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A Generalization of the FKG Inequalities

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Abstract. We generalize a theorem of Holley to include the case of continuous spins. Holley's theorem is itself a generalization of the inequalities due to Fortuin, Kastelyn and Ginibre.

1. Introduction

In the study of correlation functions for the Ising and other lattice models in statistical mechanics the inequalities of Fortuin, Kastelyn and Ginibre [2] (the FKG inequalities) play a fundamental role. The object of this paper is to give a proof of some generalized FKG inequalities which include the case of continuous spins. Results of this type have been obtained from the original FKG inequalities by using discrete approximations (see [5]); also a direct proof has been given by Cartier [6]. In this paper we will in fact generalize a result of Holley [3], which easily implies the FKG inequalities. Let Λ be a finite set and let $\mathscr{P}(\Lambda)$ denote the set of subsets of Λ. Suppose $\mu_1, \mu_2 : \mathscr{P}(\Lambda) \to \mathbb{R}$ are probability densities, i.e. $\mu_i \geqq 0$ and

$$\sum_{A \subset \Lambda} \mu_i(A) = 1 \quad \text{for} \quad i = 1, 2 .$$

Then we have:

Theorem 1 (Holley [3]). *If for all $A, B \in \mathscr{P}(\Lambda)$*

$$\mu_1(A \cup B)\, \mu_2(A \cap B) \geqq \mu_1(A)\, \mu_2(B)$$

then

$$\sum_{A \subset \Lambda} h(A)\, \mu_1(A) \geqq \sum_{A \subset \Lambda} h(A)\, \mu_2(A)$$

for any increasing $h : \mathscr{P}(\Lambda) \to \mathbb{R}$ (where by increasing we mean that $h(A) \geqq h(B)$ whenever $A \supset B$).

Using the well-known result of Birkhoff [1] that any finite distributive lattice is isomorphic to some sub-lattice of $\mathscr{P}(\Lambda)$ for some finite set Λ, it follows that Theorem 1 is true for any finite distributive lattice (where we replace $\cup$ by $\vee$ and $\cap$ by $\wedge$). From Theorem 1 we get the FKG inequalities.

Theorem 2 (FKG *inequalities*). *Let $\mu : \mathscr{P}(\Lambda) \to \mathbb{R}$ be a probability density such that for all $A, B \in \mathscr{P}(\Lambda)$*

$$\mu(A \cup B)\, \mu(A \cap B) \geqq \mu(A)\, \mu(B)\,.$$

Then for any increasing functions $f, g : \mathscr{P}(\Lambda) \to \mathbb{R}$ we have

$$\sum_{A \subset \Lambda} f(A)\, g(A)\, \mu(A) \geqq \sum_{A \subset \Lambda} f(A)\, \mu(A) \sum_{A \subset \Lambda} g(A)\, \mu(A)\,.$$

Proof. By adding a constant we can assume that $g > 0$. Define $\mu_2 = \mu$ and

$$\mu_1 = \left[\sum_{B \subset \Lambda} g(B)\, \mu(B)\right]^{-1} g\mu\,.$$

Then μ_1, μ_2 satisfy the hypotheses of Theorem 1 and thus

$$\sum_{A \subset \Lambda} f(A)\, \mu_1(A) = \left[\sum_{B \subset \Lambda} g(B)\, \mu(B)\right]^{-1} \sum_{A \subset \Lambda} f(A)\, g(A)\, \mu(A) \geqq \sum_{A \subset \Lambda} f(A)\, g(A)\,.$$

