the lemma is proved. □

Theorem 3 requires that g have compact support and that Λ be sufficiently large, containing the support of g. In order to handle $g \in \mathcal{S}$ we consider those regions Λ which are a finite union of unit squares Λ_{ij}. Then

$$|\langle e^{:\phi^2:(g)}\rangle_\Lambda^{HD}| = |\langle e^{:\phi^2:(g_\Lambda)}\rangle_\Lambda^{HD}| \leqq \prod_{ij} (\langle e^{:\phi^2:(2p_{ij}g_{+ij})}\rangle_\Lambda^{HD})^{\frac{1}{2p_{ij}}} (\langle e^{-:\phi^2:(2p_{ij}g_{-ij})}\rangle_\Lambda^{HD})^{\frac{1}{2p_{ij}}},$$

where g_+, g_- are the positive and negative parts of $\mathrm{Re}\, g_\Lambda$, and g_Λ is the restriction of g to Λ. The first term is estimated as in Theorem 3, the second term satisfies

$$\langle e^{-:\phi^2:(2p_{ij}g_{-ij})}\rangle_\Lambda^{HD} \leqq \langle e^{-:\phi^2:(2p_{ij}g_{-ij})}\rangle_{\Lambda_{ij}}^{HD}\,.$$

This holds because, due to the special form assumed for Λ, $\Lambda \supset \Lambda_{ij}$. The right hand side is estimated uniformly as in Theorem 3. Thus we have

Theorem 3′. *Let $g \in \mathcal{S}$. If Λ is of the form $\bigcup_{\text{finite}} \Lambda_{ij}$ then*

$$\|g\|_s \leqq c < \infty \quad \textit{implies} \quad |\langle e^{:\phi^2:(g)}\rangle_\Lambda^{HD}| \leqq c' < \infty\,,$$

where c' depends only on c and is independent of Λ.

Remark. Theorems 3 and 3′ are also true for $\langle e^{:\phi^2:(g)}\rangle_\Lambda^D$ for a ϕ^4 interaction. The proof is the same, using Theorems 1′, 2′ in place of Theorems 1, 2.

Theorem 3 implies that $\langle e^{z:\phi^2:(g)}\rangle_\Lambda^{HD}$ is an *entire* analytic function of z, since no restriction on the magnitude of the norm $\|g\|_s$ was imposed. For the purpose of studying the existence of the time-zero field $\int :\phi(x,0)^2:\, f(x)\, dx$ it is useful to have a norm $\|\ \|_s'$ which remains bounded as $f \otimes \omega_n \to f \otimes \delta$. Lemma 2 gives such a norm (see the discussion in Section 4) although in this case a restriction on the magnitude of the norm $\|g\|_s'$ is imposed.

Lemma 2. *Let $g(x) \in \mathcal{D}(R^1)$ be real. Then*

i) *$H_0 - \int :\phi(x,0)^2:\, g(x)\, dx \geqq -a$ uniformly in g if $\|g\|_1 \leqq b$ where b is sufficiently small.*

ii) *As a consequence, if $f(x,t) \in \mathcal{D}$ is real,*

$$\int e^{\int :\phi(x,s)^2:\, f(x,s)\, dx\, ds}\, d\mu_0 \leqq A$$

if $|||f|||' \leqq B$ where B is sufficiently small and $|||f|||' = \sup_t \int |f(x,t)|\, dx$, A depending only on B and the support of f.

