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The Connection between the Energy-Momentum Tensor and the Tensor Field in Presence of a Mass Gap

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Abstract. The conditions under which a tensor field can be regarded as an energy-momentum tensor are discussed. The problem connected with dilatational and conformal symmetries are exhibited.

1. Introduction

In most of the Lagrangean theories of relativistic fields it is assumed that the trace of the energy-momentum tensor is proportional to the square of the mass of the particle. Therefore for a massless field the trace should vanish. The validity of this assumption in the case of free fields can be explicitly verified.

In this note we are going to show – without having recourse to Lagrangean theory – that in a theory of strictly interacting, quantal fields with a mass gap a symmetrical, local, Poincaré covariant, locally conserved tensor field with a vanishing trace can not be used as an energy-momentum tensor since it gives rise to vanishing generators of the Poincare group.

This result can be used as an criterion to detect massless particles in the theory: the vanishing of the trace of the energy-momentum tensor entails the existence of massless particles.

We are going also to show that the dilatational current built with help of this tensor field yields a vanishing charge.

Contrary to the naive intuitive judgment the conformal current gives rise to a vector charge which does not always vanish and coincides with the energy-momentum vector. This result is verified by inspection on the model of scalar fields. It gives a prescription how to build an energy-momentum vector out of traceless tensor fields.

Finally we give the necessary and sufficient condition which have to be fulfilled by a tensor field in order that it can be used as an energy-momentum tensor.

2. Preliminaries

We list the relevant assumptions.

We shall deal with a Quantum Field Theory described by Wightman's axioms.

In particular, the Poincaré group is unitarily implemented in the Hilbert space $\mathscr{H}$ by the operators $U(A, a)$, where A are the 2×2 matrices belonging to $SL(2, C)$ and $a = (a_0, a_1, a_2, a_3)$. The generators of this group satisfy the Lie-Cartan relations

$$[P_\mu, P_\nu] = 0\,, \tag{2.1a}$$

$$[M_{\alpha\beta}, P_\mu] = i(g_{\beta\mu} P_\alpha - g_{\alpha\mu} P_\beta)\,, \tag{2.1b}$$

$$\begin{aligned}[M_{\alpha\beta}, M_{\gamma\delta}] = i(&g_{\alpha\delta} M_{\beta\gamma} + g_{\beta\gamma} M_{\alpha\delta} \\ &+ - g_{\beta\delta} M_{\alpha\gamma} - g_{\alpha\gamma} M_{\beta\delta})\,.\end{aligned} \tag{2.1c}$$

We assume that the mass spectrum of P^2 contains a discrete eigenvalue $m^2 \neq 0$ separated by a gap from the eigenvalue zero of the unique vacuum state Ω and the continuous part of the spectrum.

The operator algebra is spanned by a finite set of local, Poincaré covariant, strictly interacting fields $\{\varphi\}$ acting as operator-valued distributions in $\mathscr{H}$ and in the 4-dimensional Minkowski space. By strict interaction we mean that each field of the set interacts with another either directly or through the mediation of the rest of the fields [1].

We assume that the states $\varphi(f)\Omega$, $\varphi \in \{\varphi\}$ for $f \in \mathscr{S}_4$ with non-vanishing support for $p^2 = m^2$ in the momentum space contribute on the mass shell m^2. Then the free asymptotic fields $\{\varphi_{in}\}$ and $\{\varphi_{\text{out}}\}$ exist and are different from zero and we assume additionally that they form an irreducible set.

Let us take any local, Poincaré covariant, hermitean tensor field $T_{\mu\nu}(x)$, where $x = (x_0, x_1, x_2, x_3)$, is a point in the Minkowski space, which is local with respect to $\{\varphi\}$ [2], with the properties

$$T_{\mu\nu} = T_{\nu\mu}\,, \tag{2.2a}$$

$$\partial^\mu T_{\mu\nu} = 0\,, \tag{2.2b}$$

$$T^\mu_\mu = 0\,, \tag{2.2c}$$

$$U(A, a)\, T_{\mu\nu}(x)\, U(A, a) = \Lambda_\mu^{-1\sigma}(A)\, \Lambda_\nu^{-1\varrho}(A)\, T_{\sigma\varrho}(\Lambda(A)\, x + a)\,, \tag{2.2d}$$

where $\Lambda(A)$ is the representation of the group $SL(2, C)$ in the Minkowski space. The relations (2.2a–c) are the standard ones satisfied by an ir-

[1] The meaning of strict interaction among the fields is explained in more detail in references [11], [5], and [6].