We will now state our generalization of Theorem 1. The setting will be a finite product of totally ordered measure spaces. Let Λ again be a finite set and for each $t \in \Lambda$ let $(X_t, \mathscr{F}_t, \omega_t)$ be a measure space with ω_t a non-negative σ-finite measure. Suppose that X_t is equipped with a total order $\geqq$ that is $\mathscr{F}_t$-measurable, i.e. $\{(x, y) \in X_t \times X_t : x \geqq y\} \in \mathscr{F}_t \times \mathscr{F}_t$. Let us denote $\prod_{t \in \Lambda} X_t$ by X and the corresponding product σ-algebra $\prod_{t \in \Lambda} \mathscr{F}_t$ by $\mathscr{F}$, and let $\omega = \prod_{t \in \Lambda} \omega_t$. Suppose $f_1, f_2 : X \to \mathbb{R}$ are $\mathscr{F}$-measurable with the properties (1) $f_1, f_2 \geqq 0$; (2) $\int f_1\, d\omega = \int f_2\, d\omega = 1$. For $i = 1, 2$ let μ_i denote the probability measure $f_i \omega$ on $(X, \mathscr{F})$.

Theorem 3. *Suppose f_1, f_2 satisfy*

$$f_1(x \vee y)\, f_2(x \wedge y) \geqq f_1(x)\, f_2(y) \quad \text{for all} \quad x, y \in X$$

(where if $x = \{x_t\}_{t \in \Lambda}$, $y = \{y_t\}_{t \in \Lambda}$ then $x \vee y = \{\max(x_t, y_t)\}_{t \in \Lambda}$ $x \wedge y = \{\min(x_t, y_t)\}_{t \in \Lambda}$). If $h : X \to \mathbb{R}$ is bounded, $\mathscr{F}$-measurable and increasing (i.e. $h(x) \geqq h(y)$ if $x_t \geqq y_t$ for all $t \in \Lambda$) then

$$\int_X h\, d\mu_1 \geqq \int_X h\, d\mu_2\,.$$

Remarks. (1) Theorem 1 follows of course from Theorem 3 by taking $X_t = \{0, 1\}$ for all $t \in \Lambda$ and letting ω_t be counting measure on $\{0, 1\}$.

(2) Nothing would probably be lost if we replaced each X_t by $\mathbb{R}$; we use the present set-up to emphasize that the result only depends on the properties of a total order.

2. Proof of the Theorem

The proof of Theorem 3 is based on a proof of Theorem 1 due to Holley [4]. (Holley's original proof of Theorem 1 in [3] was based on the coupling of two Markov chains whose equilibrium distributions were μ_1 and μ_2.) The first step is to change the problem and consider the following:

Proposition 1. *Suppose* f_1, f_2 *satisfy*

$$f_1(x \vee y)\, f_2(x \wedge y) \geqq f_1(x)\, f_2(y) \quad \textit{for all} \quad x, y \in X\,.$$

Then there exists a probability measure ν *on* $(X \times X,\ \mathscr{F} \times \mathscr{F})$ *such that*

$$\nu(A \times X) = \mu_1(A) \quad \textit{for all} \quad A \in \mathscr{F} \tag{1}$$

$$\nu(X \times B) = \mu_2(B) \quad \textit{for all} \quad B \in \mathscr{F} \tag{2}$$

$$\nu(\{(x, y) \in X \times X : x \geqq y\}) = 1, \ (\textit{where } x \geqq y \textit{ means that } x_t \geqq y_t \quad \textit{for all} \quad t \in \Lambda)\,. \tag{3}$$

(1) and (2) say that the projection of ν onto the first (resp. second) factor is μ_1 (resp. μ_2). Theorem 3 is an immediate consequence of Proposition 1, since if $h : X \to \mathbb{R}$ is as in Theorem 3 and if we write $E = \{(x, y) \in X \times X : x \geqq y\}$ then we have

$$\begin{aligned}\int_X h\, d\mu_1 - \int_X h\, d\mu_2 &= \int_{X \times X} (h(x) - h(y))\, d\nu(x, y)\\ &= \int_E (h(x) - h(y))\, d\nu(x, y) \geqq 0\,,\end{aligned}$$

because $h(x) - h(y) \geqq 0$ if $(x, y) \in E$.

