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Existence of Time Evolution for ν-dimensional Statistical Mechanics

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Abstract. Infinite systems of particles in ν-dimensions are considered. The pair interaction is assumed to be C^2, finite range and superstable. The existence of a time evolution which satisfies the infinite equations of motion in a set of full equilibrium measure is proved. This measure is proved to be invariant; so a dynamical system is obtained.

1. Introduction

The first step in the construction of rigorous statistical dynamics is to prove an existence theorem for the time evolution itself. Let $\mathfrak{X} = \{(q_i), (p_i)\}$ be the phase space of the system of particles. The formal infinite equations of motion read as

$$m\ddot{q}_i = \sum_{j \neq i} \frac{\partial}{\partial q_i} \Phi(|q_i - q_j|), \tag{1.1a}$$

$$q_i(0) = q_i \qquad m\dot{q}_i(0) = p_i. \tag{1.1b}$$

The problem is to find a function $x(t): \mathbb{R} \to \mathfrak{X}$ such that when inserted into Eq. (1.1) both sides of the equation make sense and are equal. To obtain dynamics significant in statistical mechanics we need that the allowed set of initial conditions $\hat{\mathfrak{X}} \subset \mathfrak{X}$ be of full equilibrium measure.

This problem has been solved by Lanford [1] and [2] for one dimensional particles interacting pairwise via a C^2 finite range potential. By a different approach Sinai [3], has obtained similar results for singular interactions. He considers one-dimensional hard-cores interacting via a finite range pair potential possibly diverging at the hard core length. He proved that in a set of full measure the particles fall into finite clusters and that for a finite time the clusters do not interact with each other. A similar technique has been used in the ν-dimensional case for small enough densities. In general in more dimensions the cluster structure cannot exist for all values of temperature and chemical potentials. For the one dimensional case the extension to long range interactions has been obtained in [5].

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In this paper we solve the problem for ν-dimensional systems. The pair interaction is assumed to be C^2, finite range and superstable. It is proved the existence of a time evolution which satisfies the infinite equations of motion in a set of full equilibrium measure. This measure is proved to be invariant, so a dynamical system is obtained.

We sketch the proof[1]. We construct $S(t)$ for $|t| \leqq \tau$ as the limit (coordinate by coordinate) of the partial flows $S^\alpha(t)$. $S^\alpha(t)$ is the time evolution in which the particles outside the sphere $\Sigma(2^\alpha, 0)$, with center in the origin and radius 2^α are frozen, those inside move interacting pairwise and in the field of the external ones with elastic collisions on the boundary of the sphere. An estimate on the possible displacements of the particles under the partial flows is obtained (Theorem 2.1). Then by use of an iterative procedure applied to the integral equations of motion it can be shown that the sequence $S^\alpha(t)\,x$ is Cauchy (in α)-coordinate by coordinate, (Theorem 2.2). Then it is possible to show that the limit

$$S(t)\,x = \lim_{\alpha \to \infty} S^\alpha(t)\,x$$

satisfies the equations of motion.

In Section 2 we give notations and definitions used throughout the paper and then we state the results. In Section 3 we give the proofs.

2. Definition and Results

We first give some notations and definitions.

D 2.1. Phase Space. Let

$$\mathfrak{X} = \{x = (\hat{q}, \hat{p}) \,|\, \hat{q} = (\ldots, q_i, \ldots),\, q_i \in \mathbb{R}^\nu, \text{ such that for every bounded } \Delta \subset \mathbb{R}^\nu, \operatorname{Card}(\hat{q} \cap \Delta) < +\infty;\, \hat{p} : \hat{q} \subset \mathbb{R}^\nu \to \mathbb{R}^\nu\}\,.$$

In $\mathfrak{X}$ we shall identify points which differ only for permutations. Then $\mathfrak{X}$ becomes the physical phase space of an infinite system of particles moving in $\mathbb{R}^\nu$; for every $x \in \mathfrak{X}$, $\hat{q}$ denotes the spatial configuration and $\hat{p}(q)$ the momentum of the particle $q \in \hat{q}$. For every Lebesgue measurable bounded $\Delta \subset \mathbb{R}^\nu$ we define analogously the finite phase space $\mathfrak{X}(\Delta)$. We denote by Q the set of spatial configurations associated to $\mathfrak{X}$.

The natural topology on $\mathfrak{X}$ is the local topology defined by the following requirement. The net $x_\alpha \to x$ iff for every open bounded set $\Delta \subset \mathbb{R}^\nu$ such that $\hat{q} \cap \partial\Delta = \emptyset$ we have

$$\hat{q}_\alpha \cap \Delta \to \hat{q} \cap \Delta \qquad \text{point by point in } \mathbb{R}^\nu$$
$$\hat{p}_\alpha|_{\hat{q}_\alpha \cap \Delta} \to \hat{p}|_{\hat{q} \cap \Delta} \qquad \text{pointwise.}$$

D 2. 2. Interactions. Let $\Phi : \mathbb{R}^+ \to \mathbb{R}$ such that

(i) Φ is C^2.

