Commun. math. Phys. 43, 109—120 (1975)


# Scattering Theory of the Linear Boltzmannoperator

J. Hejtmanek

Math. Institut der Universität Wien, Wien, Austria

Received April 14, 1975

**Abstract.** In time dependent scattering theory we know three important examples: the wave equation around an obstacle, the Schrödinger and the Dirac equation with a scattering potential. In this paper another example from time dependent linear transport theory is added and considered in full detail.

First the linear Boltzmann operator in certain Banach spaces is rigorously defined, and then the existence of the Møller operators is proved by use of the theorem of Cook-Jauch-Kuroda, that is generalized to the case of a Banach space.

## § 1. The Abstract Cauchy Problem of Linear Transport Theory

In Statistical Mechanics transport phaenomena of neutrons and photons are decribed by the linear Boltzmann equation ([1–4]). One has to start from the formal Cauchy problem:

$$\frac{\partial n}{\partial t} = -v \operatorname{grad}_x n + \int_{R^3} k(x, v', v) n(x, v', t) dv' - \sigma(x, v) n \tag{1.1}$$

$$n(x, v, 0) = f(x, v).$$

$x$ and $v$ are three dimensional vectors: $x=(x_1, x_2, x_3)$ and $v=(v_1, v_2, v_3)$. $w=(x,v)$ is a six dimensional vector in the $\mu$-space of Statistical Mechanics.

$n(w, t)$ is a real valued function with $\operatorname{dom} n = R^6 \times R$.

$k(x, v', v)$ and $\sigma(x, v)$ are non-negative, bounded and measurable functions with $\operatorname{dom} k = R^9$ and $\operatorname{dom} \sigma = R^6$. Both functions vanish for $x \in R^3 \sim D$, where $D$ is a compact and convex subset of $R^3$. In neutron transport theory $D$ stands for reactor and in radiation transfer theory for star. In transport theory sometimes $k(x, v' \to v)$ is preferred to $k(x, v', v)$. Physically $k(x, v', v)$ is the number of particles with final velocity $v$, that are generated after one particle with initial velocity $v'$ has suffered a collision in $x$. In neutron transport theory $\sigma(x, v)$ has the meaning of a reaction rate, it actually equals $|v| \cdot \Sigma_t(x, v)$, $\Sigma_t(x, v)$ being the total macroscopic cross section. $\sigma(x, v)$ has the dimension of an inverse time. Later we also need

$$\sigma_s(x, v) := \int_{R^3} k(x, v, v') dv'. \tag{1.2}$$

Note that $k(x, v, v')$ appears in (1.2), but that $k(x, v', v)$ appears in (1.1)!

The formal Cauchy problem can also be written in the following form:

$$\frac{\partial n}{\partial t} = -v \operatorname{grad}_x n + \chi_D(x) \cdot \left[\int_{R^3} k(x, v', v) n(x, v', t) dv' - \sigma(x, v) n\right], \tag{1.3}$$

where $\chi_D(x)$ is the characteristic function of $D$. One is interested in the solution of the initial value problem, i.e. a non-negative function $n(x, v, t)$, that equals a prescribed non-negative initial distribution $f(x, v)$ at time $t=0$.

The structure of this paper is the following: First the abstract Cauchy problem will be solved in case of the Banach space $n \in X = L^1(R^6)$ and in case of the Hilbert space $n \in X = L^2(R^6)$. It can be written in the following form:

$$\begin{aligned} \dot{n} &= -(T+A)n \\ n(0) &= f \in D(T+A) \end{aligned} \tag{1.4}$$

$n(t)$ is a vector valued function for $t \in R$ into the Banach space $X$, $\dot{n}(t)$ is the strong derivative of $n(t)$. One must define the collision free linear Boltzmann operator $T$ and the linear collision operator $A$ by suitable extension of the formal operators:

$$v \operatorname{grad}_x n$$

$$-\int_{R^3} k(x, v', v) n(x, v') dv' + \sigma(x, v) n.$$

## § 2. The Collision Free Linear Boltzmann Operator in $X = L^1(R^6)$

The physicist will say, that the Banach space $X = L^1(R^6)$ is good for linear transport processes, because the norm in $L^1$ equals the total number of particles in $\mu$-space. First the minimal collision free linear Boltzmann Operator is defined:

*Definition.* Let $X = L^1(R^6)$. The operator $T_0$

$$(T_0 n)(x, v) := v \operatorname{grad}_x n$$

$$D(T_0) := C_0^\infty(R^6)$$

is called minimal collision free linear Boltzmann operator. One gets at once: $\overline{D(T_0)} = X$ and $R(T_0) \subseteqq C_0^\infty(R^6) = D(T_0)$.