We will prove Proposition 1 by induction on $|\Lambda|$, the cardinality of Λ. The following notation will be useful: for $A \subset \Lambda$ let

$$X(A) = \prod_{t \in A} X_t\,, \qquad \mathscr{F}(A) = \prod_{t \in A} \mathscr{F}_t\,, \qquad \omega_A = \prod_{t \in A} \omega_t\,.$$

Suppose for the moment that $|\Lambda| \geqq 2$, let $t \in \Lambda$ and put $\Lambda' = \Lambda - \{t\}$. For $i = 1, 2$ let $\varrho(\mu_i)$ denote the projection of μ_i onto $X(\Lambda')$. Then we have $\varrho(\mu_i) = g_i \omega_\Lambda$, where $g_i : X(\Lambda') \to \mathbb{R}$ is given by

$$g_i(x) = \int_{X_t} f_i(x, \xi)\, d\omega_t(\xi)\,.$$

Lemma 1. *Suppose that for all* $x, y \in \Lambda$

$$f_1(x \vee y)\, f_2(x \wedge y) \geqq f_1(x)\, f_2(y)\,.$$

Then for all $x', y' \in \Lambda'$ *we have*

$$g_1(x' \vee y')\, g_2(x' \wedge y') \geqq g_1(x')\, g_2(y')\,.$$

Proof. Let $G=\{(\xi,\eta)\in X_t\times X_t:\xi>\eta\}$, $E=\{(\xi,\eta)\in X_t\times X_t:\xi=\eta\}$, $L=\{(\xi,\eta)\in X_t\times X_t:\xi<\eta\}$. Then

$$g_1(x'\vee y')\,g_2(x'\wedge y')=\iint\limits_{G\cup E\cup L} f_1(x'\vee y',\xi)\,f_2(x'\wedge y',\eta)\,d\omega_t(\xi)\,d\omega_t(\eta)$$

$$=\iint\limits_{E} f_1(x'\vee y',\xi)\,f_2(x'\wedge y',\eta)\,d\omega_t(\xi)\,d\omega_t(\eta)$$

$$+\iint\limits_{G}\{f_1(x'\vee y',\xi)\,f_2(x'\wedge y'\,\eta)+f_1(x'\vee y',\eta)\,f_2(x'\wedge y',\xi)\}\,d\omega_t(\xi)\,d\omega_t(\eta).$$

Similarly

$$g_1(x')\,g_2(y')=\iint\limits_{E} f_1(x',\xi)\,f_2(y',\eta)\,d\omega_t(\xi)\,d\omega_t(\eta)$$

$$+\iint\limits_{G}\{f_1(x',\xi)\,f_2(y',\eta)+f_1(x',\eta)\,f_2(y',\xi)\}\,d\omega_t(\xi)\,d\omega_t(\eta)\,.$$

But by hypothesis we have

$$f_1(x'\vee y',\xi)\,f_2(x'\wedge y',\xi)\geqq f_1(x',\xi)\,f_2(y',\xi)$$

and thus we can ignore the terms involving integrations over E. The proof of the lemma would therefore be complete if we could show that

$$f_1(x'\vee y',\xi)\,f_2(x'\wedge y',\eta)+f_1(x'\vee y',\eta)\,f_2(x'\wedge y',\xi)$$
$$\geqq f_1(x',\xi)\,f_2(y',\eta)+f_1(x',\eta)\,f_2(y',\xi)$$

whenever $\xi>\eta$. Let us write

$$a=f_1(x'\vee y',\xi)\,f_2(x'\wedge y',\eta)\,,$$
$$b=f_1(x'\vee y',\eta)\,f_2(x'\wedge y',\xi)\,,$$
$$c=f_1(x',\xi)\,f_2(y',\eta)\,,$$
$$d=f_1(x',\eta)\,f_2(y',\xi)\,.$$

It is easily checked that if $\xi>\eta$ then by hypothesis we have $a\geqq c$, $a\geqq d$ and $ab\geqq cd$. We want, of course, to show that $a+b\geqq c+d$, and this follows from Lemma 2.