[2] We could as well consider $T_{\mu\nu}$ quasilocal with respect to $\{\varphi\}$.

reducible $(\frac{1}{2}, \frac{1}{2})$ representation of the Lorentz group with spin one. From (2.2c–d) and

$$U(A, a)\,\Omega = \Omega \tag{2.3}$$

follows immediately that

$$(\Omega, T_{\mu\nu}(f)\,\Omega) = 0\,. \tag{2.2e}$$

Our goal is to disclose some properties of the field $T_{\mu\nu}$ as well as of the translationally non-covariant, locally conserved currents

$$D_\mu(x) = x^\lambda T_{\lambda\mu}\,, \tag{2.4}$$

$$M_{\mu\lambda\nu} = x_\lambda T_{\mu\nu} - x_\nu T_{\mu\lambda}\,, \tag{2.5}$$

$$K_{\mu\nu} = (2x_\nu x_\lambda - x^2 g_{\nu\lambda})\, T^\lambda_\mu\,, \tag{2.6}$$

which we are going to call dilational, rotational and conformal currents resp.

3. The Dilatational and Translational Symmetries

To begin with let us examine the case of the dilatational current (2.4). $D_\mu(x)$ is a local field operator, local with respect to $T_{\mu\nu}$ as well as to $\{\varphi\}$.

We have the

Statement 1. *The charge D induced by the current $D_\mu(x)$ exists and is a self-adjoint operator. D and P_μ have a common dense domain of analytic vectors.*

The proof of the existence of the charge D induced by the translationally non-covariant current

$$D_\mu(x) = x^\lambda T_{\lambda\mu}(x)$$

runs essentially along the same lines as that of Kastler *et al.* [3] for translationally covariant currents and was given by Reeh [12].

To show that D is an essentially self-adjoint operator we observe that

$$i[D, \varphi]$$

for each $\varphi \in \{\varphi\}$, is a local field, relatively local with respect to $\{\varphi\}$. Notice that D is time-independent, in spite of the fact that it does not commute, in general, with the translation generators; we have namely

$$\frac{dD}{dx_0} = \frac{\partial D}{\partial x_0} + i[P_0, D] = 0\,.$$

Hence the field $i[D, \varphi]$ has the asymptotic form

$$i[D, \varphi^{(a)}_{\mathrm{ex}}] = \sum_b c^{(a,b)}\, \varphi^{(b)}_{\mathrm{ex}}$$

where $\varphi_{\text{ex}}^{(a)}$ and $\varphi_{\text{ex}}^{(b)}$ belong to $\{\varphi_{\text{ex}}\}$ and $c^{(a,b)}$ are numerical coefficients, which may also depend on ∂. Following Kraus and Landau [4] as well as Snellman [14] it is easy to show that D acts additively and is bounded on the dense set G_{ex} of asymptotic states describing a finite number of particles with the wave functions in the momentum space belonging to $\mathscr{D}$ and these states are analytic vectors for D with an infinite radius of convergence. Since G_{ex} is dense in $\mathscr{H}$ we may apply to D the well known theorem of Nelson [9] stating that if an operator, hermitean on a dense set G, possesses a dense set G^{an} of analytic vectors and $G^{\text{an}} \subset G$, then this operator is essentially self-adjoint.

The vectors G^{an} are also analytic for P_μ.

This accomplishes the proof.

In view of this statement we shall not go wrong by using in the computations the formal expression

$$\int d^3 x x^\lambda T_{\lambda 0} \tag{3.1}$$

instead of D; everything said below can be put in a rigorous form, if needed.

Since

$$[T_{\lambda\varkappa}, P_\mu] = i\partial_\mu T_{\lambda\varkappa} \tag{3.2}$$

we have

$$\begin{aligned}[D, P_\mu] &= \int d^3 x x^\lambda [T_{\lambda 0}, P_\mu] \\ &= i x_0 \int d^3 x \partial_\mu T_{00} + i \int d^3 x x^i \partial_\mu T_{i0}.\end{aligned} \tag{3.3}$$

For $\mu = 0$

$$[D, P_0] = i\partial_0 \int d^3 x x^i T_{i0} \tag{3.4}$$

in virtue of (see [11])

$$\int d^3 x T_{0\mu} = \alpha P_\mu \tag{3.5}$$

$\alpha - a$ real number.