Proof.

i) $$e^{-E[H_0 - \int :\phi(x,0)^2: g(x)dx]} \leqq \left\| e^{\int_0^1 ds \int dx g(x) :\phi(x,s)^2:} \right\|_{\frac{2}{1-e^{-m}}}$$

(see [7] or [11]). Using Jensen's inequality the right hand side is

$$\leqq \left\| e^{\|g\|_1 \int_0^1 ds :\phi(0,s)^2:} \right\|_{\frac{2}{1-e^{-m}}}$$

which is $\leqq c_1$ if $\|g\|_1 \leqq c_2$ for c_2 sufficiently small.

ii) Follows immediately from

$$\int e^{:\phi^2:(f)} d\mu_0 \leqq e^{-\int E[H_0 - :\phi^2:(f_t)]dt}. \quad \square$$

Corollary 1. *Let $g \in \mathscr{D}(R^2)$, $\operatorname{supp} g \subset \Lambda_0$, where the distance from the support of g to the boundary of Λ_0 is $\geqq \varepsilon > 0$. Let $\Lambda \supset \Lambda_0$. Then*

$$|\langle e^{:\phi^2:(g)} \rangle_\Lambda^{HD}| \leqq A < \infty$$

if $\|g\|_s' \leqq B$ for B sufficiently small, where

$$\|g\|_s' = \Sigma\, p_{ij} |||g_{ij}|||', \qquad |||h|||' = \sup_t \int |h(x,t)|\, dx\,.$$

Proof. The proof proceeds as in Theorem 3, using Lemma 2 in appropriate places. Note that because of the assumptions on Λ_0 and $\operatorname{supp} g$, $S(x,x) - S_{\Lambda_0,D}(x,x)$ is continuous on the support of g, so that in Eq. (4) the norm $\|\ \|_2$ can be replaced by $\|\ \|_1$. $\square$

4. The Infinite Volume Limit

Theorems 1–3 lead to the infinite volume Schwinger functions via Vitali's theorem, as discussed by Frohlich [6] (see also [4]).

Theorem 4. *Let $j_1, \ldots, j_n, g_1, \ldots, g_m \in \mathscr{D}$. Then*

$$\langle \phi(j_1) \ldots \phi(j_n) :\phi^2: (g_1) :\phi^2: (g_m) \rangle_\Lambda^{HD}$$

converges as $\Lambda \nearrow R^2$, and there exists a norm $\|\ \|_s$ on $\mathscr{S}$ such that the infinite volume Schwinger functions satisfy the bound

$$|\langle \phi(j_1) \ldots :\phi^2: (g_m) \rangle^{HD}| \leqq c^{n+m}(n+m)!\ \|j_1\|_s \cdots \|g_m\|_s$$

for some constant c. Moreover they satisfy the Osterwalder-Schrader axioms and thus can be analytically continued to Lorentz invariant Wightman functions.

Proof. The proof is as in Frohlich [6]. Take the norm $\|\ \|_s$ as defined on p. 149 It is known [6, 7] that $|\langle e^{\phi(j)} \rangle_\Lambda^{HD}| \leqq c$ if $\|j\|_s \leqq d$. We may conclude by the Schwartz inequality and Theorem 3, that

$$|\langle e^{\phi(j) + :\phi^2:(g)} \rangle_\Lambda^{HD}| \leqq c \quad \text{if} \quad \|j\|_s + \|g\|_s \leqq c'\,,$$

where c depends only on c'. Let $j_1 \ldots j_n, g_1 \ldots g_m \geqq 0$.

$$\langle e^{\lambda_1 \phi(j_1) + \cdots + \lambda_n \phi(j_n) + z_1 :\phi^2: (g_1) \cdots + z_m :\phi^2: (g_m)} \rangle_\Lambda^{HD}$$

is an entire function of $\lambda_1, \ldots, z_m$ which is bounded on any ball, and monotone increasing in Λ if all λ_i, z_j are non-negative and $j_1, \ldots, g_m \in \mathscr{D}(\Lambda)$. By Vitali's theorem the limit as $\Lambda \nearrow R^2$ exists and is an entire function. The bounds on the Schwinger functions come from the Cauchy integral formula. See [6] for details. □

Remark. Theorem 4 is also true for the full Dirichlet Schwinger functions for a ϕ^4 interaction.