Lemma 2. *Let a, b, c, d be non-negative real numbers with $a\geqq c$, $a\geqq d$ and $ab\geqq cd$. Then $a+b\geqq c+d$.*

Proof. If $a=0$ then $c=d=0$ and the result is true; thus we can assume that $a>0$. Now $(a-c)(a-d)\geqq 0$ which gives $aa+cd\geqq ac+ad$ and since $cd\leqq ab$ we get $aa+ab\geqq ac+ad$. Hence dividing by a gives the result.

At this point it is worth outlining how the proof of Proposition 1 will proceed. Suppose the proposition is true for all sets with cardinality less than $|\Lambda|$; then from Lemma 1 there exists a probability measure ν' on $(X(\Lambda') \times X(\Lambda'), \mathscr{F}(\Lambda') \times \mathscr{F}(\Lambda'))$ such that

$$\nu'(A \times X(\Lambda')) = \varrho(\mu_1)(A) \quad \text{for all} \quad A \in \mathscr{F}(\Lambda')\,; \tag{1}$$

$$\nu'(X(\Lambda') \times B) = \varrho(\mu_2)(B) \quad \text{for all} \quad B \in \mathscr{F}(\Lambda')\,; \tag{2}$$

$$\nu'(\{(x', y') \in X(\Lambda') \times X(\Lambda') : x' \geqq y'\}) = 1\,. \tag{3}$$

Now we can write $\mu_i(x', \xi) = F_i(x', \xi)\, \varrho(\mu_i)(x') \times \omega_t(\xi)$ where $F_i(x', \xi)$ as a function of ξ is the conditional density (with respect to ω_t) of μ_i on X_t given the event x' on $X(\Lambda')$. (Equivalently F_i is the Radon Nikodym derivative of μ_i with respect to $\varrho(\mu_i) \times \omega_t$.) We will show that if $x' \geqq y'$ then

$$F_1(x', \xi \vee \eta)\, F_2(y', \xi \wedge \eta) \geqq F_1(x', \xi)\, F_2(y', \eta) \quad \text{for all} \quad \xi, \eta \in X_t$$

and thus from Proposition 1 for the case of cardinality 1 we have there exists a probability measure $M(x', y')$ on $(X_t \times X_t, \mathscr{F}_t \times \mathscr{F}_t)$ such that

$$M(x', y')(A \times X_t) = \int_A F_1(x', \xi)\, d\omega_t(\xi) \quad \text{for all} \quad A \in \mathscr{F}_t\,; \tag{1}$$

$$M(x', y')(X_t \times B) = \int_B F_2(y', \eta)\, d\omega_t(\eta) \quad \text{for all} \quad B \in \mathscr{F}_t\,; \tag{2}$$

$$M(x', y')(\{(\xi, \eta) \in X_t \times X_t : \xi \geqq \eta\}) = 1\,. \tag{3}$$

Then if we define a probability measure ν on $(X \times X, \mathscr{F} \times \mathscr{F})$ by

$$\nu(x', y', \xi, \eta) = \nu'(x', y')\, M(x', y'; \xi, \eta)$$

it is not difficult to show that ν has the right properties. Of course, the above recipe for a proof raises some problems, the most serious of which is whether the measures $M(x', y')$ can be chosen to depend in a measurable way on x' and y'. We get round this problem by giving an explicit formula for $M(x', y')$.

