Commun. math. Phys. 43, 39—40 (1975)

On Commutative Normal $*$-Derivations*

Shôichirô Sakai
Department of Mathematics, University of Pennsylvania, Philadelphia, Pa. 19104, USA

Received February 10, 1975

Abstract. The notion of commutative derivations is introduced into the theory of unbounded derivations in operator algebras. A useful result for the phase transition theory will be shown for these derivations.

§ 1. Introduction

In the papers [1, 2], normal $*$-derivations in uniformly hyperfinite C^*-algebras have been studied. One of the main purposes of those studies is to develop the phase transition theory in general settings and is hopefully to contribute to unsolved problems in the theory.

As the first step to this goal, it is important to study commutative normal $*$-derivations, since they include the Ising models. In the present paper, we shall show a useful result for commutative normal $*$-derivations. We shall discuss applications of this result to the phase transition theory in later papers.

§ 2. A Theorem

Let $\mathfrak{A}$ be a uniformly hyperfinite C^*-algebra and let δ be a normal $*$-derivation in $\mathfrak{A}$ – i.e., there is an increasing sequence $\{\mathfrak{A}_n \mid n=1,2,\ldots\}$ of finite type I-subfactors in $\mathfrak{A}$ such that $\bigcup_{n=1}^{\infty} \mathfrak{A}_n$ is dense in $\mathfrak{A}$ and the domain $\mathscr{D}(\delta)$ of $\delta = \bigcup_{n=1}^{\infty} \mathfrak{A}_n$. Then there is a sequence (h_n) of self-adjoint elements in $\mathfrak{A}$ such that $\delta(a) = i[h_n, a]$ $(a \in \mathfrak{A}_n)$ for $n = 1, 2, 3, \ldots$.

Definition. A normal $*$-derivation δ in $\mathfrak{A}$ is said to be commutative if we can choose (h_n) as a mutually commuting family. Then we shall show

Theorem. *Suppose that δ is a commutative normal $*$-derivation in $\mathfrak{A}$ and (h_n) is a corresponding family of self-adjoint elements in $\mathfrak{A}$ to δ such that (h_n) is mutually commutative.*

Then there exists a strongly continuous one-parameter subgroup $\{\varrho(t) \mid -\infty < t < +\infty\}$ of $*$-automorphisms on $\mathfrak{A}$ such that $\varrho(t)(a) = \exp t\delta_{ih_n}(a)$ for $a \in \mathfrak{A}_n (n = 1, 2, 3, \ldots)$ and $-\infty < t < +\infty$, where $\delta_{ih_n}(x) = [ih_n, x]$ $(x \in \mathfrak{A})$. Moreover let δ_1 be the infinitesimal generator of $\{\varrho(t) \mid -\infty < t < +\infty\}$; then $\delta_1 = \delta$ on $\bigcup_{n=1}^{\infty} \mathfrak{A}_n$.

Proof. Since $h_n - h_{n_0} \in \mathfrak{A}'_{n_0}$ $(n \geqq n_0)$, where $\mathfrak{A}'_{n_0}$ is the commutant of $\mathfrak{A}_{n_0}$ and since $h_n - h_{n_0}$ commutes with h_{n_0}, $h_n - h_{n_0}$ commutes with $\delta^m_{ih_{n_0}}(\mathfrak{A}_{n_0})$ $(m = 1, 2, 3, \ldots)$.

* This research is supported by N.S.F.

Hence $[ih_n, [ih_{n_0}, a]] = [ih_{n_0}, [ih_{n_0}, a]]$ and so

$$P_n[ih_n, [ih_{n_0}, a]] = P_n[ih_{n_0}, [ih_{n_0}, a]] \to [ih_{n_0}, [ih_{n_0}, a]] \, (n \to \infty) .$$

Therefore $[ih_{n_0}, a] = \delta(a) \in \mathcal{D}(\tilde{\delta})$ and $\tilde{\delta}\delta(a) = [ih_{n_0}, [ih_{n_0}, a]] = \delta^2_{ih_{n_0}}(a)$, where $\tilde{\delta}$ is the greatest regular extension of δ (cf. [2]). Hence we have $\tilde{\delta}^2(a) = \delta^2_{ih_{n_0}}(a)$ $(a \in \mathfrak{A}_{n_0})$. Quite similarly, we have $[ih_n, [ih_{n_0}, [ih_{n_0}, a]]] = [ih_{n_0}, [ih_{n_0}, [ih_{n_0}, a]]]$ $(a \in \mathfrak{A}_{n_0})$. Therefore by the same discussion with the above, $\delta^2_{ih_{n_0}}(a) \in \mathcal{D}(\tilde{\delta})$ and $\tilde{\delta}\delta^2_{ih_{n_0}} = \tilde{\delta}^3(a) = \delta^3_{ih_{n_0}}(a)$ $(a \in \mathfrak{A}_{n_0})$.