This equality should be understood in the spirit of the procedure of Kastler, Robinson and Swieca ([3], [13], [15], [10]). Of course, P_μ does not depend on time x_0. According to our Statement 1. (3.1) does not depend on time too, therefore it follows from (3.4) that

$$[D, P_0] = -i\partial_0 \int d^3 x x^0 T_{00} = -i \int d^3 x T_{00} = -i\alpha P_0$$

again because of (3.5). Proceeding in a similar way for $\mu = 1, 2, 3$ we get eventually

$$[D, P_\mu] = -i\alpha P_\mu. \tag{3.6}$$

It is easy to show that

$$[D, M_{\mu\nu}] = 0. \tag{3.7}$$

From (3.6) we get

$$[D, P^2] = -2i\alpha P^2 \tag{3.8}$$

or by exponentiating[3]

$$e^{irD} P^2 e^{-irD} = e^{2\alpha r} P^2 \tag{3.9}$$

where r is real, $-\infty < r < +\infty$.

Let us take any vector Ψ_{m^2} belonging to $\mathscr{H}$ and satisfying the eigenvalue problem

$$P^2 \Psi_{m^2} = m^2 \Psi_{m^2} .$$

We have

$$e^{irD} P^2 \Psi_{m^2} = m^2 e^{irD} \Psi_{m^2} = e^{2\alpha r} P^2 e^{irD} \Psi_{m^2} . \tag{3.10}$$

We see that $e^{irD} \Psi_{m^2}$ is an eigenstate to the eigenvalue $\exp(-2\alpha r) m^2$ of P^2. This leads us to the conclusion that for $\alpha \neq 0$ we may produce nonvanishing eigenstates of P^2 with eigenvalues arbitrary close to m^2, which contradicts our assumptions. Therefore we have $\alpha = 0$, irrespectively whether D vanishes or not.

We have the

Statement 2. *The vector charge induced by the current $T_{\mu\nu}$ vanishes.*

It is well known ([11], [5], [6]) that this vector charge is proportional to P_μ under the assumptions listed in Section 2. The immediate implication of the Statement 2 is the conclusion:

If the theory has a mass gap ($m^2 \neq 0$) the energy-momentum tensor of the field theory has a non-vanishing trace.

4. Rotational (Lorentz) Symmetry

Let us turn towards the rotational currents (2.5). It can be shown along the same lines as it was outlined in the proof of Statement 1 for the case of dilatation that these currents give rise to a selfadjoint tensor charge. If we recall the results obtained by Divgi and Woo [2], which hold true under our assumptions, this tensor charge is proportional to the tensor of the Lorentz generators $M_{\mu\nu}$.

In a formal notation we have

$$\int d^3 x\, M_{0\mu\nu} = \beta M_{\mu\nu} \tag{4.1}$$

β – a real number.

In order to show that $\beta = 0$ let us consider

$$[\beta M_{\mu\nu}, P_\lambda] = \int d^3 x (x_\mu [T_{0\nu}, P_\lambda] - x_\nu [T_{0\mu}, P_\lambda]) .$$

Since $M_{\mu\nu}$ does not depend on time and $\alpha = 0$ (see Statement 2) we get

$$[\beta M_{\mu\nu}, P_\lambda] = 0 . \tag{4.2}$$

[3] This is allowed because of Statement 1.

On the other hand it follows from (2.1b) that

$$[\beta M_{\mu\nu}, P_\lambda] = i\beta(g_{\nu\lambda} P_\mu - g_{\mu\lambda} P_\nu)\,. \tag{4.3}$$

The only way to reconcile (4.2) and (4.3) is to put β equal to zero. To summarize we have the

Statement 3. *Under the assumptions listed in Section 2 the currents $M_{\mu\lambda\nu}$, defined by* (2.5), *give rise to a vanishing tensor charge.*

Statement 3 confirms our conclusion that $T_{\mu\nu}$, satisfying (2.2c), can not be used as a energy-momentum tensor, as long as $m^2 \neq 0$.

5. The Conformal and Dilatational Symmetries

At last let us examine the currents (2.8). Also in the case of these currents we are able to show that they give rise to a time independent, self-adjoint vector charge K_μ.