The formal Cauchy problem

$$\frac{\partial n}{\partial t} = -T_0 n$$

$$n(x, v, 0) = f(x, v) \in D(T_0)$$

has the solution $n(x, v, t) = f(x - vt, v)$. This follows easily by substituting. This form of the solution expressed the fact, that particles move on straight lines with constant velocities. One introduces the following shift group:

*Definition.* Let $n \in X$ and $t \in R$.

$$[U(t)n](x, v) := n(x - vt, v)$$

$$D(U(t)) := X.$$

**Theorem.** *$\{U(t) : t \in R\}$ is an additive Abelian group of isomorphic transformations of $B(X)$.*

*Proof.* First one has to prove, that $U(t)$ is an additive Abelian group:

$$[U(t+s)n] = n(x - (t+s)v, v) = [U(s)[U(t)n]](x, v)$$

$$U(0)n = n \quad \forall n \in X$$

$$[U(-t)[U(t)n]](x, v) = n(x, v) \quad \forall n \in X, \quad \forall t \in R.$$

$U(t)$ is an isometry for all $t \in R$:

$$\|U(t)\|_1 = \int_{R^6} |n(x-vt, v)| dx dv = \int_{R^6} |n(x, v)| dx dv = \|n\|_1 \qquad \forall n \in X.$$

**Theorem.** *$\{U(t): t \in R\}$ is strongly continuous in $t$.*

*Proof.* One has to prove, that $\lim_{t\to s} \|[U(t)-U(s)]n\|_1 = 0 \ \forall n \in X$.

For all $\varepsilon > 0$ there exists a step function $g(x, v)$ such, that $\|n-g\|_1 < \varepsilon$.

$$\|[U(t)-U(s)]n\|_1 \leqq \|U(t)n - U(t)g\|_1 + \|U(t)g - U(s)g\|_1$$
$$+ \|U(s)g - U(s)n\|_1 \leqq \varepsilon + \|U(t)g - U(s)g\|_1 + \varepsilon.$$

But one has $[U(t)g - U(s)g](x, v) = g(x-tv, v) - g(x-sv, v)$. $g(x, v)$ is a step function, so one can find a $\delta(\varepsilon) > 0$ such, that

$$\|g(x-vt, v) - g(x-vs, v)\|_1 < \varepsilon \qquad \forall t \in R \quad |t-s| < \delta.$$

$$\Rightarrow \lim_{t\to s} \|U(t)n - U(s)n\|_1 = 0. \quad \square$$

**Corollary.** $\|U(t)\| = 1 \ \forall t \in R$. *One can use the theorem of Hille-Yosida* (Yosida, p. 237). *The infinitesimal generator $T$ of the group $U(t)$ exists:*

$$T := -s - \lim_{t\to 0} \frac{U(t)-1}{t}.$$

*$T$ is a closed operator with dense domain. Also the Hille-Yosida condition is satisfied for $T$. $T$ is an element of the operator class $G(1,0)$.*

See Kato, p. 485 for the definition of the operator class $G(M, \beta)$!

*Definition.* The operator $T$ is called collision free linear Boltzmann operator. $T$ is a closed extension of $T_0$, i.e. $T$ is closable.

*Definition.* Let $K$ be a subset of $R^n$. Let $\varrho$ be a positive real number

$$K(\varrho) := \bigcup_{x\in K} \{x' : |x'-x| \leqq \varrho\}.$$

One sees at once that $K(0) = K$.

*Definition.* $M_x := \{(x, v) : v = 0\} \subseteqq R^6$
$M_v := \{(x, v) : x = 0\} \subseteqq R^6$.

One sees at once that $M_x(\varrho) = \{(x, v) : |v| \leqq \varrho\}$.

*Definition.* $\hat{C}_0^\infty(R^6) := \{f(w) : f \in C_0^\infty(R^6), \operatorname{supp} f \cap M_x = \emptyset\}$.

If the particle distribution function $f \in \hat{C}_0^\infty$, then the velocities of the particles in the cloud $f$ are not too small.

**Theorem.** *$f \in \hat{C}_0^\infty(R^6) \Leftrightarrow$ 1) $f \in C_0^\infty(R^6)$ and*
*2) $\exists v_0 \in R$, $v_0 > 0$ $\forall v \in R^3$, $|v| \leqq v_0$ and $\forall x \in R^3 \Rightarrow f(x, v) = 0$.*

*Proof.* 1) $\Rightarrow$
$\operatorname{supp} f$ is compact, so there exists a closed sphere $K$ in $R^6$ with center $0 \in R^6$ such, that $\operatorname{supp} f \subseteqq K$.

