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Asymptotic Perturbation Expansion in the $P(\phi)_2$ Quantum Field Theory

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Abstract. In the $P(\phi)_2$ model it is proved that the perturbation series for the infinite volume Schwinger functions $S(\lambda)$ are asymptotic in the limit as the coupling constant λ goes to zero. We also give conditions which imply smoothness of $S(\lambda)$ at arbitrary λ.

In formal quantum field theory the low order coefficients of the perturbation series can be calculated exactly, and for quantum electrodynamics one obtains results which are in excellent agreement with experiment. It is therefore an important task of constructive field theory to establish the precise sense in which the perturbation series for a model approximates the model itself. Since the sum of the series may be expected to diverge in general [6], the best result to be hoped for is that the series is asymptotic.

In this paper we obtain such a result for the $P(\phi)_2$ model. We consider the imaginary time $P(\phi)_2$ Green's functions (Schwinger functions) as defined below, and study their dependence on the coupling constant $\lambda \in [0, \infty)$. It is proved that on an interval $[0, \lambda_0]$ including the origin they are C^∞ functions of λ. As a result the perturbation expansion for the Schwinger functions (i.e. Taylor's series around $\lambda = 0$) is asymptotic in the limit $\lambda \to 0^+$.

Nelson [9] and Osterwalder and Schrader [11] have established general techniques for passing from imaginary time to real time. It remains to show that the smoothness in λ is preserved under this analytic continuation. In particular one would like to know that the perturbation series for the real time Green's functions are asymptotic. The LSZ reduction formula [5] connects these Green's functions with the S-matrix, which is known to exist for small λ by the Haag-Ruelle theory [7] and results of Glimm, Jaffe, and Spencer on particle structure [2]. Since the series are non-trivial, one could conclude that scattering is non-trivial for the model, once the appropriate asymptotic property was established.

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A $P(\phi)_2$ model is determined by a lower semi-bounded polynomial $\mathscr{P}$ and a bare mass $m_0 > 0$, as well as the coupling constant $\lambda \geqq 0$. The Hamiltonian for the theory is formally $H_\lambda = H_0 + \lambda \int :\mathscr{P}(\phi(x)): d^1x - E_\lambda$ where H_0 is the free Hamiltonian for mass m_0, $\phi(x)$ is the time zero field operator, and E_λ is the (infinite) vacuum energy for the vacuum vector Ω_λ. Imaginary time fields are given formally by $\phi_\lambda(t, x) = e^{-H_\lambda t} \phi(x) e^{H_\lambda t}$ (instead of the usual $e^{iH_\lambda t} \phi(x) e^{-iH_\lambda t}$) and the Schwinger functions are the vacuum expectation values of time ordered products of these fields:

$$S(\lambda; t_1, x_1, \ldots, t_n, x_n) = (\Omega_\lambda, T[\phi_\lambda(t_1, x_1) \ldots \phi_\lambda(t_n, x_n)] \Omega_\lambda).$$

It is convenient to use an alternate representation in which the Schwinger functions are realized as the moments of certain probability measures [9, 14]. For no interaction we consider the probability space $\{Q, \Sigma, dq\}$ where $Q = \mathscr{S}'(R^2)$ is the nuclear space of real-valued tempered distributions, Σ is the σ-algebra of measurable sets generated by coordinate functions in $\mathscr{S}(R^2)$, and dq is the Gaussian measure with mean zero and covariance $(-\Delta + m_0^2)^{-1}$. The interaction is introduced into a bounded region $\Lambda \subset R^2$ by the function $V_\Lambda(q) = \int_\Lambda :\mathscr{P}(q(x)): d^2x$. Standard estimates (see for example [1]) show that V_Λ and $\exp(-\lambda V_\Lambda) \cdot (\lambda \geqq 0)$ are well-defined functions in $L_p(Q, dq)$ for $p < \infty$, and one may consider integrals with respect to the measure $\exp(-\lambda V_\Lambda) dq$. We use the notation

$$\langle \cdot \rangle_{\lambda, \Lambda} = \frac{\int [\cdot] e^{-\lambda V_\Lambda} dq}{\int e^{-\lambda V_\Lambda} dq}.$$