Using again the formal expressions

$$\int (2x_\mu x_\lambda T_0^\lambda - x^2 T_{\mu 0}) d^3x = \int K_{0\mu} d^3x = K_\mu \tag{5.1}$$

and taking into account that $\alpha = \beta = 0$ (Statement 2 and 3) we get that

$$[P_\mu, K_\nu] = 2i g_{\mu\nu} D \tag{5.2}$$

as well as

$$[M_{\alpha\beta}, K_\mu] = i(g_{\beta\mu} K_\alpha - g_{\alpha\mu} K_\beta)\,. \tag{5.3}$$

We are going now to show that $D = 0$ and $K_\mu = \gamma P_\mu$. We show by inspection on the model consisting only of scalar fields that γ may be different from zero.

Taking any state Ψ_{m^2}, defined in Section 3, we shall show that $K_\nu \Psi_{m^2}$ belongs to the Hilbert subspace $\mathscr{H}_{m^2}$ characterized by the mass eigenvalues m^2, and $D = 0$.

We have shown that $\alpha = 0$. Thus by (3.6) D commutes with P_μ and (5.2) yields

$$(P^2 - m^2) K_\nu \Psi_{m^2} = [P^2, K_\nu] \Psi_{m^2} = 4i P_\nu D \Psi_{m^2}\,. \tag{5.4}$$

Since the vector on the right hand side belongs to $\mathscr{H}_{m^2}$ and $(P^2 - m^2)$ is non zero on the orthogonal complement of $\mathscr{H}_{m^2}$ we conclude

$$K_\nu \Psi_{m^2} \in \mathscr{H}_{m^2} \tag{5.5}$$

and

$$D \mathscr{H}_{m^2} = 0\,. \tag{5.6}$$

From (5.6) and the fact that $i[D, \varphi]$ where $\varphi \in \{\varphi\}$ is local with respect to $\{\varphi\}$ as well as that the free fields $\{\varphi_{in}\}$ or $\{\varphi_{\text{out}}\}$, belonging to the mass m^2, form an irreducible set, follows[4]

$$D = 0\,. \tag{5.7}$$

As far as K_ν is concerned it follows from (5.2) and (5.7) that K_ν is a translationally invariant vector charge. Since $i[K_\nu, \varphi]$ is local with respect to $\{\varphi\}$ and all fields $\{\varphi\}$ interact with each other in a strict sense, the energy-momentum conservation law as well as the irreducibility of the fields $\{\varphi_{in}\}$ imply that K_ν differs from P_ν by a numerical multiplicative factor, viz.

$$K_\nu = \gamma P_\nu \tag{5.12}$$

γ – a real number.

Contrary to a naive intuitive feelings that a conformal current should give rise to vanishing generators K_μ (i.e. $\gamma = 0$) we shall show in the Appendix that γ does not need to vanish. To this end we examine the model consisting only of a set of scalar fields $\{\varphi\}$. Taking into account that the charges are determined uniquely as operator expressions bilinear in the incoming or outgoing fields [7] and with the help of the Araki and Haag asymptotic procedure [1] we are able to find the exact form of these expressions and evaluate K_μ[5].

The results of this section can be comprised in the

Statement 4. *In a field theory with a mass gap the charge D induced by the current D_μ, defined by* (2.4), *constructed with the help of a traceless tensor $T_{\mu\nu}$, vanishes.*

Statement 5. *Under the same premises as Statement* 4 *the conformal current $K_{\mu\nu}$, defined by* (2.6), *gives rise to a vector charge proportional to P_μ in general different from zero.*

Statement 5 provides us with a method in recovering the energy-momentum vector using a traceless tensor field.

6. When a Tensor Field can be a Good Candidate for an Energy-Momentum Tensor

It seems to be obvious from the preceeding discussion that every genuine tensor field is a poor candidate for a current, in particular for an energy-momentum tensor as long as a mass gap is present.

[4] From $D\Omega = 0$ and $D\varphi(f)_{\text{ex}}\,\Omega = 0$ for each $\varphi \in \{\varphi\}$ follows $i[D, \varphi(f)]_{\text{ex}}\Omega = 0$; the Reeh and Schlieder Theorem as well as the irreducibility of $\{\varphi_{\text{ex}}\}$ imply then $[D, \varphi_{\text{ex}}] = 0$ and finally (5.7).

[5] A similar result computed also for the model of scalar fields was obtained by Miss M. Karlic using different techniques (private information).

However, not every tensor field satisfying (2.2a, b, d) with a non-vanishing trace can be used as an energy-momentum tensor.