Let us assume: $\forall n \in N\ \exists v_n \in R\ |v_n| \leqq \frac{1}{n}$ and $\exists x_n \in R^3$ such, that $f(x_n, v_n) \neq 0$. Because $(x_n, v_n) \in \operatorname{supp} f$, and because $\operatorname{supp} f$ is closed, there exists a subsequence $(x'_n, v'_n)$ with $\lim_{n\to\infty}(x'_n, v'_n) = (x, 0)$.

$\Rightarrow (x, 0) \in \operatorname{supp} f$. This contradicts the fact $\operatorname{supp} f \cap M_x = \emptyset$.

2) $\Leftarrow$

$\operatorname{supp} f \cap M_x \subseteqq \operatorname{supp} f \cap M_x^0(v_0) = \emptyset$. □

**Theorem.** *$\hat{C}_0^\infty(R^6)$ is dense in $X = L^1(R^6)$.*

*Proof.* It is known, that $C_0^\infty(R^6)$ is dense in $L^1(R^6)$. Let $f \in L^1(R^6)$. First there exists a compact subset $K \subseteqq R^6$ such, that $\|f - g\|_1 < \varepsilon$, $g(w)$ being defined in the following way:

$$g(w) := f(w) \qquad w \in K$$
$$:= 0 \qquad w \notin K .$$

$\Rightarrow \operatorname{supp} g \subseteqq K$.

Then there exists a $\varrho(\varepsilon) > 0$ such, that $\|g - h\|_1 < \varepsilon$, $h(w)$ being defined in the following way:

$$h(w) := g(w) \qquad w \in K \cap \sim M_x(\varrho(\varepsilon))$$
$$:= 0 \qquad w \notin K \cap \sim M_x(\varrho(\varepsilon)) .$$

$\Rightarrow \operatorname{supp} h \subseteqq K \cap M_x(\varrho(\varepsilon))$.

The function $h(w)$ is absolutely integrable, has compact support and

$d(\operatorname{supp} h, M_x) \geqq \varrho(\varepsilon)$.

Let be $\varrho(w)$ the well known test function:

$$\varphi(w) := c \cdot e^{\frac{1}{w^2 - 1}} \qquad |w| < 1$$
$$:= 0 \qquad |w| \geqq 1 .$$

The constant $c$ has to be chosen such that $\int_{R^6} \varphi(w)dw = 1$.

Due to a theorem ([11], Hörmander, p. 3) the functions

$$h_\mu(w) := \int_{R^6} h(w - \mu w')\varphi(w')dw' \qquad \mu \in R, \mu > 0$$

are $C_0^\infty(R^6)$ functions. The net $(h_\mu)$ converges for $\mu \to 0$ in the norm of $L^1$ to $h(w)$. There exists a $\mu_0$ such, that $\|h - h_{\mu_0}\|_1 < \varepsilon$.

$$\Rightarrow \|f - h_{\mu_0}\|_1 \leqq \|f - g\|_1 + \|g - h\|_1 + \|h - h_{\mu_0}\|_1 < 3\varepsilon . \quad \square$$

*Definition.* Let $P_x$ be the projection from $(x, v) \in R^6 \to x \in R^3$. If $f \in \hat{C}_0^\infty(R^6)$, then there exist:

$$\varrho := \sup\{|x| : (x, v) \in \operatorname{supp} f\}$$
$$v_0 := \inf\{|v| : (x, v) \in \operatorname{supp} f\}$$
$$V_0 := \sup\{|v| : (x, v) \in \operatorname{supp} f\}$$

all numbers being positive.

**Theorem.** $(x, v) \in \operatorname{supp}[U(t)f] \Rightarrow -\varrho + v_0|t| \leqq |x| \leqq \varrho + V_0|t| \quad \forall t \in R$.

*Proof.* $(x, v) \in \operatorname{supp}[U(t)f]$

$$\Rightarrow |x - vt| \leqq \varrho$$

$$\Rightarrow |x| \leqq |x - vt| + |vt| \leqq \varrho + V_0|t||$$

$$\Rightarrow |x| \geqq -|x - vt| + |vt| \geqq -\varrho + v_0|t| . \quad \square$$

This theorem can also be expressed in the form

$$P_x \operatorname{supp}[U(t)f] \subseteqq \{x : x \in R^3, \; -\varrho + v_0|t| \leqq |x| \leqq \varrho + V_0|t|\} .$$

This relation reminds the famous Huyghens principle of the wave equation:

$$\operatorname{supp}[U(t)f] \subseteqq \{x : x \in R^3, \; -\varrho + c|t| \leqq |x| \leqq \varrho + c|t|\}$$

and I shall call it in this paper Pseudo Huyghens principle.

## § 3. The Collision Free Linear Boltzmann Operator in $X = L^2(R^6)$

One could use the same method to construct the collision free linear Boltzmann operator in the Hilbert space $X = L^2(R^6)$, that has been used in § 2 in the case of the Banach space $X = L^2(R^6)$. $L^2$ has the advantage of being a Hilbert space, but it is not quite appropriate for the needs of Statistical Mechanics. The $L_2$ norm has no physical meaning here. Remember, that in Quantum Mechanics $L_2$ is the appropriate space!

