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A Local Edge of the Wedge Theorem

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Abstract. A local version of the edge of the wedge theorem is proved. The boundary values are not required to be equal on a whole open neighborhood of the given point, but essentially only along a bunch of lines through this.

1. Introduction

Let g be a hyperfunction defined in a neighborhood U of the origin in $\mathbb{R}^{n+1}$, such that g can be represented as a difference of two boundary values of holomorphic functions:

$$g = \delta_{\Gamma_+}(f_+) - \delta_{\Gamma_-}(f_-),$$

where $f_\pm$ are holomorphic in $U \times i\Gamma_\pm$, $\Gamma_- = -\Gamma_+$ and Γ_+ is a proper open convex cone with vertex in the origin. In this situation the ordinary edge of the wedge theorem says that if $g=0$ on U, then there exists a function f holomorphic in a neighborhood of U in $\mathbb{C}^{n+1}$ such that f is a holomorphic continuation of f_+ and f_-. This follows simply because in that case the hyperfunction $h = \delta_{\Gamma_+}(f_+) = \delta_{\Gamma_-}(f_-)$ has its (decomposed) singular support in the set $U \times i(\Gamma_+^* \cap \Gamma_-^*)$, which is empty. ($\Gamma^*$ denotes the dual cone of Γ.) Hence h is in fact real analytic (cf. [7]).

In this paper we will try to prove a local version of this theorem under somewhat weaker conditions. In order to be able to state these, we first observe that if L is a complex line in $U \times i(\Gamma_+ \cup \Gamma_-)$ then f_+ and f_- by restriction define a hyperfunction $g|_{L \cap \mathbb{R}^{n+1}}$ in one variable. To say that this restriction vanishes means precisely that f_+ and f_- are holomorphic continuations of each other along L (cf. [5]). Our first assumption is then that $g|_{L \cap \mathbb{R}^{n+1}} = 0$ for all lines L in $U \times i(\Gamma'_+ \cup \Gamma'_-)$, where $\Gamma'_\pm$ are open subcones of $\Gamma_\pm$. If this is true, $f_\pm$ can be evaluated at the origin of each L, so that we get a function $(f_\pm|_L)(0)$. The second assumption is that this function depends analytically on L. And thirdly we require that the restrictions of all derivatives of g to one certain line in $\Gamma'_+ \cup \Gamma'_-$ vanish. The conclusion is then that there exists a unique common holomorphic continuation of $f_\pm$ to a neighborhood of the origin.

This result is rather similar to the well known Kolm-Nagel theorem (see [4] and [7]), which says that to reach the conclusion it suffices to impose a strengthened form of the third condition above. We have in fact been aiming at finding a hyperfunction version of this theorem (cf. [7], p. 77), but have not been succesful, since the second assumption above is not stated in hyperfunction language and our proof is along very classical lines.

The proof is divided into two steps. By using a theorem on separate analyticity we first get a continuation to a complex cone. Then the continuation theorem of Hartogs can be applied to give the desired conclusion.

At last we present another result which is more similar to the ordinary edge of the wedge theorem.

2. Statement of the Conditions

With g and $f_{\pm}$ as in the introduction, we may by means of a suitable coordinate transformation assume the following:

Condition A. *Let* $\Gamma_+ = \{x=(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}: x_0 > 0, \Sigma_{k=1}^n x_k^2 < x_0^2\}$, *let* $\Gamma_- = -\Gamma_+$, *let* $\hat{U} = \{z=(z_0, \ldots, z_n) \in \mathbb{C}^{n+1}: |z_k| < 1, 0 \leqq k \leqq n\}$ *and let* $U = \hat{U} \cap \mathbb{R}^{n+1}$. *Then* $f_{\pm}$ *are holomorphic in* $(U \times i\Gamma_{\pm}) \cap \hat{U}$ *and* $g = \delta_{\Gamma_+}(f_+) - \delta_{\Gamma_-}(f_-)$, *where* δ_Γ *denotes the hyperfunction boundary value.*

This condition is essentially equivalent to the condition that the (decomposed) singular support of g is contained in $U \times i(\Gamma_+^* \cup \Gamma_-^*)$ (see [7] and [8]). In fact, given such a g we can always find defining functions (or, more precisely, certain germs – see [7, 8]) f_+ and f_- as above, and by the ordinary edge of the wedge theorem these are unique modulo a function which is holomorphic in a whole complex neighborhood of U.