The condition

$$T^\mu_\mu \neq 0 \tag{6.1}$$

is a necessary but not a sufficient condition for that purpose.

To see this more clearly let us consider the following example of a tensor field $A_{\mu\nu}$ which in addition to (2.2a, b, d) and (6.1) satisfies

$$\int d^3 x A_{0\mu} = \alpha P_\mu ; \quad \alpha \neq 0 , \tag{6.2}$$

and

$$\Box A^\mu_\mu \neq 0 , \tag{6.1a}$$

$$(\Box + m^2) A^\mu_\mu \neq 0 . \tag{6.1b}$$

The relation (6.2) has to be again understood in the sense of Kastler, Robinson and Swieca.

If this field has the asymptotic free fields $A_{\mu\nu,\mathrm{ex}}$ belonging to mass $m \neq 0$ then these asymptotic fields have also the properties (2.2a, b, d) but do not need any longer to satisfy (6.1); definitely they do not satisfy (6.2), i.e.

$$\int d^3 x A_{0\mu,\mathrm{ex}} = 0 .$$

It is easy to find in the Borchers class of $A_{\mu\nu}$ a whole set of tensor fields, say $A'_{\mu\nu}$, which satisfy (2.2a, b, d) and (6.1), have the same asymptotic fields as $A_{\mu\nu}$ but violate the condition (6.2); e.g. the field

$$A'_{\mu\nu} = - \frac{\Box}{m^2} A_{\mu\nu} \tag{6.3}$$

belongs to this set [in virtue of (6.1a)].

Notice that both fields, $A_{\mu\nu}$ and $A'_{\mu\nu}$, are irreducible when $A_{\mu\nu,\mathrm{in}}$ is irreducible. Thus irreducibility of the fields is irrelevant for their ability to induce a charge.

Of course, we may find also in the same Borchers class another set of fields, say $A^{||}_{\mu\nu}$, which again satisfy the conditions (2.2a, b, d) and (6.1) and – although their asymptotic fields vanish – yield the proper charge, i.e. satisfy (6.2); e.g. such a field is

$$A^{||}_{\mu\nu} = \frac{1}{m^2} (\Box + m^2) A_{\mu\nu} \tag{6.4}$$

[again in virtue of (6.1b)].

We conclude from this simple consideration that the elements of the Borchers class are equivalent as far as the Lehmann, Symanzik, and Zimmermann limit is concerned; they yield the same S-Matrix. They are not, however, equivalent for the Araki and Haag limit [1]; not all of them yield the same charge. Thus the notion of the Borchers class is relevant for the Lehmann, Symanzik and Zimmermann limit but not for the Araki and Haag limit and for the theory of currents and charges.

We have the obvious

Statement 6. *Every field $A_{\mu\nu}$ can be split in a variety of ways into a Lehmann, Symanzik, and Zimmermann field and into an Araki and Haag field, viz.*

$$A_{\mu\nu}^{\text{L.S.Z.}}(x) = W\left(-\frac{\square}{m^2}\right) A_{\mu\nu}$$

$$A_{\mu\nu}^{\text{A.H.}}(x) = A_{\mu\nu} - W\left(-\frac{\square}{m^2}\right) A_{\mu\nu}$$

where $W(z)$ is a polynomial in z and $W(0)=0$, $W(1)=1$, both belonging to the same Borchers class as $A_{\mu\nu}$.

We return to the question what is the necessary and sufficient condition for $T_{\mu\nu}$ to be a good candidate for an energy-momentum tensor. As we mentioned already earlier, it follows from known theorems [7] [6] that in an expansion of $T_{\mu\nu}$ in terms of normal products of free asymptotic fields only the bilinear term contributes to the charge. To be more exact the relevant part of $T_{\mu\nu}$ as far as the charge is concerned is

$$\begin{aligned}(\Psi([p], \sigma; S, l), T_{\mu\nu}(x)\, \Psi([p'], \sigma'; S', l')) \\ = \exp\{i(p-p')x\}\, F_{\mu\nu\sigma\sigma'}^{SS'll'}(p, p')\,.\end{aligned} \tag{6.5}$$