The collision free linear Boltzmann operator is constructed in analogy to a a method, that is used in Mathematical Physics of Quantum Mechanics to define the Hamilton operator of a free particle (Kato, p. 300).

*Definition.* Let $X = L^2(R^6)$. The transformation $F_0$

$$(F_0 n)(x, v) := (2\pi)^{-\frac{3}{2}} \int_{R^3} e^{-ixk} n(k, v) dk$$

$$D(F_0) := S(R^6)$$

is called restricted Fourier transformation. One sees at once: $R(F_0) \subseteqq S(R^6) = D(F_0)$.

**Theorem.** *$F_0$ is an isometry.*

*Proof.* One uses the fact, that $F_0$ with fixed $v \in R^3$ is an isometry in $L^2(R^3)$ (Hewitt, Stromberg, p. 410).

$$\|F_0 n\|_2^2 = \int_{R^6} |(F_0 n)(x, v)|^2 dx dv$$

$$= \int_{R^3} [(2\pi)^{-3} |\int_{R^3} e^{-ixk} n(k, v) dk|^2 dx] dv$$

$$= \int_{R^3} [\int_{R^3} |n(x, v)|^2 dx] dv = \|n\|_2^2 . \quad \square$$

**Theorem.** *$F_0$ is bijektive from $D(F_0) = S(R^6)$ onto itself.*

*Proof.* The transformation $F_0$ with fixed $v \in R^3$ is bijective from $S(R^3)$ onto itself.

Let $n(x,v)\in S(R^6)$. $\Rightarrow n(x,v)\in S(R^3)$ with $v\in R^3$ fixed. The inverse transformation of $F_0$ has the following form ([12], Hewitt, Stromberg, p. 409):

$$(F_0^{-1}n)(x,v)=(2\pi)^{-\frac{3}{2}}\int_{R^3}e^{+ixk}n(k,v)dk\,.$$

For fixed $v\in R^3$ this function is not only in $S(R^3)$, but also in $S(R^6)$. □

*Definition.* Let $X=L^2(R^6)$. $F_0$ and $F_0^{-1}$ are densely defined in $X$. Both transformation are isometrics. There exists a unique unitrary extension $F$ of $F_0$. $F$ is called restricted Fourier-Plancherel transformation.

*Definition.* Let $X=L^2(R^6)$. The operator $T_0$

$$(T_0n)(x,v):=v\,\mathrm{grad}_x n$$

$$D(T_0):=S(R^6)$$

is called minimal collision free linear Boltzmann operator.

$$\Rightarrow \overline{D(T_0)}=X \quad \text{and} \quad R(T_0)\subseteqq S(R^6)=D(T_0)\,.$$

**Theorem.** $(FT_0n)(x,v)=i\cdot v\cdot x\cdot(Fn)(x,v)$.

*Proof.* $(FT_0n)(x,v)=(2\pi)^{-\frac{3}{2}}\int_{R^3}e^{-ixk}\sum_{n=1}^3 v_n\frac{\partial n}{\partial k_n}(k,v)dk$

$$=(2\pi)^{-\frac{3}{2}}\sum_{n=1}^3 v_n\int_{R^3}e^{-ixk}\frac{\partial n}{\partial k_n}(k,v)dk$$

$$=(2\pi)^{-\frac{3}{2}}\sum_{n=1}^3 v_n\cdot i\cdot x_n\int_{R^3}e^{-ixk}n(k,v)dk$$

$$=iv\cdot x\cdot(Fn)(x,v)\,.$$ □

The reduced Fourier-Plancherel transformed operator of the operator $T_0$ is a multiplication operator by the non bounded function $i\cdot v\cdot x$. In analogy to the procedure of defining the Hamilton operator of a free particle the collision free linear Boltzmann operator is defined now:

*Definition.* The operator $S$

$$(Sn)(x,v):=v\cdot x\cdot n(x,v)$$

$$D(S):=\{n:n\in L^2(R^6),\ Sn\in L^2(R^6)\}$$

is the maximal multiplication operator by the real valued function $v\cdot x$, so it is selfadjoint.

$$\Rightarrow FT_0n=iSFn \qquad \forall n\in D(T_0)$$

$$\Rightarrow T_0n=iF^{-1}SFn \quad \forall n\in D(T_0)\,.$$

*Definition.* The operator $T$

$$Tn:=iF^{-1}SFn$$

$$D(T):=F^{-1}D(S)$$

is called collision free linear Boltzmann operator. Because $S$ is selfadjoint, and because $T$ is unitarily equivalent $iS$, $iT$ is selfadjoint. $T$ itself is skew symmetric.