Let f be a function of the form $f = \otimes^n f_i$ with $f_i \in C_0^\infty(R^2)$ and define an associated function on Q by $F(q) = \prod_i \langle q, f_i \rangle$. Then $F \in L_p(Q, dq)$ for $p < \infty$, and cutoff Schwinger functions are defined as distributions by $S_\Lambda(\lambda, f) = \langle F \rangle_{\lambda, \Lambda}$. Infinite volume Schwinger functions are defined by $S(\lambda, f) = \langle F \rangle_{\lambda, \infty} \equiv \lim_{\Lambda \to \infty} \langle F \rangle_{\lambda, \Lambda}$ when the limit exists. The limit is known to exist if λ / m_0^2 is sufficiently small (Glimm, Jaffe, and Spencer [2, 3]) or if $\mathscr{P}$ is an even polynomial (Nelson, 1973 Erice lectures).

More generally we consider functions in $L_p(Q, dq)$ of the form

$$\psi(q) = \prod_{i=1}^{N} \left(\int :q(x)^{n_i}: w_i(x) d^2x\right) \tag{1}$$

with kernels w_i which are bounded, measurable, and have compact support. Both F and V_Λ are made up of terms of this form. We define the localization of ψ in R^2 by $\mathscr{L}(\psi)=\bigcup_i \operatorname{supp} w_i$. If E is a Euclidean transformation in R^2 (translation, rotation, reflection), then a transformed function ψ_E is defined by replacing w_i by $w_i \circ E^{-1}$. In particular ψ_x denotes translation by $x \in R^2$ and ψ_{R_s} denotes reflection through the line $t=s$.

Suppose that $I \subset [0,\infty)$ is a finite closed interval such that the following properties are satisfied by the integrals $\langle\psi\rangle_{\lambda,\Lambda}$, $\lambda \in I$.

A. Uniform boundedness. For any ψ there exists a constant M_ψ such that $\sup_{\lambda\in I} |\langle\psi\rangle_{\lambda,\Lambda}| \leqq M_\psi$ for all Λ.

B. Uniform convergence. For any ψ,

$$\sup_{\lambda\in I} |\langle\psi\rangle_{\lambda,\Lambda} - \langle\psi\rangle_{\lambda,\Lambda'}| \to 0 \quad \text{as} \quad \operatorname{dist}(0, \Lambda \Delta \Lambda') \to \infty .$$

C. Uniform cluster property. There exists a constant m_* and for any ψ, ψ' a constant $M_{\psi,\psi'}$ such that

$$\sup_{\lambda\in I} |\langle\psi', \psi_x\rangle_{\lambda,\Lambda} - \langle\psi'\rangle_{\lambda,\Lambda} \langle\psi_x\rangle_{\lambda,\Lambda}| \leqq M_{\psi\psi'} e^{-m_*|x|}$$

for all $x \in R^2$ and all Λ.

In this case we say that I is *regular* (for the particular $\mathscr{P}, m_0$). Conditions A and B imply the existence of a Euclidean invariant functional $\langle\psi\rangle_{\lambda,\infty}$ which satisfies $\sup_{\lambda\in I} |\langle\psi\rangle_{\lambda,\infty}| \leqq M_\psi$ and $\sup_{\lambda\in I} |\langle\psi\rangle_{\lambda,\infty} - \langle\psi\rangle_{\lambda,\Lambda}| \to 0$ as $\operatorname{dist}(0, \sim\Lambda) \to \infty$. Then C also holds for $\langle\cdot\rangle_{\lambda,\infty}$ because of the uniformity in Λ.

The most general result is the following.

Theorem I. *$\langle\psi\rangle_{\lambda,\infty}$ is infinitely differentiable on a regular interval, one-sided derivatives taken at the endpoints.*

The proof of this theorem is given in Lemma 1 through Lemma 5 to follow. We remark that the theorem and proof have straightforward generalizations to theories with several coupling constants.

The phenomenon of uniform exponential cluster properties implying smoothness of the N-point functions in the defining parameters also occurs in classical statistical mechanics, see Lebowitz [8]. Simon and Griffiths [13] make some use of this mechanism in their study of the $(\phi^4)_2$ model as an Ising spin system. By general analogy with statistical mechanics there may be critical points in some $P(\phi)_2$ models where the cluster property fails. This does not happen for λ small however, for we have the following key result of Glimm, Jaffe, and Spencer [2, 3].