(ii) $\Phi(x) = 0$ for $x \geqq r_0$.

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[1] An outline of these results has been given in [6].

(iii) Φ is superstable, that is $\Phi = \Phi' + \Phi''$ with Φ' stable and Φ'' a non-negative continuous function such that $\Phi''(0) > 0$, Ref. [7, 8].

For $x \in \mathfrak{X}(\Delta)$ we define

$$U(x) = \beta/2 \sum_{\substack{q \neq q' \\ q, q' \in \hat{q}_x}} \Phi(|q - q'|) , \tag{2.1a}$$

$$T(x) = \beta(2m)^{-1} \sum_{q \in \hat{q}_x} [\hat{p}(q)]^2 , \tag{2.1b}$$

and for $z = (\hat{q}', \hat{p}'), \hat{q}' \cap \Delta = \emptyset$

$$U(x|z) = \beta \left\{ U(x) + \sum_{\substack{q \in \hat{q} \\ q' \in \hat{q}'}} \Phi(|q - q'|) \right\} . \tag{2.2}$$

D 2.3. Equilibrium Measure. We define the equilibrium measure ν on $\mathfrak{X}$ by the following conditions:

(i) let π be the natural mapping of $\mathfrak{X}$ onto Q, then the image of ν induced by π on Q, $\nu^{(s)}$, is tempered (Ref. [7], Eq. (5.12); Ref. [9]).

(ii) for fixed $\hat{q}$ the conditional distribution of the momenta $\hat{p}(q)$, $q \in \hat{q}$, is the direct product of Gaussian distribution with mean 0 and dispersion $\beta(2m)^{-1}$.

As a consequence let $f \in L^1(\mathfrak{X}, \nu)$ then for every $\Delta (\subset \mathbb{R}^\nu)$ bounded and Lebesgue measurable

$$\nu(f) = \int d\nu(x) f(x) = \int_{\Delta^c} d\nu(z) \int_{\mathfrak{X}(\Delta)} dx \exp[-U(x|z)] f(x \cup z) \tag{2.3}$$

where

$$dx = 1 + \sum_{1n}^{\infty} d(q)_n \, d(p)_n \, (n!)^{-1} (2\pi\beta m)^{-n\nu/2} \exp\{-T((p)_n) + \beta\mu n\} .$$

D 2.4. Equations of Motion. For $x = (\hat{q}, \hat{p})$ the equations of motion are formally written as

$$m\ddot{q}_i = -\sum_{j \neq ij} \frac{\partial}{\partial q_i} \Phi(|q_i - q_j|) , \tag{2.4a}$$

$$q_i(0) = q_i \qquad m\dot{q}_i(0) = p_i , \tag{2.4b}$$

where (q_i), (p_i) are respectively the set $\hat{q}$ labelled in some way and $p_i = \hat{p}(q_i)$.

D 2.5. Topologies on $\mathfrak{X}$. We first denote the spheres in $\mathbb{R}^\nu$ as

$$\Sigma(r, d) = \{x \in \mathbb{R}^\nu \,|\, |x - d| < r \,; d \in \mathbb{R}^\nu, r > 0\} , \tag{2.5a}$$

$$\Sigma(r, 0) = \Sigma(r) . \tag{2.5b}$$

Given a labelled configuration (q_i, p_i) and $r > 0$ we define a seminorm on the labelled configurations $y = (q_i', p_i')$ as

$$\begin{gathered} \|y\|_{r,x} = \sup_{q_i \in \hat{q}_x \cap \Sigma(r,0)} |q_i'| \vee |p_i'| \\ a \vee b = \max(a, b) \quad a, b \in \mathbb{R} . \end{gathered} \tag{2.6}$$

The family of seminorms $\|\cdot\|_{r,x}$ for $r \in \mathbb{R}^+$ define the product topology on the labelled configurations of $\mathfrak{X}$ relatively to x.

Let $\bar{\mathfrak{X}} \subset \mathfrak{X}$ be the set such that there exist K_1 and $K_2 \in \mathbb{R}$ defined as follows:

(a) $|p_i| \leqq K_1 \log(|q_i| \vee e) \equiv K_1 \log_+ |q_i|$.

(b) $\forall d \in \mathbb{R}^\nu, L \in \mathbb{R}^+$ such that $\log_+ |d| \leqq L$ then $\mathrm{Card}(x \cap \Sigma(L, d)) \leqq K_2 L^\nu$.