The Hille-Yosida condition is satisfied by $T$. $T$ is in the operator class $G(1,0)$. Also in the case of the Hilbert space $X=L^2(R^6)$, $\hat{C}_0^\infty(R^6)$ is dense in $L^2(R^6)$.

## § 4. The Linear Collision Operator in $X=L^1(R^6)$ and $X=L^2(R^6)$

The linear collision operator $A$ is a sum of two operators.

*Definition.*

$$(A_1 n)(x,v) := -\int_{R^3} k(x,v',v)n(x,v')dv'$$

$$(A_2 n)(x,v) := \sigma(x,v)\cdot n(x,v)\,.$$

**Theorem.** *If $k(x,v',v)$ is a non-negative, bounded and measurable function with* $\operatorname{dom} k=R^9$, *if $k(x,v',v)=0\ \forall x\in R^3\sim D$ and if*

$$N(x,v') := \int_{R^3} k(x,v',v)dv$$

*is a bounded function with* $\operatorname{dom} N=R^6$, *then $A_1$ is a bounded operator of the Banach algebra $B(L^1(R^6))$.*

Note that $N(x,v')=\sigma_s(x,v')$!

*Proof.* Let $n(x,v)\in L^1(R^6)$.

$$f(x,v) := \int_{R^3} k(x,v',v)n(x,v')dv'$$

$$|f(x,v)| \leqq \int_{R^3} k(x,v',v)|n(x,v')|dv'$$

$$\int_{R^3}|f(x,v)|dv \leqq \int_{R^3}\left[\int_{R^3} k(x,v',v)dv\right]|n(x,v')|dv'$$

$$\leqq K\int_{R^3}|n(x,v')|dv'$$

$$\|f\|_1 \leqq K\|n\|_1$$

$$K := \sup\{N(x,v):(x,v)\in D\times R^3\}\,. \quad \square$$

**Theorem.** *If $k(x,v',v)$ is a non-negative, bounded and measurable function with* $\operatorname{dom} k=R^9$, *if $k(x,v',v)=0\ \forall x\in R^3\sim D$ and if*

$$M(x,v) := \int_{R^3} k(x,v',v)dv'$$

$$N(x,v') := \int_{R^3} k(v,v',v)dv$$

*with* $\operatorname{dom} M=\operatorname{dom} N=D\times R^3$ *are bounded, then $A_1$ is a bounded operator of the Banach algebra $B(L^2(R^6))$.*

*Proof.* Let $n(x,u)\in L^2(R^6)\cap L^1(R^6)$.

$$f(x,v) := \int_{R^3} k(x,v',v)n(x,v')dv'$$

$$f(x,v)\cdot(M(x,v))^{-1} = \int_{R^3} k(x,v',v)(M(x,v))^{-1}n(x,v')dv'\,.$$

Because for the non-negative function $k(x,v',v)(M(x,v))^{-1}$ one has

$$\int_{R^3} k(x,v',v)(M(x,v))^{-1}dv' = 1\,,$$

the right hand of the last equation is a weighted mean of the function $n(x, v')$ with respect to $v' \in R^3$.

$$[f(x,v)\cdot(M(x,v))^{-1}]^2 \leqq \int_{R^3} k(x,v',v)(M(x,v))^{-1}|n(x,v')|^2 dv'$$
$$|f(x,v)|^2 \leqq M(x,v)\int_{R^3} k(x,v',v)|n(x,v')|^2 dv'$$
$$\int_{R^3}|f(x,v)|^2 dv \leqq K_1\cdot\int_{R^3}[\int_{R^3}k(x,v',v)dv]|n(x,v')|^2 dv'$$
$$\leqq K_1\cdot K_2\cdot\int_{R^3}|n(x,v')|^2 dv'$$
$$\Rightarrow \|f\|_2^2 \leqq K_1\cdot K_2\cdot\|n\|_2^2$$
$$K_1 := \sup\{M(x,v):(x,v)\in D\times R^3\}$$
$$K_2 := \sup\{N(x,v):(x,v)\in D\times R^3\}\,. \quad \square$$

The conditions for the function $k$ are physically meaningful, especially the boundedness of $N(x, v')$. It expresses the fact, that the total number of neutrons generated by scattering and fission processes in $X$ from one neutron with initial velocity $v'$ is bounded.