Theorem II. *For any $(\mathscr{P}, m_0)$, the interval $[0, \lambda_0]$ is regular if λ_0 is sufficiently small.*

The convergence and cluster property are proved in [2, 3]. The uniformity in λ is not explicit, but follows by the same proof. Now we come to the main result.

Theorem III. *$S(\lambda, f)$ is C^∞ on $[0, \lambda_0]$ for λ_0 sufficiently small. The derivatives at zero $s_r(f) = (D_\lambda)^r S(\lambda, f)|_{\lambda=0}$ are the coefficients of an asymptotic series for $S(\lambda, f)$; i.e. for each N,*

$$|\lambda|^{-N} \left| S(\lambda, f) - \sum_{r=0}^{N} \lambda^r s_r(f)/r! \right| \to 0 \quad \textit{as} \quad \lambda \to 0^+ .$$

Proof. The C^∞ property is immediate from Theorem I and Theorem II. Then Taylor's theorem is used to dominate the N^{th} order difference quotient above by

$$\frac{\lambda}{(N+1)!} \sup_{\lambda' \in [0, \lambda_0]} |D_\lambda^{N+1} S(\lambda', f)|$$

and the convergence follows. ∎

We eventually identify the coefficients $s_r(f)$ with the usual perturbation series. We remark that quite a bit is known about asymptotic series for cutoff $P(\phi)_2$ theories [12].

We now begin the proof of Theorem I. Hereafter we assume that $(\mathscr{P}, m_0, I)$ are fixed and that I is regular. For Λ we take rectangles

$$\Lambda_{-\tau', \tau; \ell} = \{(t, x) \in R^2 : -\tau' < t < \tau, \ -\ell < x < \ell\}$$

and set $\Lambda_{\tau, \ell} = \Lambda_{-\tau, \tau; \ell}$ and $\Lambda_\ell = \Lambda_{\ell, \ell}$. The associated integrals are denoted $\langle \psi \rangle_{\lambda; -\tau', \tau; \ell}$, etc. If Δ_j is a unit square centered on $j \in Z^2$, then for $\ell \in Z^1 + \frac{1}{2}$ we have $V_\ell = V_{\Lambda_\ell} = \sum_{j \in \Lambda_\ell} V_{\Delta_j}$.

Lemma 1. *$\langle \psi \rangle_{\lambda, \infty}$ is continuously differentiable on I and*

$$D_\lambda \langle \psi \rangle_{\lambda, \infty} = - \sum_j [\langle \psi V_{\Delta_j} \rangle_{\lambda, \infty} - \langle \psi \rangle_{\lambda, \infty} \langle V_{\Delta_j} \rangle_{\lambda, \infty}] .$$

Proof. In general let $\langle \psi_1, \psi_2 \rangle^T = \langle \psi_1 \psi_2 \rangle - \langle \psi_1 \rangle \langle \psi_2 \rangle$. For $\ell < \infty$, it is straightforward to show that $\langle \psi \rangle_{\lambda, \ell}$ is C^1 and that

$$D_\lambda \langle \psi \rangle_{\lambda, \ell} = - \langle \psi, V_\ell \rangle^T_{\lambda, \ell} = - \sum_j \chi(j \in \Lambda_\ell) \langle \psi, V_{\Delta_j} \rangle^T_{\lambda, \ell} .$$

By condition B we have $\sup_{\lambda \in I} |\langle \psi, V_{\Delta_j} \rangle^T_{\lambda, \ell} - \langle \psi, V_{\Delta_j} \rangle^T_{\lambda, \infty}| \to 0$ as $\ell \to \infty$, and by C, $\sup_{\lambda \in I} |\langle \psi, V_{\Delta_j} \rangle^T_{\lambda, \ell}| \leqq \mathcal{O}(e^{-m_* |j|})$ uniformly in $\ell \leqq \infty$. Therefore $\sum_j \langle \psi, V_{\Delta_j} \rangle^T_{\lambda, \infty}$ is uniformly convergent on I and as $\ell \to \infty$ we have by dominated convergence