$M(x', y')$ comes from the case $|\Lambda| = 1$ of Proposition 1 and since we need to solve this case anyway to start the induction we will now look at it. Let α be a non-negative σ-finite measure on a measurable space $(Y, \mathscr{B})$ and suppose that Y is equipped with a $\mathscr{B}$-measurable total order $\geqq$. Let h_1, h_2 be the densities with respect to α of probability measures γ_1, γ_2 on $(Y, \mathscr{B})$, and let $\bar{\alpha}$ be the measure on $(Y \times Y, \mathscr{B} \times \mathscr{B})$ got by projecting α onto the diagonal of $Y \times Y$; thus if $B \in \mathscr{B} \times \mathscr{B}$ then

$$\bar{\alpha}(B) = \alpha(\{y \in Y : (y, y) \in B\})\,.$$

Define a probability measure δ on $(\mathrm{Y} \times \mathrm{Y}, \mathscr{B} \times \mathscr{B})$ by

$$\delta(x, y) = \min\{h_1(x), h_2(y)\}\, \bar{\alpha} + \left[\int h_2'(z)\, d\alpha(z)\right]^{-1} h_1'(x)\, h_2'(y)\, \alpha \times \alpha\,,$$

where $h_1'(x) = [h_1(x) - h_2(x)]^+$, $h_2'(y) = [h_2(y) - h_1(y)]^+$.

(Note that since $h_1' + h_2 = h_2' + h_1$ we have

$$\int h_2'(z)\,d\alpha(z) = \int h_1'(z)\,d\alpha(z)\,,$$

thus if $\int h_2'(z)\,d\alpha(z) = 0$ then $h_1 = h_2 = 0$ and we will leave out the second term in the definition of δ.)

Lemma 3. *Let δ be as above. Then we have*

$$\delta(A \times Y) = \gamma_1(A) \quad \text{for all} \quad A \in \mathscr{B}\,, \tag{1}$$

$$\delta(Y \times B) = \gamma_2(B) \quad \text{for all} \quad B \in \mathscr{B}\,. \tag{2}$$

Proof. This is a simple calculation.

Lemma 4. *Suppose for all $x, y \in Y$ with $x \geqq y$ we have*

$$h_1(x)\,h_2(y) \geqq h_1(y)\,h_2(x)\,.$$

Then $\delta(\{(x, y) \in Y \times Y : x \geqq y\}) = 1$.

Proof. It is sufficient to show that $h_1'(x)\,h_2'(y) = 0$ unless $x \geqq y$, thus suppose there exist x, y with $x > y$ and $h_1'(y)\,h_2'(x) > 0$. Then we have $h_1(y) > h_2(y)$, $h_2(x) > h_1(x)$, and hence

$$h_1(x)\,h_2(y) < h_1(y)\,h_2(x)$$

which contradicts the hypothesis of the lemma.

Together Lemma 3 and 4 give us Proposition 1 for the case $|\Lambda| = 1$; also the explicit expression for δ will enable us to complete the proof in general. Let $q : X_t \to \mathbb{R}$ with $q \geqq 0$ and $\int q(\xi)\,d\omega_t(\xi) = 1$ and for $i = 1, 2$ define

$$F_i(x', \xi) = \begin{cases} \dfrac{f_i(x', \xi)}{\int f_i(x', \eta)\,d\omega_t(\eta)} & \text{if} \quad \int f_i(x', \eta)\,d\omega_t(\eta) > 0\,, \\ q(\xi) \quad \text{otherwise.} \end{cases}$$

Thus F_1 (resp. F_2) is a version of the Radon-Nikodym derivative of μ_1 (resp. μ_2) with respect to $\varrho(\mu_1) \times \omega_t$ (resp. $\varrho(\mu_2) \times \omega_t$).

Define $Q, R : X(\Lambda') \times X(\Lambda') \times X_t \times X_t \to \mathbb{R}$ by

$$Q(x', y, \xi, \eta) = \min\{F_1(x', \xi), F_2(y', \eta)\}$$

$$R(x', y', \xi, \eta) = [S(x', y')]^{-1}\,[F_1(x', \xi) - F_2(y', \xi)]^+\,[F_2(y', \eta) - F_1(x', \eta)]^+\,,$$