On $\bar{\mathfrak{X}}$ we define a norm as

$$\|x\| = \sup\{\inf(K_1 | \text{Condition (a) holds}); \inf(K_2 | \text{Condition (b) holds})\}. \tag{2.8}$$

D 2.6. Partial Flows, By $S^\alpha(t)$ we denote the flow under which the particles outside $\Sigma(2^\alpha, 0)$ are frozen. Those inside move pairwise interacting in the field of the external ones with elastic collisions on the boundaries.

In the following theorem we shall state the initial conditions for which dynamics will be proven to exist.

Theorem 2.1. *Let* $\hat{\mathfrak{X}}(c)$ *be*

$$\hat{\mathfrak{X}}(c) = \{x \in \bar{\mathfrak{X}} \,|\, \forall \tau > 0 \,\exists\, \alpha(x, \tau) : \alpha \geqq \alpha(x, \tau) \sup_{|t| \leqq \tau} \sup_{(q \in \Sigma(2^\alpha) \cap \hat{q})} |S^\alpha(t)\, \hat{p}(q)| \leqq c\alpha\,.$$

Then for every $c > 0$ $\nu[\hat{\mathfrak{X}}(c)] = 1$.

Note that a stronger result than Theorem 2.1 can be obtained, in which the momenta do not grow faster than $c(\alpha)^{1/2}$ for some c. However for the sake of brevity we shall use the above formulation.

We are now able to introduce the set of initial conditions $\hat{\mathfrak{X}}$ as

$$\hat{\mathfrak{X}} = \bigcap_{c>0} \hat{\mathfrak{X}}(c)\,.$$

In fact by Theorem 2.1 $\hat{\mathfrak{X}}$ has full equilibrium measure and we can construct the global flow $S(t)$ as limit on $\hat{\mathfrak{X}}$ of the partial ones, as will be shown in the next theorem.

Theorem 2.2. *For every* $x \in \hat{\mathfrak{X}}$ *the following holds*

$$\lim_{\alpha \to \infty} S^\alpha(t)\, x = S(t)\, x \tag{2.9}$$

both in the local (D 2.1) and in the product (D 2.5) topologies of $\hat{\mathfrak{X}}$. *For every* τ *the limit is uniform in* $|t| \leqq \tau$ *and* $\|S(t)\, x\|$ *is bounded for* $|t| \leqq \tau$.

The following theorem shows that the evolution $S(t)\,x$ satisfies the equations of motion and that the equilibrium measure is $S(t)$-invariant.

Theorem 2.3. *For every* $x \in \hat{\mathfrak{X}}$.

(i) $S(t)\, x$ *satisfies Eq.* (2.4) *for* $t \in \mathbb{R}$.

(ii) $S(t)\, x$ *is the only solution of Eq.* (2.4) *in* $\bar{\mathfrak{X}}$.

(iii) $S(t) : \hat{\mathfrak{X}} \to \hat{\mathfrak{X}}$ *and* $S(-t)\, S(t) = 1$ *for every* $t \in \mathbb{R}$.

(iv) $(\hat{\mathfrak{X}}, S(t), \nu)$ *is a dynamical system.*

3. Proofs

In the sequel we shall frequently use the following theorem due to Ruelle [7].

Theorem 3.1. *Let ν be a measure on Q that is tempered and satisfies the equilibrium Eq. D.* (2.3) (*ii*), *then*

(i) *ν is regular*, [10].

(ii) *Let $\nu^{(s)}$ be the measure on Q defined in D.* (2.3) (*i*). *Then for every Lebesgue measurable set $\Lambda \subset \mathbb{R}^\nu$ with* $\operatorname{diam}\Lambda = L_\Lambda > 1$ *let*

$$P(m,\Lambda) = \nu^{(s)}[\{\hat{q} \in Q \,|\, \operatorname{Card}(\hat{q} \cap \Lambda) \geqq m\}]$$

then there exist $\gamma > 0$, $\delta < \infty$ such that

$$P(m,\Lambda) = \exp\{-\gamma m^2/L_\Lambda^\nu + \delta m\}.$$

Proof of Theorem 2.1. The proof will be obtained in the following steps.

(*a*) $\nu[M] = 1$ where

$$M = \left\{x \in \mathfrak{X} \,|\, \exists \alpha(x) : \alpha \geqq \alpha(x) \sup_{\{q \in \hat{q} \cap \Sigma(2^\alpha)\}} |\hat{p}(q)| < c\alpha\right\}.$$

(b) $\nu(N) = 1$ where $N = \bigcup_{s \geqq 0} N(s)$

$$N(s) = \{x \in \mathfrak{X} \,|\, d \in \mathbb{Z}^\nu, l \in \mathbb{Z}^+, l \geqq \log_+ |d|, \operatorname{Card}(\hat{q} \cap \Sigma(l,d) \leqq s l^\nu)\}.$$