$$\sup_{\lambda\in I}\left|D_\lambda\langle\psi\rangle_{\lambda,\ell}-(-1)\sum_j\langle\psi,V_{\Delta_j}\rangle^T_{\lambda,\infty}\right|$$
$$\leqq\sum_j\sup_{\lambda\in I}\left|\chi(j\in\Lambda_\ell)\langle\psi,V_{\Delta_j}\rangle^T_{\lambda,\ell}-\langle\psi,V_{\Delta_j}\rangle^T_{\lambda,\infty}\right|$$
$$\to 0\,.$$

The uniform convergence of $D_\lambda\langle\psi\rangle_{\lambda,\ell}$ on I implies that $\langle\psi\rangle_{\lambda,\infty}$ is C^1 on I and that $D_\lambda\langle\psi\rangle_{\lambda,\infty}=(-1)\sum\limits_j\langle\psi,V_{\Delta_j}\rangle^T_{\lambda,\infty}$. ∎

Before proceeding we review some of the probabilistic aspects of $\{Q,\Sigma,dq\}$ as formulated by Nelson [9, 10]. For $h\in\mathscr{S}(R^2)$ define $\Phi(h)(q)=\langle q,h\rangle$. The map $h\to\Phi(h)$ extends to a continuous map from the Sobolev space $H_{-1}(R^2)=\{h\in\mathscr{S}'(R^2):\int|\tilde h(k)|^2(k^2+m_0^2)^{-1}\,dk<\infty\}$ into $L_p(Q,dq)$ for all $p<\infty$. Then the functions $\Phi(h)$, $h\in H_{-1}$, form a Gaussian process indexed by H_{-1} with mean zero and covariance $(-\Delta+m_0^2)^{-1}$. For $\mathcal{O}\subset R^2$, let $\Sigma_{\mathcal{O}}$ be the smallest σ-algebra with respect to which all functions $\Phi(h)$ with $\mathit{supp}\,h\subset\mathcal{O}$ are measurable. Let $\mathscr{E}_{\mathcal{O}}$ be the conditional expectation with respect to $\Sigma_{\mathcal{O}}$, the projection onto $L_2(Q,\Sigma_{\mathcal{O}},dq)$. The Markov property is the statement that

$$\mathscr{E}_{\sim\mathcal{O}}\mathscr{E}_{\mathcal{O}}=\mathscr{E}_{\partial\mathcal{O}}\mathscr{E}_{\mathcal{O}}$$

for $\mathcal{O}$ open. By $\Sigma_{(a,b)}$, Σ_t we denote the σ-algebras associated with $(a,b)\times R^1$ and $t\times R^1$ respectively. There is a natural unitary map U_t from the Fock space $\mathscr{F}(L_2(R^1))$ to $L_2(Q,\Sigma_t,dq)$ which takes the free vacuum Ω_0 to 1 and the field operator $\phi(f)$ to multiplication by $\Phi(f\otimes\delta_t)$.

Let $H'_\ell(\lambda)=H_0+\int\limits_{-\ell}^{\ell}:\mathscr{P}(\phi(x)):d^1x$ be the cutoff Hamiltonian on Fock space with vacuum vector $\Omega_\ell(\lambda)$ and vacuum energy $E_\ell(\lambda)$. The associated semi-group can be represented by $\exp(-(b-a)H'_\ell(\lambda))=U_b^*\mathscr{E}_b\exp(-\lambda V_{a,b;\ell})U_a$ [4]. Using this identity and the Markov property, integrals over Q can be represented on Fock space. If $\theta\in L_2(Q,\Sigma_{(b,\infty)},dq)$ and $\theta'\in L_2(Q,\Sigma_{(-\infty,a)},dq)$, then for $-\tau\leqq a\leqq b\leqq\tau$ we have

$$\langle\bar\theta\theta'\rangle_{\lambda;\tau,\ell}=Z^{-1}_{\tau,\ell}(\hat\theta_{b,\tau;\ell},e^{-(b-a)H'_\ell(\lambda)}\hat\theta'_{a,-\tau;\ell})\tag{2}$$

where

$$\hat\theta_{b,\tau;\ell}=U_b^*\mathscr{E}_b\exp(-\lambda V_{b,\tau;\ell})\theta$$

$$\hat\theta'_{a,-\tau;\ell}=U_a^*\mathscr{E}_a\exp(-\lambda V_{-\tau,a;\ell})\theta'$$

$$Z_{\tau,\ell}=\int\exp(-V_{\tau,\ell})\,dq=\|e^{-\tau H'_\ell(\lambda)}\Omega_0\|^2\,.$$