where $S(x', y') = \int [F_2(y', \eta) - F_1(x'\eta)]^+\,d\omega_t(\eta)$, and as in the definition of δ we have $S(x', y') = 0$ if and only if $F_1(x', \xi) = F_2(y', \xi)$ (for ω_t – a.e. ξ) and in this case we define $R(x', y', \xi, \eta) = 0$. Let $\bar{\omega}_t$ be the measure on $(X_t \times X_t, \mathscr{F}_t \times \mathscr{F}_t)$ got by projecting ω_t onto the diagonal of $X_t \times X_t$ and define the probability measure ν on $(X(\Lambda) \times X(\Lambda), \mathscr{F}(\Lambda) \times \mathscr{F}(\Lambda))$ by

$$\nu = Q\,\nu' \times \bar{\omega}_t + R\,\nu' \times \omega_t \times \omega_t\,.$$

Lemma 5. *ν satisfies* (1) *and* (2) *of Proposition 1.*

Proof. This is a straightforward calculation.

Finally we complete the proof of Proposition 1 with:

Lemma 6. *ν satisfies* (3) *of Proposition 1.*

Proof. For $i=1,\ 2$ let $B_i=\{x'\in X(\Lambda'):\int f_i(x',\xi)\,d\omega_t(\xi)=0\}$. If $x'\notin B_1$, $y'\notin B_2$ and $x'\geqq y'$ then

$$F_1(x',\xi)\,F_2(y',\eta)\geqq F_1(x',\eta)\,F_2(y',\xi)$$

whenever $\xi\geqq\eta$ and exactly as in Lemma 4 we have $R(x',y',\xi,\eta)=0$ unless $\xi\geqq\eta$. Therefore we are finished provided we can show that $\nu(B_1\times X_t\times X(\Lambda))=\nu(X(\Lambda)\times B_2\times X_t)=0$. But

$$\nu(B_1\times X_t\times X(\Lambda))=\mu_1(B_1\times X_t)=\int\limits_{B_1}\int\limits_{X_t} f_1(x',\xi)\,d\omega_t(\xi)\,d\omega_{\Lambda'}(x')=0$$

and similarly $\nu(X(\Lambda)\times B_2\times X_t)=0$.

3. Some Remarks on the Theorem

Remark 1. For the case $|\Lambda|=1$ there is a simple direct proof of Theorem 3. Let $(Y,\mathscr{B})$, α, h_1, h_2, γ_1, γ_2 be as before for the cardinality 1 case. If for all $x,y\in Y$ with $x\geqq y$ we have $h_1(x)\,h_2(y)\geqq h_1(y)\,h_2(x)$ then for any $\mathscr{B}$-measurable, bounded, increasing $f: Y\to\mathbb{R}$ we have $\int f\,d\gamma_1\geqq f\,d\gamma_2$ because

$$\int f\,d\gamma_1-\int f\,d\gamma_2=\tfrac{1}{2}\iint[f(x)-f(y)]\,[h_1(x)\,h_2(y)-h_1(y)\,h_2(x)]\cdot d\alpha(x)\,d\alpha(y)\geqq 0$$

(since the integrand is always non-negative).

Remark 2. At least for the case when each X_t is a finite set we have that Theorem 3 and Proposition 1 are equivalent, because of the following result:

Proposition 2. *Let S be a finite partially ordered set, and let $\mu_1,\mu_2: S\to\mathbb{R}$ be probability densities. The following are equivalent:*

(1) For any increasing $h: S\to\mathbb{R}$ $\sum\limits_{t\in S} h(t)\,\mu_1(t)\geqq\sum\limits_{t\in S} h(t)\,\mu_2(t)$.

(2) There exists a probability density $\nu: S\times S\to\mathbb{R}$ such that

(a) $\sum\limits_{t\in S}\nu(s,t)=\mu_1(s)$ for all $s\in S$;

(b) $\sum\limits_{s\in S}\nu(s,t)=\mu_2(t)$ for all $t\in S$;

(c) $\nu(s,t)=0$ unless $s\geqq t$.