(c) For every $\tau \in \mathbb{R}^+$ and $c > 0$, $\nu[\mathfrak{X}(\tau)] = 1$ where

$$\mathfrak{X}(\tau) = \left\{x \in \mathfrak{X} \,|\, \exists \alpha(x,\tau) : \alpha \geqq \alpha(x,\tau) \sup_{|t| \leqq \tau} \sup_{q \in \hat{q} \cap \Sigma(2^\alpha)} |S^\alpha(t)\,\hat{p}(q)| < c\alpha\right\}.$$

Conditions (a) and (b) together imply that $\nu(\bar{\mathfrak{X}}) = 1$. Since

$$\hat{\mathfrak{X}}(c) = \bar{\mathfrak{X}} \cap \left\{\bigcap_{n=1}^{\infty} \mathfrak{X}(n)\right\}$$

by Condition (c) we complete the proof. Therefore we only have to prove (a), (b), and (c).

(a) The proof is completely analogous to the one of Ref. [3] and here we only sketch it. Let

$$M_\alpha = \{x \in \mathfrak{X} \,|\, \exists\, q_i \in \Sigma(2^\alpha) : |\hat{p}(q_i)| > c\alpha\}.$$

By use of the equilibrium equations we have

$$\begin{aligned}\nu[M_\alpha] \leqq \int_{\Sigma^c(2^\alpha)} d\nu(z) \sum_{n=0}^{\varrho 2^{\alpha\nu}} \int_{[\Sigma(2^\alpha)]^n} d(q)_n \int_{(\mathbb{R})^{n\nu}} d(p)_n (n!)^{-1} (2\pi\beta^{-1}m)^{-n\nu/2} \cdot [\chi_{M_\alpha}] \\ \cdot \exp\{-U((q)_n|z) - T((p)_n) + \beta\mu n\} + P(\varrho 2^{\alpha\nu}, \Sigma(2^\alpha))\end{aligned}$$

where

$$\varrho > \delta^\nu 2/\gamma. \tag{3.1}$$

By use of Theorem 3.1 we have

$$\nu(M_\alpha) \leqq \varrho 2^{\alpha\nu} \exp(-c^2\alpha^2\beta(2m)^{-1}) + g_\alpha$$

where

$$g_\alpha = \exp\{-\varrho 2^{\alpha\nu}(\gamma\varrho 2^{-\nu} + \delta)\} \tag{3.2}$$

and therefore the Condition (a) is proved.

(b) We estimate $\nu[N^c(s)]$ by use of Theorem 3.1. We have

$$\nu[N^c(s)] \leqq \sum_{d \in \mathbb{Z}^\nu} \sum_{l \geqq \log_+ |d|} \exp\{-\gamma s^2 l^{2\nu}/(2l)^\nu + \delta s l^\nu\} \xrightarrow[s\to\infty]{} 0.$$

(c) By (a) it is sufficient to show that $\Sigma_\alpha \nu(R_\alpha) < \infty$ where

$$R_\alpha = \left\{x \in \mathfrak{X} \mid \exists q_i \in \Sigma(2^\alpha) \text{ and } t, |t| \leqq \tau, : |S^\alpha(t)\,\hat{p}(q_i)| > c\alpha;\ \sup_{q_j \in \Sigma(2^\alpha)} |\hat{p}(q_j)| \leqq c\alpha\right\}.$$

We choose ϱ according to Eq. (3.1) and we write

$$\begin{aligned}\nu(R_\alpha) \leqq g_\alpha + \int_{\Sigma^c(2^\alpha)} d\nu(z) \sum_{n=1}^{\varrho 2^{\alpha\nu}} \int_{[\Sigma(2^\alpha)]^n} d(q)_n \int_{[\mathbb{R}]^{n\nu}} d(p)_n (n!)^{-1} (2\pi\beta^{-1}m)^{-n\nu/2} \\ \cdot \exp\{-U((q)_n|z) - T((p)_n) + \beta\mu n\}\end{aligned} \tag{3.3}$$

where g_α is defined in Eq. (3.2).

We shall now estimate Eq. (3.3). We fix n in the sum of Eq. (3.3). Let $\mathcal{M}_n$ be the corresponding phase space. $\mathcal{M}_n$ is a Riemannian manifold with boundary in the usual metric $ds^2 = \Sigma_{i=1}^n (dp_i^2 + dq_i^2)$. In $\mathcal{M}_n$ we consider the open submanifolds of codimension 1

$$\Delta_{i,\pm}^s = \{(q)_n, (p)_n \mid p_i^s = \pm c\alpha; i = 1, \ldots, n; s = 1, \ldots, \nu\}$$

where p_i^s is the s-coordinate of the vector $p_i \in \mathbb{R}^\nu$. Proceeding as in Ref. [4], we generalize the Lemma of p. 492 of Ref. [3] where the one-parametric group T_t must be read as $S^\alpha(t)$. Here we only give the main steps. We obtain