Let $H_\ell(\lambda) = H'_\ell(\lambda) - E_\ell(\lambda)$ so that $H_\ell(\lambda)\,\Omega_\ell(\lambda) = 0$. The cluster property C implies that $H_\ell(\lambda)$ has no spectrum in $(0, m_*)$ for all $\ell < \infty$ and $\lambda \in I$, i.e. there is a uniform mass gap [3]. Here we reverse the argument and use the uniform mass gap to determine the constant $M_{\psi\psi'}$ in the infinite volume cluster property. The proof that follows would be easier to carry out without reference to the cutoff theory, once one has enough structure (e.g. measure, Markov property) in the infinite volume limit. Such information is not known in general, and so to keep the assumptions minimal we use cutoffs.

Lemma 2. *Let $\lambda \in I$ and suppose $\mathscr{L}(\psi)$ and $\mathscr{L}(\psi')$ can be separated by a strip of width d. Then*

$$|\langle \psi, \psi' \rangle^T_{\lambda,\infty}| \leqq \langle |\psi|^2 \rangle^{1/2}_{\lambda,\infty} \langle |\psi'|^2 \rangle^{1/2}_{\lambda,\infty} e^{-m_* d}.$$

Proof. Because of the Euclidean invariance it is sufficient to assume that the strip is $[-d, 0] \times R^1$ and that $\mathscr{L}(\psi') \subset (-\infty, -d) \times R^1$ and $\mathscr{L}(\psi) \subset (0, \infty) \times R^1$. Then $\psi \in L_2(Q, \Sigma_{(0,\infty)}, dq)$ etc., and suppressing λ we have by (2):

$$\begin{aligned}\langle \bar{\psi}, \psi' \rangle^T_{\tau,\ell} = {} & Z^{-1}_{\tau,\ell}(\hat{\psi}_{0,\tau;\ell}, e^{-dH'_\ell}\hat{\psi}'_{-d,-\tau;\ell}) \\ & - Z^{-2}_{\tau,\ell}(\hat{\psi}_{0,\tau;\ell}, e^{-\tau H'_\ell}\Omega_0)\,(e^{-(\tau+d)H'_\ell}\Omega_0, \hat{\psi}'_{-d,-\tau;\ell}).\end{aligned} \tag{3}$$

We now take the limit $\tau \to \infty$. Since $(\Omega_0, \Omega_\ell) \neq 0$ and Ω_ℓ is a simple eigenvector of H_ℓ, we have $e^{-\tau H_\ell}\Omega_0 \to (\Omega_\ell, \Omega_0)\,\Omega_\ell$ and hence $Z^{-1/2}_{\tau,\ell} e^{-\tau H'_\ell}\Omega_0 \to \Omega_\ell$. By time reversal invariance $\hat{\psi}_{0,\tau,\ell} = (\psi_{R_0})\hat{}_{0,-\tau,\ell}$ and thus

$$\begin{aligned}&\|Z^{-1/2}_{\tau,\ell}\hat{\psi}_{0,\tau;\ell} - Z^{-1/2}_{\tau',\ell}\hat{\psi}_{0,\tau'\ell}\|^2 \\ &= \langle \bar{\psi}_{R_0}\psi \rangle_{\tau,\ell} + \langle \bar{\psi}_{R_0}\psi \rangle_{\tau',\ell} \\ &\quad - Z^{-1/2}_{\tau,\ell} Z^{-1/2}_{\tau',\ell} Z_{(\tau+\tau')/2,\ell}(\langle \bar{\psi}_{R_0}\psi \rangle_{-\tau,\tau';\ell} + \langle \bar{\psi}_{R_0}\psi \rangle_{-\tau',\tau;\ell}).\end{aligned}$$