Let $u=(u_1, \ldots, u_n)$ be real parameters such that the line $\{x \in \mathbb{R}^{n+1}: x=(x_0, u_1 x_0, \ldots, u_n x_0), |x_0| < 1\}$ is contained in $(\Gamma_+ \cup \Gamma_-) \cap U$. As mentioned in the introduction, the restriction g_u of g to this line can then be defined.

Condition B. $g_u = 0$ *for all* $u=(u_1, \ldots, u_n)$ *such that* $\Sigma_{k=1}^n u_k^2 < \varepsilon$ *for some* ε, $0 < \varepsilon < 1$.

Let L_u be the complex line $L_u = \{z \in \mathbb{C}^{n+1}: z=(z_0, u_1 z_0, \ldots, u_n z_0), |z_0| < 1\}$. Then the restrictions of $f_{\pm}$ to L_u define holomorphic functions in the upper and lower half planes of L_u. Condition B means precisely that these functions are holomorphic continuations of each other. In particular the value of $f_{\pm}|_{L_u}$ at the origin is then well-defined. This value of course depends on u, and in this way a function $\Psi(u) = \Psi(u_1, \ldots, u_n)$ is defined.

Condition C. $\Psi(u)$ *is real analytic in the region* $\{u \in \mathbb{R}^n: \Sigma_{k=1}^n u_k^2 < \varepsilon'\}$ *for some* ε', $0 < \varepsilon' \leqq \varepsilon$.

The derivatives of g are defined by

$$D^\alpha g = \delta_{\Gamma_+}(D^\alpha f_+) - \delta_{\Gamma_-}(D^\alpha f_-)$$

for $\alpha = (\alpha_0, \ldots, \alpha_n)$, where α_i are non-negative integers. The restriction of $D^\alpha g$ to $L_u \cap U$ is denoted by $g_u^{(\alpha)}$.

Condition D. $g_0^{(\alpha)} = 0$ *for all* $\alpha = (\alpha_0, \ldots, \alpha_n)$.

3. The First Step

We first recall the following theorem due to Siciak ([9]):

Theorem (Siciak). *Let* D *be a domain in* $\mathbb{R}^n$ *and let* f *be a function defined in* D *so that for every* $x^0 \in D$ *there exists a polydisc* $P(x^0, r) = \{z \in \mathbb{C}^n: |z_k - x_k^0| < r_k, k=1, \ldots, n\}$ *such that for fixed* ξ_k, *where* $x_k^0 - r_k < \xi_k < x_k^0 + r_k$ $(k=1, \ldots, n)$, $k \neq j$, *the function* $f(\xi_1, \ldots, \xi_{j-1}, x_j, \xi_{j+1}, \ldots, \xi_n)$, $x_j^0 - r_j < x_j < x_j^0 + r_j$, *is continuable to an analytic function in the disc* $|z_j - x_j^0| < r_j (j=1, \ldots, n)$. *Then* f *is analytic in* D.

For other (and easier) versions of this theorem we refer to [3] and [6] for instance.

Theorem 1. *Suppose that g and $f_\pm$ satisfy the conditions A, B, and C. Then there is a positive number d and a function F(z) which is holomorphic in the complex cone* $\mathscr{C}=\{z\in\mathbb{C}^{n+1}: |z_0|<d,\ |z_i|<d|z_0|$ *for* $i=1,\dots,n\}$ *such that* $F=f_\pm$ *on* $\mathscr{C}\cap(U\times i\Gamma_\pm)\cap\tilde{U}$.

Proof. For each $u=(u_1,\dots,u_n)$ with $\Sigma_{k=1}^n u_k^2<\varepsilon$ condition B says that f_+ and f_- define a function

$$h_u(z_0)=f_\pm(z_0,u_1z_0,\dots,u_nz_0)$$

which is holomorphic in $\{z_0\in\mathbb{C}: |z_0|<1\}$.