$$\begin{aligned}\nu(R_\alpha) &\leqq g_\alpha + 2 \int_{\Sigma^c(2^\alpha)} d\nu(z) \sum_{n=1}^{\varrho 2^{\alpha\nu}} \sum_{s=1}^{\nu} \sum_{i=1}^{n} \int_{\mathcal{M}_n} d(q)_n\, d(p)_n (n!)^{-1} (2\pi\beta^{-1}m)^{-n\nu/2} \\ &\quad \cdot \exp\{-U[(q)_n|z] - T((p)_n) + \beta n\mu\} X \bigcup_{|t|\leqq\tau} S^\alpha(t)\,\Delta_i^s \\ &\leqq g_\alpha + 4\tau \int_{\Sigma c(2^\alpha)} d\nu(z) \sum_{n=1}^{\varrho 2^{\alpha\nu}} n\nu \int_{\Delta_1^1} d(q)_n\, d(p)_n^1 (n!)^{-1} (2\pi\beta^{-1}m)^{-n\nu/2} \\ &\quad \cdot \exp\{-U[(q)_n|z] - T((p)_n) + \beta\mu n\} \cdot v((q)_n)\end{aligned} \tag{3.4}$$

where

$$\begin{aligned} d(p)_n^1 &= \left\{\prod_{j=2}^n dp_j\right\} dp_1^2 \ldots dp_1^\nu \\ v((q)_n) &= \left|\sum_{j \neq 1} \frac{\partial}{\partial q_1} \Phi(|q_1 - q_j|)\right|. \end{aligned} \tag{3.5}$$

Inserting Eq. (3.5) into Eq. (3.4) we have

$$\nu(R_\alpha) \leqq g_\alpha + \exp(-c^2\alpha^2)\, \tau\nu (2\pi\beta^{-1}m)^{-1/2} \varrho 2^{\alpha\nu} \int_{\mathfrak{X}} d\nu(x)\, F_\alpha(x) \tag{3.6}$$

where

$$F_\alpha(x) = \sup_{\{q_0 \in \Sigma(2^\alpha) \cap \hat{q}\}} \partial/\partial q_0 \sum_{q_j \in \hat{q}} \Phi(|q_0 - q_j|).$$

We introduce the following partition of the space $\Gamma(s)$, $s \in \mathbb{R}^\nu$

$$\Gamma(s) = \{x \in \mathbb{R}^\nu \mid (s_k - 1/2)\, r_0 \leqq x_k < (s_k + 1/2)\, r_0\,;\ k = 1, \ldots, \nu\}\,.$$

s_k and x_k are the coordinates of s and k respectively.

Let $\mathscr{S}_0 \subset \mathbb{Z}^\nu$ be the minimum set for which

$$\bigcup_{s \in \mathscr{S}_0} \Gamma(s) \supset \Sigma(2^\alpha + r_0)$$

where r_0 was defined in D. (2.2) (ii).

There exists $a < \infty$ such that

$$\operatorname{Card}(\mathscr{S}_0) = a 2^{\alpha\nu} \qquad \forall \alpha \in \mathbb{Z}^+\,.$$

We have

$$F_\alpha(x) \leqq F \sum_{S \in \mathscr{S}_0} \operatorname{Card}(x \cap \Gamma(s))$$

where

$$F = \max_{x \geqq 0} \left| \frac{\partial}{\partial x} \Phi(x) \right|.$$

Then using Theorem 3.1 we have

$$\begin{aligned} \int d\nu(x)\, F_\alpha(x) &\leqq \sum_{S \in \mathscr{S}_0} F \int d\nu(x) \operatorname{Card}(x \cap \Gamma(s)) \\ &\leqq F \operatorname{Card} \mathscr{S}_0 \sum_m m \exp[-\gamma m^2 (r_0 \sqrt{\nu})^{-\nu} + \delta m]\,. \end{aligned}$$

Inserting this estimate into Eq. (3.6) the thesis is proved. □

Proof of Theorem 2.2. We first show that the limit (2.9) exists in the product topology. We consider $|t| \leqq 1$ and $x \in \hat{\mathfrak{X}}$. Then for every $R_0 > 0$ we shall prove that

$$\sum_{\alpha=2}^{\infty} \sup_{|t| \leqq 1} \|S^\alpha(t)\, x - S^{\alpha-1}(t)\, x\|_{R_0} < \infty \tag{3.7}$$

where $\|\cdot\|_{R_0}$ means $\|\cdot\|_{R_0,x}$ and was defined in *D* (2.5). Equation (3.7) proves that $S^\alpha(t)\,x$ is a Cauchy sequence in the product topology.