**Theorem.** *If $\sigma(x, v)$ is a non-negative, bounded and measurable function with* dom$\sigma = R^6$, *then $A_2$ is a bounded operator in both Banach algebras $B(L^1(R^6))$ and $B(L^2(R^6))$.*

*Proof.* Let $n(x, v) \in L^1(R^6) \cap L^2(R^6)$.

$$f(x,v) := \sigma(x,v)\cdot n(x,v)$$
$$\|f\|_1 \leqq K\|n\|_1$$
$$\|f\|_2 \leqq K\|n\|_2$$
$$K := \sup\{\sigma(x,v):(x,v)\in D\times R^3\}\,. \quad \square$$

If

$$\int_{R^3} k(x,v',v)dv = \sigma(x,v') \qquad \forall (x,v')\in R^6\,,$$

then no absorptions and no fission processes occour, only scattering processes. In this case one has:

$$\int_{R^3}[A_1 n + A_2 n](x,v)dxdv$$
$$= \int_{R^3}[-\int_{R^3} k(x,v',v)n(x,v')dv' + \sigma(x,v)\cdot n(x,v)]dv = 0\,.$$

Note also that the operator norm of $A_1 + A_2$ does not vanish!

## § 5. The Linear Boltzmann Operator

The linear Boltzmann operator $T + A$ is a generalized partial differential operator, that is additively perturbed by the linear collision operator. So the problem should be solved within the frame work of perturbation theory of linear operator [5].

In both cases $X = L^1(R^6)$ and $X = L^2(R^6)$ $T$ is in the operator class $G(1, 0)$, $t \in R$ and generates an additive group of bounded operators $\{e^{-Tt} : t \in R\}$.

The abstract Cauchy problem

$$\dot{n} = -Tn$$
$$n(0) = f \in D(T)$$

has a unique solution $n(t) = e^{-Tt} f \; \forall t \in R$. $\{e^{-Tt} : t \in R\}$ is the shift group.

**Theorem.** *The linear Boltzmann operator is in the operator class* $G(1, \|A\|)$.

*Proof.* One can use a theorem of Phillips (Kato, p. 495). If $T \in G(M, \beta)$ and if $A \in B(X)$, then $T + A \in G(M, \beta + M\|A\|)$. In this case one has $T \in G(1, 0)$, and so $T + A \in G(1, \|A\|)$.

The abstract Cauchy problem

$$\dot{n} = -(T + A)n$$
$$n(0) = f \in D(T + A) = D(T)$$

has a unique solution $n(t) = e^{-(T+A)t} f$. The group $\{e^{-(T+A)t} : t \in R\}$ has the following series expansion (Kato, p. 495):

$$e^{-(T+A)t} = \sum_{n=0}^{\infty} U_n(t)$$
$$U_0(t) := e^{-Tt}$$
$$U_{n+1}(t) := -\int_0^t U(t-s) A U_n(s) ds \qquad n = 0, 1, 2, \ldots .$$

This series is ordered with respect to powers of the collision operator $A$. $U_1(t)$ is the contribution to the neutron density at time $t$ from all neutrons, that have suffered exactly one collision during the time interval $[0, t]$.

*Definition.* Let $X$ be a Banach space. A subset $K \subseteq X$ is called convex cone, iff

1) $\forall x \in M \;\; \forall \alpha \in R, \alpha > 0 \Rightarrow \alpha x \in M$ and
2) $\forall x, y \in M \;\; \forall \alpha \in R, 0 \leqq \alpha \leqq 1 \Rightarrow \alpha x + (1 - \alpha) y \in M$.

$L^{1+}(R^6) := \{f(w) : f \in L^1(R^6), f(w)$ is real valued and non-negative$\}$ is a convex cone in $X = L^1(R^6)$.

$$L^1 = L^{1+} - L^{1+} + i(L^{1+} - L^{1+}).$$

*Definition.* $k, \sigma$ is called a scattering system, iff

1) $\forall n \in L^{1+}(R^6) \Rightarrow e^{-(T+A)} n \in L^{1+}(R^6) \;\; \forall t \in R, t \geqq 0$ and
2) $\forall n \in L^{1+}(R^6) \Rightarrow \|e^{-(T+A)t} n\|_1 = \|n\|_1 \;\; \forall t \in R, t \geqq 0$.

The trivial transport system $k = 0$, $\sigma = 0$ is a scattering system.

**Theorem.** *In a scattering system* $\|e^{-(T+A)t}\| = 1 \;\; \forall t \in R, t \geqq 0$.

*Proof.* Let $f$ be a real valued function in $L^1(R^6)$. $f$ can be written in the form $f = f^+ - f^-$, $f^+(w) := \max(0, f(w))$ and $f^{-1}(w) := \max(0, -f(w))$. $f^+$ and $f^-$ are elements of the convex cone $L^{1+}$, $|f| = f^+ + f^-$.

$$\|e^{-(T+A)t} f\|_1 \leqq \|e^{-(T+A)t} f^+\|_1 + \|e^{-(T+A)t} f^-\|_1$$
$$= \|f^+\|_1 + \|f^-\|_1 = \|f\|_1 .$$

Now let $f$ be a complex valued function in $L^1(R^6)$. $f$ can be written in the form $f = g + ih$, $g(w) := \operatorname{Re} f(w)$ and $h(w) = \operatorname{Im} f(w)$.