As $\tau, \tau' \to \infty$ the product of Z's converges to one, and by condition B each factor $\langle \bar{\psi}_{R_0}\psi \rangle_{\tau,\ell}$, etc. converges to the same limit $\langle \bar{\psi}_{R_0}\psi \rangle_{\infty,\ell}$. Thus the above expression converges to zero, and so $Z^{-1/2}_{\tau,\ell}\bar{\psi}_{0,\tau;\ell}$ converges to a limit Ψ_ℓ. Similarly we obtain $\|\Psi_\ell\| = \langle \bar{\psi}_{R_0}\psi \rangle^{1/2}_{\infty,\ell}$. Repeating the argument gives that $Z^{-1/2}_{\tau-d,\ell}\psi'_{-d,-\tau;\ell}$ has a limit Ψ'_ℓ with norm $\langle \bar{\psi}'_{R_{-d}}\psi' \rangle^{1/2}_{\infty,\ell}$. Finally using $Z^{-1/2}_{\tau,\ell} Z^{1/2}_{\tau-d,\ell} \to e^{dE_\ell}$ and $Z^{-1}_{\tau,a} Z^{1/2}_{\tau-d,\ell} Z^{1/2}_{\tau+d,\ell} \to 1$ we obtain the convergence of the right side of (3).

If P_ℓ is the projection onto $\Omega_\ell^\perp$, then by the mass gap,

$$\begin{aligned}|\langle \bar{\psi}, \psi' \rangle^T_{\infty,\ell}| &= |(\Psi_\ell, e^{-dH_\ell}\Psi'_\ell) - (\Psi_\ell, \Omega_\ell)\,(\Omega_\ell, \Psi'_\ell)| \\ &= |(\Psi_\ell, P_\ell e^{-dH_\ell} P_\ell \Psi'_\ell)| \\ &\leqq \|\Psi_\ell\|\,\|\Psi'_\ell\|\, e^{-dm_*} \\ &= \langle \bar{\psi}_{R_0}\psi \rangle^{1/2}_{\infty,\ell} \langle \bar{\psi}'_{R_{-d}}\psi' \rangle^{1/2}_{\infty,\ell} e^{-dm_*}\end{aligned}$$

Again by condition B it is straightforward to show $\langle\cdot\rangle_{\infty,\ell}\to\langle\cdot\rangle_\infty$ as $\ell\to\infty$, and this proves the lemma. In the last step we use $|\langle\bar{\psi}_R\psi\rangle_\infty| \leqq \langle|\psi|^2\rangle_\infty$; this follows by the Schwarz inequality (which is easily proved by limits) and the Euclidean invariance. ∎

Corollary. *Given $\psi_1, \ldots, \psi_N$ and $\psi'_1, \ldots, \psi'_M$ there exists a constant K such that for all $x_i, y_j \in R^2$ and $d \geqq 0$*

$$\sup_{\lambda\in I}\left|\left\langle \prod_{i=1}^{N}\psi_{i,x_i}, \prod_{j=1}^{M}\psi'_{j,y_j}\right\rangle^T_{\lambda,\infty}\right| \chi_d(\{x_i\},\{y_j\}) \leqq K e^{-m_* d}$$

where $\chi_d(\{x_i\},\{y_j\})$ equals one if $\bigcup_i x_i$ and $\bigcup_j y_j$ can be separated by a strip of width d, and equals zero otherwise.

Proof. Choose ϱ so that $\mathscr{L}(\psi_i)$, $\mathscr{L}(\psi'_j) \subset \{x\in R^2 : |x| \leqq \varrho\}$. Then the above expression can be dominated by

$$2\sup_{\lambda\in I}\left|\left\langle\left|\prod_i \psi_{i,x_i}\right|^2\right\rangle^{1/2}_{\lambda,\infty}\left\langle\left|\prod_j \psi'_{j,y_j}\right|^2\right\rangle^{1/2}_{\lambda,\infty}\right| e^{-m_*(d-2\varrho)}.$$

For $d>2\varrho$ this follows from the Lemma, and for $d\leqq 2\varrho$ from the Schwarz inequality. Continuing to apply the Schwarz inequality and using translation invariance gives bounds which are independent of $\{x_i\}$, $\{y_j\}$. ∎