Next fix $y_0\neq 0$ with $|y_0|<1$. Then the function

$$h_{y_0}(w)=f_\pm(iy_0,iy_0w_1,\dots,iy_0w_n)\quad\text{for}\quad y_0\gtrless 0$$

is holomorphic in $\{w\in\mathbb{C}^n: \Sigma_{k=1}^n|w_k|^2<1\}$ since $(iy_0,iy_0w_1,\dots,iy_0w_n)\in(U\times i\Gamma_\pm)\cap\tilde{U}$ for these y_0 and w.

For $y_0=0$ we set $h_0(u)=\Psi(u)$, which is real analytic in $\{u\in\mathbb{R}^n: \Sigma_{k=1}^n u_k^2<\varepsilon'\}$ by condition C.

Now let $D=\{(y_0,u)=(y_0,u_1,\dots,u_n)\in\mathbb{R}^{n+1}: |y_0|<1,\ \Sigma_{k=1}^n u_k^2<\varepsilon'\}$ and define the function $\tilde{h}(y_0,u)$ on D by $\tilde{h}(y_0,u)=h_u(y_0)$ for fixed u and $\tilde{h}(y_0,u)=h_{y_0}(u)$ for fixed y_0. Then $\tilde{h}(y_0,u)$ satisfies the hypotheses in Siciak's theorem, so $\tilde{h}(y_0,u)$ is in fact analytic in D. Hence there is a $d>0$ such that $\tilde{h}$ has a holomorphic continuation to the polydisc $Q=\{(z_0,w)\in\mathbb{C}^{n+1}: |z_0|<d,\ |w_k|<d$ for $k=1,\dots,n\}$.

By the biholomorphic mapping

$$\phi:\mathscr{C}\to\{(z_0,w)\in Q: z_0\neq 0\}$$

$$(z_0,\dots,z_n)\mapsto\left(z_0,\frac{z_1}{z_0},\dots,\frac{z_n}{z_0}\right)$$

$$(z_0,w_1z_0,\dots,w_nz_0)\leftarrow\!\dashv(z_0,w_1,\dots,w_n)$$

$\tilde{h}$ defines a holomorphic function $F=\tilde{h}\circ\phi$ on $\mathscr{C}$.

In order to see that $F=f_\pm$ on $\mathscr{C}\cap(U\times i\Gamma_\pm)\cap\tilde{U}$, it is enough to observe that $F=f_\pm$ for $x_0=0$ and $y_0\gtrless 0$, i.e. on a piece of a hyperplane of real codimension 1. □

Remark 1. Condition C seems to be rather superfluous, since after all $z_0\neq 0$ in $\mathscr{C}$. But this is not quite true because of the following reason. Let g be a bounded holomorphic function on $\mathscr{C}$. Then $g\circ\phi^{-1}$ is bounded and holomorphic on $\{(z_0,w)\in Q: z_0\neq 0\}$ and hence, by the Riemann extension theorem, has a holomorphic continuation G to all of Q. Then $G(0,w)$ is the holomorphic extension of the corresponding $\Psi(u)$.

Remark 2. The proof gives the following result if we forget about the edge of the wedge situation. Let $f(z_0,\dots,z_n)$ satisfy the following conditions:

1) $f(z_0,u_1z_0,\dots,u_nz_0)$ is holomorphic in z_0 for fixed real u;
2) $\Psi(u)$, which can be defined according to 1), is real analytic;
3) $f(iy_0,iy_0w_1,\dots,iy_0w_n)$ is holomorphic in $w=(w_1,\dots,w_n)$ for fixed real y_0.

Then f can be continued holomorphically to $\mathscr{C}$.

As an example, let $f(z_0, z_1) = \frac{z_0 z_1}{z_0^2 + z_1^2}$. Then $f(z_0, w z_0) = \frac{w}{1+w^2}$, so the conditions above are clearly satisfied. $\Psi(w) = \frac{w}{1+w^2}$ is holomorphic for $w \neq \pm i$. In this case f is holomorphic in $\{(z_0, z_1) \in \mathbb{C}^2 : z_1 \neq \pm i z_0\} \supset \{|z_0| < |z_1|\} \cup \{|z_1| < |z_0|\}$ (since f is symmetric in z_0 and z_1, we get two cones here).

4. The Second Step

In order to get a continuation from $\mathscr{C}$ to a whole neighborhood of the origin we use condition D and the continuation theorem of Hartogs.