Continuing this process, we have that $\tilde{\delta}^m(a) = \delta^m_{ih_{n_0}}(a)$ for $a \in \mathfrak{A}_{n_0}$ and $m = 1, 2, 3, \ldots$. Hence $\exp t\tilde{\delta}(a) = \exp t\delta_{ih_{n_0}}(a)$ for $a \in \mathfrak{A}_{n_0}$ $(n_0 = 1, 2, \ldots)$ and $-\infty < t < +\infty$. Put $\varrho_n(t)(x) = \exp t\delta_{ih_n}(x)$ $(x \in \mathfrak{A})$; then $\varrho_n(t)$ is a uniformly continuous one-parameter subgroup of $*$-automorphisms on $\mathfrak{A}$ for each n. $\varrho_n(t)(a) \to \exp t\tilde{\delta}(a)$ (uniformly) $(n \to \infty)$ for each $a \in \bigcup_{n=1}^{\infty} \mathfrak{A}_n$ and each $t \in (-\infty, \infty)$, and $\|\varrho_n(t)\| = 1$: hence $\{\varrho_n(t)(x)\}$ is a Cauchy sequence for each $x \in \mathfrak{A}$ and each $t \in (-\infty, \infty)$, Therefore there is a limit "$\lim_{n \to \infty} \varrho_n(t)(x)$".

Put $\varrho(t)(x) = \lim_{n \to \infty} \varrho_n(t)(x)$ $(x \in \mathfrak{A})$; then clearly $\varrho(t)$ is a $*$-isomorphism of $\mathfrak{A}$ into $\mathfrak{A}$ for each t. Moreover $\varrho(t)(a) = \exp t\delta_{ih_n}(a)$ for $a \in \mathfrak{A}_n (n = 1, 2, 3, \ldots)$: hence $t \to \varrho(t)$ is a strongly continuous one-parameter subgroup. Therefore $x = \varrho(t-t)(x) = \varrho(t)\varrho(-t)(x)$ $(x \in \mathfrak{A})$ and so the range of $\varrho(t) = \mathfrak{A}$; hence $\varrho(t)$ is a $*$-automorphism of $\mathfrak{A}$.

The remaining part of the theorem is almost clear. This completes the proof.

Remark. Let $\mathfrak{L}_{n_0}$ be the C^*-subalgebra of $\mathfrak{A}$ generated by $\mathfrak{A}_{n_0}$ and h_{n_0}; then it is easily seen that $\varrho(t)\mathfrak{L}_{n_0} = \mathfrak{L}_{n_0}$. If $h_{n_0} \in \bigcup_{n=1}^{\infty} \mathfrak{A}_n$, then $\mathfrak{L}_{n_0}$ is finite-dimensional. In this case, let $\mathfrak{L}_{n_0} = \sum_j \oplus \mathfrak{L}_{n_0} p_{n_0,j}$ be the central decomposition of $\mathfrak{L}_{n_0}$, where $(p_{n_0,j})$ is the family of all minimal projections in the center of $\mathfrak{L}_{n_0}$. Then for every KMS state φ on $\mathfrak{A}$ with respect to $\{\varrho(t)\}$, φ can be written as follows: $\varphi = \sum_j \lambda_j \varphi_j$ on $\mathfrak{L}_{n_0}$, where (λ_j) is a family of positive numbers such that $\Sigma \lambda_j = 1$, and φ_j is the unique KMS state on $\mathfrak{L}_{n_0} p_{n_0,j}$ with respect to the automorphisms $(y \to \varrho(t)(y))$ $\cdot (y \in \mathfrak{L}_{n_0} p_{n_0,j})$. Therefore the problem on the phase transition can be reduced to the problem how freely we can choose (λ_j). If they are unique for each n_0, then there is no phase transition.

References

1. Powers, R., Sakai, S.: Existence of ground states and KMS states in approximately inner dynamics (to appear in Communication in Mathematical Physics)
2. Powers, R., Sakai, S.: Unbounded derivations in operator algebras (to appear in Journal of Functional Analysis)

Communicated by R. Haag

Shôichirô Sakai
Department of Mathematics
University of Pennsylvania
Philadelphia, Pa. 19174, USA