Proof. This result seems to be quite well-known, but it is difficult to find out where it first appeared. It can be found, for example, in Holley [4]. Clearly (2) $=>$ (1); to prove the converse consider the following network flow:

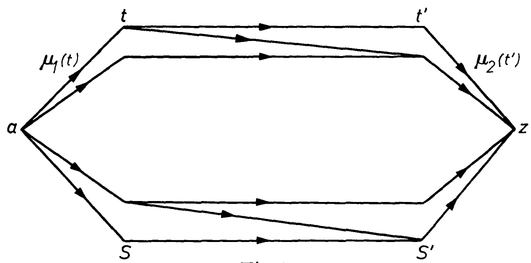

Fig. 1

Here S' is a copy of S; for each $t \in S$ there is an edge from the source a to the point t with capacity $\mu_1(t)$; for each $t' \in S'$ there is an edge from t' to the sink z with capacity $\mu_2(t')$, and for $t \in S$, $t' \in S'$ with $t \geqq t'$ there is an edge from t to t' with unlimited capacity. The maximum flow through this network is clearly $\leqq 1$ and it is also clear that (2) holds if and only if the maximum flow is exactly 1, [and $\nu(t, t')$ is then the amount assigned to the edge from t to t' in some optimal flow]. But it is not difficult to show that (1) implies that the flow through any cut is $\geqq 1$, and hence (1) $=>$ (2) by the min-cut max-flow theorem.

Remark 3. In the case in which each X_t is a finite set we can prove Proposition 1 without explicitly writing down any measures. This is because there are no measurability problems with a finite set and thus for each $x', y' \in X(\Lambda')$ with $x' \geqq y'$ we need only know that $M(x', y')$ exists with the right properties. But the existence of $M(x', y')$ follows since Proposition 2 and Remark 1 imply that Proposition 1 is true for $|\Lambda| = 1$.

Remark 4. The only property of a total order used in the proof of Proposition 1 is that if $x \neq y$ then exactly one of $x \geqq y$ and $y \geqq x$ is true; the transitivity of a total order is never used. It is thus perhaps worth writing down exactly what has been proved For each $t \in \Lambda$ let $D_t = \{(x, x) : x \in X_t\}$ and let $E_t \subset X_t \times X_t - D_t$ have the properties:

(a) $E_t \in \mathscr{F}_t \times \mathscr{F}_t$.

(b) If $x, y \in X_t$ with $x \neq y$ then exactly one of (x, y) and (y, x) is in E_t.

Let $\bar{E}_t = E_t \cup D_t$ and for $x, y \in X_t$ define

$$x \uparrow y = \begin{cases} x & \text{if } (x, y) \in \bar{E}_t, \\ y & \text{otherwise}, \end{cases}$$

$$x \downarrow y = \begin{cases} y & \text{if } (x, y) \in \bar{E}_t, \\ x & \text{otherwise}. \end{cases}$$

If $x=\{x_t\}_{t\in\Lambda}$, $y=\{y_t\}_{t\in\Lambda}$ then define

$$x\uparrow y=\{x_t\uparrow y_t\}_{t\in\Lambda},\ x\downarrow y=\{x_t\downarrow y_t\}_{t\in\Lambda}\,.$$

Suppose for all $x, y\in X(\Lambda)$ we have

$$f_1(x\uparrow y)\,f_2(x\downarrow y)\geqq f_1(x)\,f_2(y)\,.$$

Then the proof of Proposition 1 shows that there exists a probability measure ν on $(X(\Lambda)\times X(\Lambda),\ \mathscr{F}(\Lambda)\times\mathscr{F}(\Lambda))$ satisfying (1) and (2) of Proposition 1 and also $\nu(\bar{E}_\Lambda)=1$ where

$$\bar{E}_\Lambda=\{(x,y)\in X(\Lambda)\times X(\Lambda):(x_t,y_t)\in\bar{E}_t\quad\text{for all}\quad t\in\Lambda\}\,.$$

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