Truncated N-point functions $\langle\psi_1,\ldots,\psi_N\rangle^T$ are defined by $\langle\psi\rangle^T=\langle\psi\rangle$ for $N=1$, and then inductively by

$$\langle\psi_1,\ldots,\psi_N\rangle^T = \langle\psi_1\ldots\psi_N\rangle - \sum_{\pi\in\Pi_N}\prod_{p\in\pi}\langle\{\psi_i\}_{i\in p}\rangle^T$$

where Π_N is the set of all partitions of the set $(1,\ldots,N)$. By a standard argument we now pass from the cluster property of the corollary to the decay of the general N-point function, c.f. [5].

Lemma 3. *Given $\psi_1,\ldots,\psi_N$ there exists a constant K such that for all $x_1,\ldots,x_N\in R^2$*

$$\sup_{\lambda\in I}|\langle\psi_{1,x_1},\ldots,\psi_{N,x_N}\rangle^T_{\lambda,\infty}| \leqq K\exp(-m_*\delta(x_1,\ldots,x_N)/N)$$

where $\delta(x_1,\ldots,x_N)=\sup_{1\leqq i\leqq j\leqq N}|x_i-x_j|$ is the diameter of the set $\{x_i\}$.

Proof. It is sufficient to prove that for any proper subset σ of $(1,\ldots,N)$

$$\sup_{\lambda\in I}|\langle\psi_{1,x_1},\ldots,\psi_{N,x_N}\rangle^T_{\lambda,\infty}|\,\chi_d(x_\sigma,x_{\sim\sigma})\leqq K' e^{-m_* d} \tag{4}$$

where $x_\sigma=\{x_i\}_{i\in\sigma}$.

The Lemma then follows by restricting to the submanifold $d=\delta(x_1,\ldots,x_N)/N$ and summing over σ. Here we use $1\leqq\sum_\sigma \chi_{\delta(x)/N}(x_\sigma,x_{\sim\sigma})$, that is for any $x=(x_1,\dots,x_N)$ there exists a partition $(\sigma, \sim\sigma)$, such that $\bigcup_{i\in\sigma} x_i$ and $\bigcup_{i\in\sim\sigma} x_i$ can be separated by a strip of width $\delta(x_1,\dots,x_N)/N$. This follows by a simple geometric argument ([5], p. 176).

The proof of (4) is by induction on N. Suppose it has been established for $1,\dots,N-1$. In the definition of the truncated N-point function we break the sum over Π_N into partitions which are finer than $(\sigma, \sim\sigma)$, denoted $\Pi_N(\sigma)$, and those which are not, denoted $\sim\Pi_N(\sigma)$. The sum over $\Pi_N(\sigma)$ may be identified with $\left\langle\prod_{i\in\sigma}\psi_{i,x_i}\right\rangle_{\lambda,\infty}\left\langle\prod_{i\in\sim\sigma}\psi_{i,x_i}\right\rangle_{\lambda,\infty}$ and so

$$\begin{aligned}
&\sup_{\lambda\in I}|\langle\psi_{1,x_1},\dots,\psi_{N,x_N}\rangle^T_{\lambda,\infty}|\\
&\leqq \sup_{\lambda\in I}\left|\left\langle\prod_{i=1}^N\psi_{i,x_i}\right\rangle_{\lambda,\infty}-\left\langle\prod_{i\in\sigma}\psi_{i,x_i}\right\rangle_{\lambda,\infty}\left\langle\prod_{i\in\sim\sigma}\psi_{i,x_i}\right\rangle_{\lambda,\infty}\right|\\
&\quad+\sum_{\pi\in\sim\Pi_N(\sigma)}\prod_{p\in\pi}\sup_{\lambda\in I}|\langle\{\psi_{i,x_i}\}_{i\in p}\rangle^T_{\lambda,\infty}|\,.
\end{aligned}$$