Let us write the integral equations for

$$S^\alpha(t)\, x = x + \zeta^\alpha(t) \qquad \zeta^\alpha(t) = (\xi^\alpha(t), \eta^\alpha(t))\,.$$

$\zeta^\alpha(t) = \int_0^t d\tau\, A_i^\alpha[\zeta^\alpha(\tau)]$ + elastic collision on the boundary of $\Sigma(2^\alpha)$ where

$$A_i^\alpha(\zeta^\alpha) = \left(p_i + \eta_i^\alpha, \sum_{j \neq i} \frac{\partial}{\partial q_i} \Phi(|q_i + \xi_i^\alpha - q_j - \xi_j^\alpha|\right) \qquad q_i \in \Sigma(2^\alpha)$$

$$A_i^\alpha(\zeta^\alpha) = 0 \qquad\qquad q_i \notin \Sigma(2^\alpha)$$

so that $\zeta_i^\alpha(t) = 0$ for $|t| \leqq 1$ whenever $q_i \notin \Sigma(2^\alpha)$.

For $q_i \in \Sigma(R_0)$ and for $\alpha \geqq \alpha(x; 1)$ sufficiently large so that Eq. (3.10) below holds

$$|\zeta_i^\alpha(t) - \zeta_i^{\alpha-1}(t)| \equiv |\xi_i^\alpha(t) - \xi_i^{\alpha-1}(t)| \vee |\eta_i^\alpha(t) - \eta_i^{\alpha-1}(t)|$$
$$\leqq \int_0^t dt_1 \{|\eta_i^\alpha(t_1) - \eta_i^{\alpha-1}(t_1)| \vee h(t_1)\}$$

where

$$h(t_1) = \sum_{i \neq j} {}_j \frac{\partial}{\partial q_i}$$
$$\{\Phi(|q_i + \xi_i^\alpha(t_1) - q_j - \xi_j^\alpha(t_1)|) - \Phi(q_i + \xi_i^{\alpha-1}(t_1) - q_j - \xi_j^{\alpha-1}(t_1)|)\}.$$

We easily have

$$h(t_1) \leqq F' \operatorname{Card} G_i 2 \|\zeta^\alpha(t_1) - \zeta^{\alpha-1}(t_1)\|_{R_1} \tag{3.8}$$

where

$$G_i = \{j \in \mathbb{Z} \| q_i + \xi_i - q_j - \xi_j| \leqq r_0 \quad \text{for} \quad |\xi_i| \leqq c\alpha; |\xi_j| \leqq c\alpha\}, \tag{3.9a}$$

$$F' = \max_{x \geqq 0} \left| \frac{d^2}{dx^2} \Phi(x) \right|, \tag{3.9b}$$

$$2c\alpha + R_0 + r_0 \leqq c'\alpha + R_0 = R_1 . \tag{3.9c}$$

Equation (3.8) was obtained by the same estimate on the momenta given in Theorem 2.1 and in the hypothesis that

$$2^{\alpha-1} \geqq R_0 + c\alpha . \tag{3.10}$$

For $c \geqq 1$ we have

$$\operatorname{Card} G_i \leqq \|x\| (2c\alpha + r_0)^\nu = \|x\| (c'\alpha)^\nu . \tag{3.11}$$

Therefore using Eqs. (3.8) and (3.11) we obtain

$$\sup_{q_i \in \Sigma(R_0)} |\zeta_i^\alpha(t) - \zeta_i^{\alpha-1}(t)|$$
$$= \|\zeta^\alpha(t) - \zeta^{\alpha-1}(t)\|_{R_0} \leqq 2F'(c'\alpha)^\nu \int_0^t dt_1 \|\zeta^\alpha(t_1) - \zeta^{\alpha-1}(t_1)\|_{R_1} .$$

We define $n(\alpha)$ as

$$n(\alpha) = \text{Integer part of} \left[\frac{2^{\alpha-1} - R_0}{c'\alpha}\right] \geqq 1 \tag{3.12}$$

and we iterate $n(\alpha)$ times the above procedure to obtain

$$\|\zeta^\alpha(t) - \zeta^{\alpha-1}(t)\|_{R_0}$$
$$\leqq (2F')^{n(\alpha)} (c'\alpha)^{\nu n(\alpha)} \int_0^t dt_1 \ldots \int_0^{t_{n(\alpha)-1}} dt_{n(\alpha)} \|\zeta^\alpha(t_{n(\alpha)}) - \zeta^{\alpha-1}(t_{n(\alpha)})\|_{R_{n(\alpha)}}$$
$$\leqq (2F')^{n(\alpha)} (c'\alpha)^{\nu n(\alpha)} c'\alpha [n(\alpha)!]^{-1} . \tag{3.13}$$

Equation (3.7) is then verified by the estimate (3.13). Therefore we have proved that uniformly in $|t| \leqq 1$

$$\lim_{\alpha \to \infty} \zeta_i^\alpha(t) = \zeta_i(t) < \infty \qquad \forall i \in \mathbb{Z} .$$