The Schwinger functions obtained in Theorem 4 can be extended by continuity to functions $j_1 \ldots g_m \in \mathscr{S}$. On the other hand Theorem 3′ enables us to take the limit $\Lambda \nearrow R^2$ for $j_i, g_j \in \mathscr{S}$ in case Λ is a union of unit squares, and we obtain the same sort of bounds as in Theorem 4. A $3\in$ argument shows that the Schwinger functions for j_i, g_j obtained either by extension by continuity or by taking the limit $\Lambda \nearrow R^2$ for Λ a union of unit squares coincide.

Finally, we remark that the time zero Wick square is well-defined in the infinite volume limit: Let $f \in \mathscr{D}(R^1)$ and let $\phi_0(f)$ denote $\int \phi(x,0) f(x)\,dx$. Likewise $:\phi_0^2:(f) = \int :\phi(x,0)^2: f(x)\,dx$.

Theorem 5. *Let $f \in \mathscr{D}(R^1)$. The time zero field $:\phi_0^2(f):$ is well-defined in the infinite volume limit. Let $f_1, \ldots f_n, h_1, \ldots h_m \in \mathscr{D}(R^1)$. Then*

$$\langle \phi_0(f_1) \ldots \phi_0(f_n) :\phi_0^2:(h_1) \ldots :\phi_0^2:(h_m) \rangle_\Lambda^{HD}$$

converges as $\Lambda \nearrow R^2$.

Proof. The proof follows from the fact that, according to Corollary 1, the norm $\| \ \|_s$ may be replaced by the norm $\| \ \|_s'$ which is bounded as $\omega_n \otimes F \to \delta \otimes F$ and from Euclidean covariance: As we have defined the norm $\| \ \|_s'$ it is clear that $\int :\phi(0,t)^2: f(t)\,dt$ is well-defined. A rotation brings this to the form $:\phi_0^2:(f)$. □

5. Positivity of $\langle :\phi(x)^2: \rangle$

Theorem 6. *For a $\lambda\phi^4$ interaction, in the infinite volume limit*

$$\langle :\phi(x)^2: \rangle^{HD} \geqq 0, \qquad \langle :\phi(x)^2: \rangle^{D} \geqq 0.$$

Proof. The proof follows from the existence of the above infinite volume limits, as discussed in Section 4, plus integration by parts. If q_i, $i = 1, \ldots, n$ are Gaussian random variables with covariance $C_{ij} = \langle q_i q_j \rangle$, and $F = F(q_1, \ldots q_n)$ then [2]

$$\langle q_i F \rangle = \sum_{j=1}^{n} C_{ij} \left\langle \frac{\partial}{\partial q_j} F \right\rangle.$$

For the Half-Dirichlet theory in the Lattice approximation the above formula (used twice) gives

$$\left\langle :q_i^2: e^{-\lambda\delta^2 \sum_k :q_k^4:} \right\rangle_\Lambda^{HD} + 12\lambda\delta^2 \sum_j S_{\Lambda,D}^\delta(i,j)^2 \left\langle :q_j^2: e^{-\lambda\delta^2 \sum_k :q_k^4:} \right\rangle_\Lambda^{HD}$$
$$= 16\lambda^2\delta^4 \left\langle \left(\sum_j S_{\Lambda,D}^\delta(i,j) :q_j^3: \right)^2 e^{-\lambda\delta^2 \sum_k :q_k^4:} \right\rangle_\Lambda^{HD} + [S_{\Lambda,D}^\delta(i,i) - S^\delta(i,i)] \langle e^{-\lambda\delta^2 \sum :q_k^4:} \rangle_\Lambda^{HD}.$$

Dividing through by $\langle e^{-\lambda\delta^2 \sum :q_k^4:} \rangle_\Lambda^{HD}$ and taking the lattice spacing $\delta \to 0$ gives

$$\langle :\phi(x)^2: \rangle_\Lambda^{HD} + 12\lambda \int d^2(y)\, S_{\Lambda,D}(x,y)^2 \langle :\phi(y)^2: \rangle_\Lambda^{HD} \geqq S_{\Lambda,D}(x,x) - S(x,x). \tag{5}$$