Multiplying by $\chi_d(x_\sigma, x_{\sim\sigma})$ the first term is $\mathcal{O}(e^{-m_*d})$ by the Corollary. In the second term for each π in $\sim\Pi_N(\sigma)$ there is a $p^*\in\pi$ such that $p^*\cap\sigma\neq\emptyset$ and $p^*\cap\sim\sigma\neq\emptyset$ (since otherwise $\pi\in\Pi_N(\sigma)$). Then by the induction hypothesis the expression

$$\sup_{\lambda\in I}|\langle\{\psi_{i,x_i}\}_{i\in p^*}\rangle^T_{\lambda,\infty}|\,\chi_d(x_\sigma x_{\sim\sigma})$$

is also $\mathcal{O}(e^{-m_*d})$ once we note that $\chi_d(x_\sigma, x_{\sim\sigma})\leqq\chi_d(x_{\sigma\cap p^*}, x_{\sim\sigma\cap p^*})$.

The remaining terms in the product over π are bounded by constants and so (4) is proved. ∎

Lemma 4. *For* $\lambda\in I$

$$D_\lambda\langle\psi_1,\dots,\psi_N\rangle^T_{\lambda,\infty}=-\sum_j\langle\psi_1,\dots,\psi_N, V_{\Delta_j}\rangle^T_{\lambda,\infty}\,.$$

Proof. Note that the sum converges (uniformly on I) by the previous lemma. The proof of differentiability is by induction on N. The case $N=1$ was proved in Lemma 1, and we now assume the lemma is true for $1,\dots,N-1$. Returning to the definition of the truncated function we have

$$\begin{aligned}
D_\lambda\langle\psi_1,\dots,\psi_N\rangle^T_{\lambda,\infty}=\sum_j\Bigg[&-\langle\psi_1\dots\psi_N V_{\Delta_j}\rangle_{\lambda,\infty}+\langle\psi_1\dots\psi_N\rangle_{\lambda,\infty}\langle V_{\Delta_j}\rangle_{\lambda,\infty}\\
&+\sum_{\pi\in\Pi_N}\sum_{p'\in\pi}\langle\{\psi_i\}_{i\in p'}, V_{\Delta_j}\rangle^T_{\lambda,\infty}\prod_{\substack{p\in\pi\\ p\neq p'}}\langle\{\psi_i\}_{i\in p}\rangle^T_{\lambda,\infty}\Bigg]\,.
\end{aligned}$$

Comparing the terms in the brackets with the definition of $-\langle\psi_1,\dots,\psi_N, V_{\Delta_j}\rangle^T_{\lambda,\infty}$ we obtain equality by identifying our second term with the sum over $\{\pi \in \Pi_{N+1} : \{N+1\} \in \pi\}$ and our third term with the sum over $\{\pi \in \Pi_{N+1} : \{N+1\} \notin \pi\}$. ∎

Lemma 5. *$\langle \psi \rangle_{\lambda,\infty}$ is C^∞ on I and*

$$D_\lambda^r \langle \psi \rangle_{\lambda,\infty} = (-1)^r \sum_{j_1,\ldots,j_r} \langle \psi, V_{\Delta_{j_1}}, \ldots, V_{\Delta_{j_r}} \rangle_{\lambda,\infty}^T .$$

Proof. The sums converge uniformly on I by Lemma 3 since $\delta(0, j_1, \ldots, j_r) \geqq \frac{1}{r}(|j_1| + \cdots + |j_r|)$. The result then follows by repeatedly applying Lemma 4. In each case the infinite series can be differentiated term by term because of the uniform convergence of the resulting series. This proves the lemma and Theorem I. ∎

Final Remark. The derivatives may also be expressed as

$$D_\lambda^r \langle \psi \rangle_{\lambda,\infty} = \lim_{\ell \to \infty} (-1)^r \langle \psi, V_\ell, \ldots, V_\ell \rangle_{\lambda,\infty}^T .$$

Then since $\langle \cdot \rangle_{0,\infty} = \langle \cdot \rangle_0 \equiv \int [\cdot]\, dq$, the coefficients of the perturbation series are given by $s_r(f) = \lim_{\ell \to \infty} (-1)^r \langle F, V_\ell, \ldots, V_\ell \rangle_0^T$. These Gaussian integrals can be evaluated and the results expressed as a sum over connected Feynman diagrams (see [1], for example). Thus we identify the usual perturbation series.

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