We shall now prove that the solution $x+\zeta(t)$ is bounded in the norm of Eq. (2.8) The essential point is that there exists $m\in\mathbb{Z}$ so that for every $R_0=2^{\alpha_0}$ the following holds

$$\sup_{|t|\leqq 1}\|\zeta^\alpha(t)-\zeta^\beta(t)\|_{R_0}\leqq 1 \qquad \beta\geqq\gamma\geqq m\alpha_0\,. \tag{3.14}$$

This because by Eqs. (3.13) and (3.12)

$$\|\zeta^\gamma(t)-\zeta^\beta(t)\|_{R_0}\leqq \sum_{m\alpha_0^\alpha}^{\infty}(2F')^{n(\alpha)}(c'\alpha)^{\nu n(\alpha)}(c'\alpha)(n(\alpha)!)^{-1}\leqq 1$$

for m sufficiently large and every $\alpha_0\in\mathbb{Z}^+$. By Eq. (3.14) we have that

$$\|\zeta^\beta(t)\|_{R_0}\leqq 1+cm\alpha_0\leqq 2cm\alpha_0 \tag{3.15a}$$

and therefore

$$\|\zeta(t)\|_{2^{\alpha_0}}\leqq 2cm\alpha_0\,. \tag{3.15b}$$

By use of Eq. (3.15b) and by proceeding as in the proof of Lemma 3.2 of Ref. [1] we obtain that $\|x+\zeta(t)\|$ is bounded for $|t|\leqq 1$.

To complete the proof of Theorem 2.2 we prove that

$$\lim_{\alpha\to\infty}\zeta^\alpha(t)+x=\zeta(t)+x$$

in the local topology of $\mathfrak{X}$. That is, for every $R_0>0$

$$\lim_{\alpha\to\infty}(\hat{q}+\xi^\alpha(t))\cap\Sigma(R_0)=(\hat{q}+\xi(t))\cap\Sigma(R_0) \tag{3.16}$$

and the same for the momenta. This is ensured by the convergence of $\zeta^\alpha(t)$ in the product topology and by the estimate (3.15).

Define $\alpha(R_0)$ as

$$\alpha(R_0)=\inf\{\beta\in\mathbb{Z}^+|2^\beta-2cm\beta\geqq R_0\}<+\infty\,.$$

Then for $\alpha>\alpha(R_0)$ the particles of $x+\zeta^\alpha(t)$ and $x+\zeta(t)$ which are in $\Sigma(R_0)$ are those initially inside $\Sigma(2^{\alpha(R_0)})$, therefore Eq. (3.16) is implied by

$$\lim_{\alpha\to\infty}\|\zeta^\alpha(t)-\zeta(t)\|_{2^{\alpha(R_0)}}=0\,. \quad \square$$

Proof of Theorem 2.3. (i) We have by Theorem 2.2 that

$$\eta_i(t)=\lim_{\alpha\to\infty}\eta_i^\alpha(t)=\lim_{\alpha\to\infty}-\int_0^t d\tau\sum_{j\neq i}\frac{\partial}{\partial q_i}\Phi(|q_i+\xi_i^\alpha(\tau)-q_j-\xi_j^\alpha(\tau)|)$$

$$=\lim_{\alpha\to\infty}-\int_0^t d\tau\sum_{q_j\in\Sigma(2^{\alpha(R_i)})}\frac{\partial}{\partial q_i}\Phi(|q_i+\xi_i^\alpha(\tau)-q_j-\xi_j^\alpha(\tau)|)$$

$$=-\int_0^t d\tau\sum_{q_j\in\Sigma(2^{\alpha(R_i)})}\lim_{\alpha\to\infty}\frac{\partial}{\partial q_i}\Phi(|q_i+\xi_i^\alpha(\tau)-q_j-\xi_j^\alpha(\tau)|)$$

$$=-\int_0^t d\tau\sum_{j\neq i}\frac{\partial}{\partial q_i}\Phi(|q_i+\xi_i(\tau)-q_j-\xi_j(\tau)|)$$

where

$$\begin{aligned} \alpha_i &= \inf\{\beta \in \mathbb{Z}^+ | q_i \in \Sigma(2^\beta)\}\,; \qquad R_i = 2^{\alpha_i} \\ \alpha(R_i) &= \inf\{\beta \in \mathbb{Z}^+ | 2^\beta - 4cm\beta > r_0 + R_i\}\,. \end{aligned} \tag{3.17}$$

Further

$$\xi_i(t) = \lim_{\alpha\to\infty} \xi_i^\alpha(t) = \lim_{\alpha\to\infty} \int_0^t d\tau(p_i + \eta_i^\alpha(\tau)) = \int_0^t d\tau(p_i + \eta_i(\tau))\,.$$