By the results of Section 4 we can take the limit $\Lambda \nearrow R^2$ where the Λ are finite unions of unit squares:

Let $0 \leqq f \in \mathscr{D}$. Define $g(x) = 12\lambda \int S(x-y)^2 f(y) d^2(y)$,

$$h(x) = 12\lambda \int [S_{\Lambda,D}(x,y)^2 - S(x-y)^2] f(y)\, d^2(y), \quad \text{and}$$

$$F_\Lambda = \int [S_{\Lambda,D}(x,x) - S(x,x)] f(x)\, d^2(x).$$

Then Eq. (5) becomes

$$\langle :\phi^2: (f)\rangle_\Lambda^{HD} + \langle :\phi^2: (g)\rangle_\Lambda^{HD} + \langle :\phi^2: (h_\Lambda)\rangle_\Lambda^{HD} \geqq F_\Lambda .$$

The first two terms converge as $\Lambda \nearrow R^2$ and $F_\Lambda \to 0$. By Theorem 4 and Lemma 3 below, $\langle :\phi^2: (h_\Lambda)\rangle_\Lambda^{HD} \to 0$ as $\Lambda \to R^2$.

Thus in the infinite volume limit

$$\langle :\phi^2: (f)\rangle^{HD} + \langle :\phi^2: (g)\rangle^{HD} \geqq 0 .$$

Using the translation invariance of the infinite volume Schwinger functions gives

$$[\int (f+g)\, d^2(y)]\, \langle :\phi(x)^2:\rangle^{HD} \geqq 0 .$$

Thus $\langle :\phi(x)^2:\rangle^{HD} \geqq 0$.

A similar discussion for the Full-Dirichlet theory leads to

$$\langle :\phi(x)^2:_D\rangle_\Lambda^D + 12\lambda \int d^2(y)\, S_{\Lambda,D}(x,y)^2 \langle :\phi(y)^2:_D\rangle_\Lambda^D \geqq 0$$

and to $\langle :\phi(x)^2:\rangle^D \geqq 0$ as before. □

Lemma 3. *Let* $h_\Lambda(x) = \int [S(x-y)^2 - S_{\Lambda,D}(x,y)^2] f(y)\, d^2(y)$. *Then* $\|h_\Lambda\|_s \to 0$ *as* $\Lambda \nearrow R^2$.

Proof. Consider points $x = (z,t) \in \Lambda_{ij}$ for some unit square Λ_{ij}.

$$h_\Lambda(x) \searrow 0 \quad \text{as} \quad \Lambda \nearrow R^2 \quad \text{since} \quad S_{\Lambda,D}(x,y) \nearrow S(x-y) \quad [3] .$$

Also, $h_\Lambda \in \mathscr{S}$ so that $\int dz |h_\Lambda(z,t)|^2$ is continuous in t and monotone decreases to zero. Therefore, by Dini's theorem,

$$\sup_{t \in \Lambda_{ij}} \int_{\Lambda_{ij}} dz |h_\Lambda(z,t)|^2 \searrow 0 .$$

Thus $|||h_{\Lambda_{ij}}||| \searrow 0$. Finally $\|h_\Lambda\|_s \searrow 0$ follows from

$$|||h_{\Lambda_{ij}}||| \leqq |||H_{ij}||| \quad \text{and} \quad \Sigma\, p_{ij} |||H_{ij}||| < \infty ,$$

where $H(x) = \int S(x-y)^2 f(y)\, d^2(y)$ and H_{ij} is the restriction of H to Λ_{ij}. □

Remarks. 1. Our proof that $\langle :\phi(x)^2:\rangle \geqq 0$ depends crucially on the translation invariance of the infinite volume expectation. The same proof gives the result for a finite volume theory with full periodic boundary conditions (i.e. the Wick ordering is with respect to the periodic measure) since one again has the required "translation invariance". This has been obtained by Baumel [13] by a different method.

2. Theorem 6 applies also for free boundary conditions and small coupling. The required infinite volume limits are in Schrader [12].

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