Here

$$\Psi([p], \sigma; S, l) \tag{6.6}$$

denotes a one particle state of momentum p, $p^2 = m^2$, in the Wigner basis, where $[p]$ indicates that p is taken in an arbitrary but fixed Lorentz tetrad system $\{1/mp, n_1, n_2, n_3\}$ and σ is the spin variable $\sigma = -S$, $-S+1, \ldots, S$ while l labels the fields belonging to the same spin index S (see e.g. [8]). The state (6.6) transforms under the Poincaré transformation according to

$$\begin{aligned}&U(A, a)\, \Psi([p], \sigma; S, l) \\ &\quad = e^{i\Lambda(A)p \cdot a} \sum_{\sigma'} \Psi([Ap], \sigma'; S, l)\, \mathscr{D}_{\sigma'\sigma}^{S}([Ap]^{-1} A[p])\end{aligned}$$

with $\mathscr{D}_{\sigma\sigma'}^{S}(A)$ being the $(2S+1)$-dimensional representation of the rotation group ($A \in SU(2)$). Let us introduce the variables

$$p - p' = s\,, \qquad \frac{p+p'}{2} = r\,, \tag{6.7a}$$

$$\frac{s^2}{4} + r^2 = m^2\,, \qquad sr = 0\,, \tag{6.7b}$$

$$s^2 \leqq 0\,; \qquad r \in V_+\,; \qquad r^2 \geqq m^2\,. \tag{6.7c}$$

Using these variables we can write

$$(\Psi([p],\sigma;S,l),\, T_{\mu\nu}(x)\,\Psi([p'],\sigma';S',l'))
= e^{isx}\,\hat{F}^{SS'll'}_{\mu\nu\sigma\sigma'}(s,r) \tag{6.5a}$$

where $\hat{F}$ is a polynomial in r_μ. Then

$$\int d^3x \int dp \int dp'\,\theta(p)\,\theta(p')\,\delta(p^2-m^2)\,\delta(p'^2-m^2)
\cdot \sum_S \sum_{S'} \sum_\sigma \sum_{\sigma'} \sum_l \sum_{l'} a^{+Sl}_\sigma(p)\, a^{S'l'}_\sigma(p')\,(\Psi(p;\sigma;S,l),\, T_{0\mu}(x)\,\Psi(p',\sigma',S',l')) \tag{6.8a}$$

$$= (2\pi)^3 \int \frac{d^3p}{2p_0}\,\frac{d^3p'}{2p'_0}\, a^+(p)\,a(p')\,\delta(\mathbf{s})\,\hat{F}_{0\mu}(s,r)\Big|_{p_0=p'_0=\omega} \tag{6.8b}$$

$$= \beta P_\mu \qquad \omega = +\sqrt{\mathbf{p}^2+m^2} \tag{6.8c}$$

where β is some constant. In (6.8b) we omit the summation indices. (6.8a) is a shorthand notation for a properly smeared out operator expression depending on the parameter R ($R>0$) taken between two quasilocal states. The exchange of the integrations in (6.8b) is legitimate since the expression decreases sufficiently fast when $R\to\infty$ as the consequence of $m^2 \neq 0$.

If

$$\hat{F}_{0\mu}(s,r) \doteq s_\lambda f^\lambda_{0\mu}(s,r)$$

and f^λ is not singular for $s\to 0$, then (6.8) vanishes, i.e. $\beta=0$.

If

$$\hat{F}_{0\mu}(s,r)|_{s=0} \neq 0$$

then

$$(2\pi)^3 \int \frac{d^3p}{4p_0^2}\, a^+(p)\,a(p')\,\hat{F}_{0\mu}(0,p)\Big|_{p_0=\omega} = \beta P_\mu \tag{6.9}$$

with $\beta \neq 0$.

Thus we have the

Statement 7. *The necessary and sufficient condition for $T_{\mu\nu}(x)$ to yield a non-vanishing charge is that*

$$F^{SS'll'}_{\mu\nu\sigma\sigma'}(p,p') = \hat{F}^{SS'll'}_{\mu\nu\sigma\sigma'}(s,r)$$

defined in (6.5), *has the properties*

$$\hat{F}_{0\nu}(0,p)|_{p_0=\omega} \neq 0$$

and is a polynomial in p_1, p_2, p_3 as well as

$$\hat{F}^\mu_\mu(s,r) \neq 0\,.$$

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Appendix

The computations reported below, although straightforward, are lengthy. Therefore we shall only outline the main steps in obtaining the final results.