(ii) The unicity theorem reads as follows. Let $x(t): (-\tau, \tau) \to \hat{\mathfrak{X}}$ which is a bounded [in the norm of Eq. (2.8)] solution of the equation of motion (2.4). Let $x(t) = x(0) + \zeta'(t)$ then unicity means that

$$\zeta'(t) = \zeta(t) = \lim_{\alpha\to\infty} \zeta^\alpha(t)\,.$$

We first note that there exists $\lambda < \infty$ such that

$$|\zeta_i'(t)| \leqq \lambda\alpha_i\,, \tag{3.18a}$$

$$|\zeta_i(t)| \leqq \lambda\alpha_i\,, \tag{3.18b}$$

where α_i was defined in Eq. (3.17). The proof of Eq. (3.18a) follows from the assumed boundedness of $\|x(t)\|$. Since it is analogous to the one of Lemma 3.2 of Ref. [1] we shall omit it. We proceed now in a way completely similar to the one of Theorem 2.2. We consider $\|\zeta(t) - \zeta'(t)\|_{R_0}$ and we iterate n-times the procedure of Eq. (3.13). Therefore the particles we consider will be in the sphere $\Sigma(R_n)$ where

$$R_n = R_0 + n(r_0 + 2\lambda\alpha_n)$$

with

$$\alpha_n = \inf\{\beta \in \mathbb{Z}^+ | n \leqq \text{Integer part of } \{(2^\beta - R_0)(r_0 + 2\lambda\beta)^{-1}\}\}\,.$$

Therefore as in Eq. (3.13)

$$|\zeta_i(t) - \zeta_i'(t)| \leqq (2F')^n (c'\alpha_n)^{n+1} (n!)^{-1} \tag{3.19}$$

with vanishes as n diverges.

(iii) It follows directly from (i).

(iv) It is sufficient to show that for every ν-measurable set $E \subset \hat{\mathfrak{X}}$ and every fixed t, also $S(-t)E$ is ν-measurable and that $\nu(S(-t)E) = \nu(E)$. This amounts to say that for every $\varepsilon > 0$ there exist ν-measurable sets $\bar{A}_\varepsilon$ and $\bar{F}_\varepsilon$ such that

$$\bar{A}_\varepsilon \supset S(-t)E \supset \bar{F}_\varepsilon \qquad \nu(\bar{A}_\varepsilon - \bar{F}_\varepsilon) < \varepsilon\,. \tag{3.20}$$

To prove Eq. (3.20) we use the regularity of the measure ν [see Theorem 3.1 (i)] to find an open set A_ε and a closed set F_ε so that

$$A_\varepsilon \supset E \supset F_\varepsilon \qquad \nu(A_\varepsilon - F_\varepsilon) < \varepsilon\,. \tag{3.21}$$

We then have

$$A_\varepsilon^* = \liminf_{\alpha\to\infty} S^\alpha(-t)\{A_\varepsilon \cap \hat{\mathfrak{X}}\} \supset S(-t)\{A_\varepsilon \cap \hat{\mathfrak{X}}\} \supset S(-t)E \tag{3.22a}$$

because of Eq. (2.9) read in the local topology D. (21) and because A_ε is open. By complementarity in $\hat{\mathfrak{X}}$ from Eq. (3.22a) we obtain

$$F_\varepsilon^* = \limsup_{\alpha\to\infty} S^\alpha(-t)\{F_\varepsilon \cap \hat{\mathfrak{X}}\} \subset S(-t)\{F_\varepsilon \cap \hat{\mathfrak{X}}\} \subset S(-t)E. \quad (3.22b)$$

By Liouville theorem in $\mathfrak{X}[\Sigma(2^\alpha)]$ and the equilibrium equations

$$v(F_\varepsilon) = v[S^\alpha(-t)\{F_\varepsilon \cap \hat{\mathfrak{X}}\}] \qquad v(A_\varepsilon) = v[S^\alpha(-t)\{A_\varepsilon \cap \hat{\mathfrak{X}}\}] \quad (3.23)$$

since $v(\hat{\mathfrak{X}}) = 1$. By a classical theorem in measure theory, [10] III.4.8, and by Eq. (3.23)

$$v(A_\varepsilon^*) \leqq \liminf_{\alpha\to\infty} v[S^\alpha(-t)\{A_\varepsilon \cap \hat{\mathfrak{X}}\}] = v(A_\varepsilon), \quad (3.24a)$$

$$v(F_\varepsilon^*) \geqq \limsup_{\alpha\to\infty} v[S^\alpha(-t)\{F_\varepsilon \cap \hat{\mathfrak{X}}\}] = v(F_\varepsilon). \quad (3.24b)$$

Finally Eqs. (3.21)–(3.23), and (3.24) together imply Eq. (3.20) with $A_\varepsilon^* = \bar{A}_\varepsilon$ and $F_\varepsilon^* = \bar{F}_\varepsilon$. □

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