Theorem 2. *Suppose that g and $f_\pm$ satisfy the conditions A, B, C, and D. Then there is a positive number d (the same as in Theorem 1) and a function f holomorphic in the polydisc $P = \{z \in \mathbb{C}^{n+1} : |z_0| < d,\ |z_k| < d^2 \text{ for } k = 1, \ldots, n\}$ such that $f = f_\pm$ in $P \cap (U \times i\Gamma_\pm) \cap \tilde{U}$.*

Proof. For a positive integer m, the function f_m is defined by the Cauchy integral

$$f_m(z_0, \ldots, z_n) = \frac{1}{2\pi i} \int_{|z_0| = d - \frac{1}{m}} \frac{F(\zeta, z_1, \ldots, z_n)}{\zeta - z_0} d\zeta,$$

where F is the function given by Theorem 1. If m is large, f_m is well-defined, continuous and holomorphic in each variable for $|z_0| < d - \frac{1}{m}$ and $|z_k| < d \cdot \left(d - \frac{1}{m}\right)$, $k = 1, \ldots, n$. Therefore f_m is holomorphic. Moreover f_m is independent of m. Set $f = f_m$, then f is defined in P and is holomorphic there.

To see that $f = f_\pm$ on the common domain of definition, it suffices to show that all derivatives coincide in one point.

Now we can express $D^\alpha f_\pm$ near $z_0 = 0$ on the line $\{z \in \mathbb{C}^{n+1} : z_1 = \cdots = z_n = 0\}$ by means of the Cauchy integral thanks to condition D. It is clear that this gives the same result as when differentiating f above and setting $z_1 = \cdots = z_n = 0$ afterwards.

Hence f is really a continuation of $f_\pm$. □

5. Another Consequence of the Theorem on Separate Analyticity

At last we want to mention a rather immediate corollary to Siciak's theorem, which is essentially contained in Browders paper [2].

Let U be an open set in $\mathbb{R}^n$, let $\tilde{U}$ be open in C^n such that $U = \tilde{U} \cap \mathbb{R}^n$, let Γ_+ be a proper open convex cone in $\mathbb{R}^n$ with vertex in the origin, set $\Gamma_- = -\Gamma_+$ and let f_+, f_- be holomorphic in $(U \times i\Gamma_\pm) \cap \tilde{U}$ respectively. Then pick n linearly independent unit vectors $e_1, \ldots, e_n$ in Γ_+ and a unit vector e such that $e \neq e_k$, $k = 1, \ldots, n$. e is the normal vector of a hyperplane H. Without loss of generality we may assume that $H = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_n = 0\}$.

Let V be an open set in $H \cap U$. Through each point $x' = (x_1, \ldots, x_{n-1}) \in V$ there passes n complex lines $L_k^{x'}$, $k = 1, \ldots, n$, the directions of which are determined by the unit vectors e_k. Let $\Gamma(V)=\{x \in U$: for each $k=1, \ldots, n$ exists $x'_k \in V$ such that $x \in L_k^{x'_k}\}$. $\Gamma(V)$ is then a kind of Γ-convex envelope of V.

Now assume the following:

1) The restrictions of f_+ and f_- are holomorphic continuations of each other on $L_k^{x'} \cap \tilde{U}$ for every $k=1, \ldots, n$ and each $x' \in V$, and hence define a holomorphic function (in one variable) $f_k^{x'}$ on $L_k^{x'} \cap \tilde{U}$.

2) For each $x \in \Gamma(V)$ the n values $f_k^{x'_k}(x)$ are equal.

By introducing new coordinates such that the coordinate axes are given by $e_1, \ldots, e_n$, Siciak's theorem can be applied to give:

Theorem 3. *With hypotheses as above there is a function f which is holomorphic in a complex neighborhood of $\Gamma(V)$ and is a holomorphic continuation of $f_\pm$.* □

In this paper the condition that one-variable functions f_+ and f_-, holomorphic in the upper and lower half plane respectively, are holomorphic continuations of each other, has been used repeatedly. In order to determine if this is the case, a good Painlevé theorem is needed. As far as we know the best version of this is due to Beurling in [1]. In particular he has shown that it is not necessary to assume the boundary values to exist as distributions for the Painlevé theorem to hold.

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