For $T_{\mu\nu}$, which meets the requirements listed in Section 2, we get

$$\begin{aligned} e^{-i(p-p')x}(\Psi(p;j), T_{\mu\nu}(x)\,\Psi(p';k)) &\equiv F_{\mu\nu jk}(p,p') \\ &= A_{jk}(s^2)\,\{3s^2 r_\mu r_\nu + r^2(s_\mu s_\nu - s^2 g_{\mu\nu})\} \end{aligned} \tag{B.1}$$

where

$$p, p' \in V_+\,, \qquad p^2 = p'^2 = m^2 \neq 0\,. \tag{B.1a}$$

j and k label the underlying scalar fields, s and r are given by (6.7), Ψ stands for a one particle state, A_{jk} is a nonsingular matrix, a function of s^2 for which

$$A_{jk} = A_{kj} = \overline{A_{jk}}\,. \tag{B.1b}$$

We assume that

$$A_{jk}(0) \neq 0\,. \tag{B.1c}$$

To obtain (B.1) and (B.1b) we used the properties of $T_{\mu\nu}$ [for (B.1) translational and Lorentz covariance, the symmetry, tracelessness and local conservation of the tensor, for (B.1b) – the hermiticity and the TCP covariance]. If we drop (2.2c) we get for $T'_{\mu\nu}$ (with $T'^{\mu}_{\mu} \neq 0$)

$$\begin{aligned} (\Psi(p;j), T'_{\mu\nu}(x)\,\Psi(p';k))\,e^{-i(p-p')x} &\equiv F'_{\mu\nu jk}(p,p') \\ = B_{jk}(s^2)\,r_\mu r_\nu + C_{jk}(s^2)\,(s_\mu s_\nu - s^2 g_{\mu\nu}) & \end{aligned} \tag{B.2}$$

with

$$B_{jk} = B_{kj} = \overline{B_{jk}}\,; \qquad C_{jk} = C_{kj} = \overline{C_{jk}}\,. \tag{B.2a}$$

If $B_{jk}(0) \neq 0$ and $C_{jk}(0) \neq 0$ it is obvious from the Statement 7 that only the term $B_{jk} r_\mu r_\nu$ in (B.2) is responsible for inducing the energy-momentum vector P_μ. It is also obvious from (B.1) that $T_{\mu\nu}$ can never induce an energy momentum vector different from zero.

We get the local Araki and Haag tensor fields $T^{\mathrm{A.H.}}_{\mu\nu,\mathrm{ex}}$ and $T'^{\mathrm{A.H.}}_{\mu\nu,\mathrm{ex}}$ corresponding to $T_{\mu\nu}$ and $T'_{\mu\nu}$ in (B.1) and (B.2) resp. by the substitution

$$p \to -i\frac{\partial}{\partial x}; \qquad p' \to +i\frac{\partial}{\partial y}; \qquad A_{jk}(s^2) \to A_{jk}(0)$$

in $F_{\mu\nu jk}(p, p')$ etc., by applying the action of this expression on $:\varphi_{\mathrm{ex}}(x)\,\varphi_{\mathrm{ex}}(y):$ and finally by putting $x = y$. Notice that

$$(\partial^x_\mu + \partial^y_\mu) :\varphi_{\mathrm{ex}}(x)\,\varphi_{\mathrm{ex}}(y):|_{x=y} = \partial^x_\mu :\varphi^2:(x)\,.$$

Hence we get

$$\begin{aligned} T^{\mathrm{A.H.}}_{\mu\nu,\mathrm{ex}} = \Sigma a_{jk} :\{ & 3\Box_x[(\partial^x_\mu - \partial^y_\mu)(\partial^x_\nu - \partial^y_\nu)\,\varphi_j(x)\,\varphi_k(y)]_{x=y} \\ & + \partial^x_\mu \partial^x_\nu [(\partial^x - \partial^y)^2\,\varphi_j(x)\,\varphi_k(y)]_{x=y} \\ & - \Box_x[(\partial^x - \partial^y)^2\,\varphi_j(x)\,\varphi_k(y)]_{x=y}\, g_{\mu\nu}\}: \\ & a_{jk} = \text{constant matrix} \end{aligned} \tag{B.3}$$

and

$$\begin{aligned} T'^{\mathrm{A.H.}}_{\mu\nu,\mathrm{ex}} = {} & \Sigma\, b_{jk} :[(\partial^x_\mu - \partial^y_\mu)(\partial^x_\nu - \partial^y_\nu)\,\varphi_j(x)\,\varphi_k(y)]:_{x=y} \\ & + \Sigma\, c_{jk}[\partial^x_\mu \partial^x_\nu - \Box_x g_{\mu\nu}] :\varphi_j \varphi_k:(x) \end{aligned} \tag{B.4}$$

(with $\varphi_j \equiv \varphi_{j,\mathrm{ex}}$). If we put in (B.4) $b_{jk} = -4c_{jk} = \delta_{jk}$ we get the standard free field energy-momentum tensor.