The identity

$$\sqrt{a^2+b^2}=D\int_0^{2\pi}|a\cos\theta+b\sin\theta|d\theta \qquad a,b\in R$$
$$D:=[\int_0^{2\pi}|\cos\theta|d\theta]^{-1}$$

can be proved easily. Define $U:=e^{-(T+A)t}$!

$$\begin{aligned}\|U(g+ih)\|_1:&=\int_{R^6}\sqrt{(Ug)^2(w)+(Uh)^2(w)}dw\\ &=D\cdot\int_{R^6}[\int_0^{2\pi}|(Ug)(w)\cos\theta+(Uh)(w)\sin\theta|d\theta]dw\\ &=D\cdot\int_0^{2\pi}[\int_{R^6}|(Ug)(w)\cos\theta+(Uh)(w)\sin\theta|dw]d\theta\\ &=D\cdot\int_0^{2\pi}[\int_{R^6}|U(g(w)\cos\theta+h(w)\sin\theta)|dw]d\theta\\ &=D\cdot\int_0^{2\pi}[\int_{R^6}|g(w)\cos\theta+h(w)\sin\theta|dw]d\theta\\ &=D\cdot\int_{R^6}[\int_0^{2\pi}|g(w)\cos\theta+h(w)\sin\theta|d\theta]dw\\ &=\int_{R^6}\sqrt{g^2(w)+h^2(w)}dw=:\|g+ih\|_1=\|f\|_1\,.\end{aligned}$$

*Remark.* The last part of this proof was suggested to me by B. Simon during the Symposium on Scattering Theory in Oberwolfach, West Germany, August 1974.

*Remark.* In a scattering system the linear Boltzmann operator $T+A$ is in the operator class $G(1,0)$ for $t\geqq 0$. For all $t\in R$ $T+A\in G(1,\|A\|)$, but for negative $t$, $T+A$ is in general not in the operator class $G(1,0)$. Such a counterexample was communicated to me by B. Simon during this Symposion.

The reason for this stem from the fact, that $\{e^{-(T+A)t}:t\geqq 0\}$ is positivity preserving, but $\{e^{-(T+A)t}:t\leqq 0\}$ is in general not.

## § 6. The Møller Operators of the Linear Boltzmann Operator

In quantum mechanical scattering theory the solutions of the unperturbed and the perturbed abstract Cauchy problems are compared:

$$\begin{aligned}&\dot n=-Tn &&\dot n=-(T+A)n\\ &n(0)=f\in D(T) &&n(0)=f\in D(T+A)=D(T)\,.\end{aligned}$$

$T=i\Delta$, $T+A=i(\Delta+V)$. $T$ and $T+A$ are skew symmetric operators in a Hilbert space, so the conditions of the Hille-Yosida theorem are satisfied. The solutions of both abstract Cauchy problems are unitary transformation groups $\{e^{-Tt}:t\in R\}$ and $\{e^{-(T+A)t}:t\in R\}$. The Møller operators are defined the strong limits

$$W_{\pm}(T+A,T):=\underset{t\to\pm\infty}{s\text{-}\lim}\,e^{(T+A)t}e^{-Tt}\,,$$

if these limits exist.

Cook ([7–10], Kato, p. 533) has proved a theorem, that gives a sufficient condition for the existence of the Møller operators.

This theorem has been proved for quantum mechanical scattering theory in the case of a Hilbert space, but it can be generalized without any difficulty to the case of a Banach space.

**Theorem.** *Let $X$ be a Banach space, let $D_0 \subseteqq D(T)$ be a dense linear subspace of $X$. Let $T+A \in G(M,0)$, $t \geqq 0$.*

*If for all $u \in D_0$, there exists a $s \in R$ such, that $Ae^{-Tt}u$ is continuous for all $t \leqq s$ and $\|Ae^{-Tt}u\|$ is $(-\infty, s)$ integrable, then $W_-(T+A, T)$ exists.*

*Proof.* Let $u \in D_0$.

$$h(t) := e^{(T+A)t} e^{-Tt}$$

$$\dot{h}(t) := e^{(T+A)t} A e^{-Tt}$$

$$\Rightarrow h(t_2) - h(t_1) = \int_{t_1}^{t_2} \dot{h}(t)dt \quad t_1 < t_2$$

$$\|h(t_2) - h(t_1)\| \leqq \int_{t_1}^{t_2} \|\dot{h}(t)\| dt = \int_{t_1}^{t_2} \|e^{(T+A)t} A e^{-Tt} u\| dt$$

$$\leqq M \int_{t_1}^{t_2} \|Ae^{-Tt}u\| dt .$$

So one gets, that $(h(t))$ is a Cauchy net for $t \to -\infty$. □

**Theorem.** *Let $T$ be the collision free linear Boltzmann operator. If $T+A \in G(M,0)$, $t \geqq 0$, then $W_-(T+A, T)$ exists.*

*Proof.* $D_0 := \hat{C}_0^\infty(R^6)$ is a dense linear subspace of $X$ with $D_0 \subseteqq D(T)$. Let $u \in D_0 \cdot Ae^{-Tt}u$ is continuous for all $t \in R$, because $A$ is a bounded operator and $e^{-Tt}$ is strongly continuous. Because of the Pseudo Huyghens principle there exists a $s \in R$ such, that $P_x \operatorname{supp}(e^{-Tt}u) \cap D = \emptyset \ \forall t \leqq s$.

$$\Rightarrow Ae^{-Tt}u = 0 \qquad \forall t \leqq s$$

$\Rightarrow \|Ae^{-Tt}u\| = 0$ and $(-\infty, s)$ integrable. □

*Remark.* For a scattering system $T+A \in G(1,0)$ $t \geqq 0$, so $W_-(T+A, T)$ exists. Because $\{e^{-(T+A)t} : t \leqq 0\}$ is in general not uniformly bounded so one cannot prove existence of $W_+(T+A, T)$ by Cook's theorem.

**Theorem.** *Let $T$ be the collision free linear Boltzmann operator. If $T+A \in G(M,0)$ $t \geqq 0$, and if $\exists \alpha \in R$, $\alpha > 0$ such, that $\forall t \in R$, $t > 0$ and $\forall u \in D_0$*

$$\int_{D \times R^3} |(e^{-(T+A)t}u)(x,v)| dx dv \leqq K \cdot e^{-\alpha t} \|u\| ,$$

*then $W_+(T, T+A)$ exists.*

*Proof.* One has to show, that $Ae^{-(T+A)t}u$ satisfies the theorem of Cook. Let $u \in D_0 \cdot Ae^{-(T+A))t}u$ is continuous for all $t \in R$, because $A$ is a bounded operator and $e^{-(T+A)t}u$ is strongly continuous.

$$(Ae^{-(T+A)t}u)(x,v) = \chi_D(x) \cdot (Ae^{-(T+A)t})(x,v)$$

$$\Rightarrow \|Ae^{-(T+A)t}u\| = \int_{D \times R^3} |(Ae^{-(T+A)t})u(x,v)| dx dv$$

$$\leqq \|A\| \cdot K \cdot e^{-\alpha t} \|u\|$$

i.e. $Ae^{-(T+A)t}u$ is $(0, \infty)$ integrable. □

*Remark.* Physically the last condition means exponential leakage of the particles out of the set $D$. It would be very interesting to learn something about sufficient conditions for this behavior.

*Remark.* This paper was presented during the Seminar on Spectral and Scattering Theory in Oberwolfach, BRD, August 1974. B. Simon, who attended this

Seminar too, gave many suggestions to me concerning minor errors, and he himself used my ideas to write a paper on the Existence of the Scattering Matrix for the Linearized Boltzmann Equation, that has already been published in Commun. math. Phys. **41**, 99 (1975).

## References

1. Cohen, E., Thirring, W.: The Boltzmann equation. Berlin-Heidelberg-New York: Springer 1973
2. Bell, I., Glasstone, S.: Nuclear reactor theory. Princeton, N.Y.: Van Nostrand Reinhold Company 1970
3. Davison, B., Sykes, B.: Neutron transport theory. Oxford: University Press 1958
4. Mihalas, D.: Stellar atmospheres.: W. H. Freeman and Company 1970
5. Kato, T.: Perturbation theory for linear operators. Berlin-Heidelberg-New York: Springer 1966
6. Yosida, K.: Funktional analysis. Berlin-Heidelberg-New York: Springer
7. Cook, J.: Convergence to the Møller wave matrix. J. Math. Phys. **36**, 82—87 (1957)
8. Jauch, J.: Theory of the scattering operator. Helv. Phys. Acta **31**, 127—158 (1958)
9. Kuroda, S.: On the existence and the unitary property of the scattering operator. Nuovo Cimento **12**, 431—454 (1959)
10. Dollard, J.: Quantum mechanical scattering theory. Rocky Mountain J. of Math. **1** (1971)
11. Hörmander, L.: Linear partial differential operators. Berlin-Heidelberg-New York: Springer 1969
12. Hewitt, E., Stromberg, K.: Real and abstract analysis. Berlin-Heidelberg-New York: Springer 1965

Communicated by W. Hunziker

J. Hejtmanek
Mathematisches Institut
der Universität
Strudlhofgasse 4
A-1090 Wien, Austria