To evaluate K_μ it is enough to compute only one of its components, because of the covariance properties of K_μ. Thus we shall concentrate on the evaluation of K_0. Notice that formally

$$\begin{aligned} \int d^3x\, K_{00} &= x_0 \int d^3x\, x^\lambda\, T_{\lambda 0} - x_0^2 \int d^3x\, T_{00} \\ &\quad + \int d^3x\, \boldsymbol{x}^2\, T_{00} = \int d^3x\, \boldsymbol{x}^2\, T_{00} \end{aligned} \tag{B.5}$$

in virtue of Statement 2 and 4. The existence of the vector charge K_μ is ensured by our considerations in Appendix A, therefore we do not need to bother about smearing the operator over x with proper test functions. Let us examine the expression

$$\begin{aligned} & \int d^3x (\Psi, K_{00}\Phi) = (\Psi, K_0 \Phi) \\ & = -\int d^3x \int dp\,dp'\, \theta(p_0)\,\theta(p'_0)\,\delta(p^2 - m^2)\,\delta(p'^2 - m^2) \\ & \quad \cdot \Sigma\, \overline{\Psi}(p; j)\,\Phi(p'; k)\,F_{00jk}(p, p')\,(\nabla^p)^2\, e^{i(p-p')x} \end{aligned} \tag{B.6}$$

where Ψ and Φ are arbitrary one particle states in $\mathscr{H}$ and

$$\Phi(p; k) = (\Psi(p; k), \Phi)\,, \quad \text{etc.} \tag{B.6a}$$

A straightforward computation yields

$$(\Psi, K_0 \Phi) = -(2\pi)^3 \int dp\, \Sigma \frac{\Phi(\boldsymbol{p}, \omega; k)}{2\omega} e^{i(p_0-\omega)x_0}$$
$$(\nabla^p)^2 \left[\frac{\delta(p_0-\omega)}{2\omega} \bar{\Psi}(p;j)\, F_{00jk}(p,p') \right]_{\substack{\boldsymbol{p}'=\boldsymbol{p}\\ p_0'=\omega}} \tag{B.7}$$

where $\omega = \sqrt{\boldsymbol{p}^2 + m^2}$. On the r.h.s. of (B.7) the surface terms like

$$\int dp (\nabla^p)^2 (\cdots) \quad \text{or} \quad \int dp\, \partial_\mu^p(\cdots)$$

were left out as it can be shown by power counting of $\boldsymbol{p}$ that they do not contribute to the expression.

If we take into account the functional form of F_{00jk} presented in (B.1) as well as that

$$F_{00jk}(p,p) = \frac{\partial F_{00jk}}{\partial p_0}(p,p) = \frac{\partial F_{00jk}}{\partial p_i}(p,p) = \frac{\partial^2 F_{00jk}}{\partial p_0\, \partial p_i}(p,p) = 0$$

the relevant terms in (B.7) yield the result

$$\begin{aligned}(\Psi, K_0 \Phi) &\\ = -(2\pi)^3 &\int \frac{d^3 p}{2\omega} \overline{\Sigma\, \Psi(p;j)}\, A_{jk}(0)\, \Phi(p;k) \left[\frac{3\boldsymbol{p}^2}{\omega} - 9\omega + \frac{3p^2}{\omega} \right] \\ = (2\pi)^3\, 6 &\int \frac{d^3 p}{2\omega}\, \omega\, \Sigma\, A_{jk}(0)\, (\Psi, a_{\text{ex}}^+(p;j)\, a_{\text{ex}}(p';k)\, \Phi) \Big|_{p_0=p_0'=\omega} .\end{aligned} \tag{B.8}$$

Now $A_{jk}(0)$ can be diagonalized (it is a real, symmetrical matrix) and the assumption about strict interaction among the fields can be exploited, which leads us to

$$A_{jk}(0) = \delta_{jk} \gamma \frac{1}{6(2\pi)^3}$$

or, inserted in (B.8), yields (5.12) with $\gamma \neq 0$.

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