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Golden-Thompson and Peierls-Bogolubov Inequalities for a General von Neumann Algebra

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Abstract. Some inequalities for a general von Neumann algebra, which reduces to Golden-Thompson and Peierls-Bogolubov inequalities when the von Neumann algebra has a trace, are proved.

§ 1. Main Results

Golden-Thompson and Peierls-Bogolubov inequalities are extended to von Neumann algebras, which have traces, by Ruskai [5]. We shall extend them to a general von Neumann algebra. Because a von Neumann algebra does not necessarily have a trace, we use the notion of relative Hamiltonian [3] instead of the trace.

Let $\mathfrak{M}$ be a von Neumann algebra and Ψ be a cyclic and separating vector. Let $\psi(x) = (x\Psi, \Psi)$ for $x \in \mathfrak{M}$. For a self-adjoint h in $\mathfrak{M}$, a vector $\Psi(h)$ is defined by

$$\Psi(h) \equiv \sum_{n=0}^{\infty} \int_0^{1/2} \mathrm{d}t_1 \ldots \int_0^{t_{n-1}} \mathrm{d}t_n \, \Delta_\Psi^{t_n} h \, \Delta_\Psi^{t_{n-1}-t_n} h \ldots \Delta_\Psi^{t_1-t_2} h \Psi \,, \quad (1.1)$$

where Δ_Ψ is the modular operator for Ψ. (As we shall see in (3.6), it is also possible to write $\Psi(h) = e^{(H+h)/2}\Psi$ where $H = \log \Delta_\Psi$.)

Theorem 1. *If* $\|\Psi\| = 1$, *then*

$$\|\Psi(h)\|^2 \geqq \exp \psi(h) \,. \quad (1.2)$$

Theorem 2.

$$\psi(e^h) \geqq \|\Psi(h)\|^2 \,. \quad (1.3)$$

The connection with Golden-Thompson and Peierls-Bogolubov inequalities for finite matrices can be seen as follows.

Let $\mathfrak{M}$ be a finite matrix algebra and Ω be a cyclic and separating vector such that $(x\Omega, \Omega) = \operatorname{tr} x$ for $x \in \mathfrak{M}$. Let $\Psi = (\operatorname{tr} e^A)^{-1/2} e^{A/2}\Omega$ for a self-adjoint A in $\mathfrak{M}$. Then Ψ is a unit cyclic and separating vector.

We have $\Delta_\Psi^t h \Delta_\Psi^{-t} = e^{tA} h e^{-tA}$ and

$$(\operatorname{tr} e^A)^{1/2}\ \Psi(h) = \operatorname{Exp}_r\left(\int_0^{1/2};\ e^{tA} h e^{-tA}\,\mathrm{d}t\right)\Psi$$
$$= e^{(A+h)/2}\Omega$$

due to formulas (5.4) and (2.8) of [2]. Hence we have

$$\|\Psi(h)\|^2 = \operatorname{tr} e^{(A+h)}/(\operatorname{tr} e^A),$$
$$\psi(h) = \operatorname{tr}(e^A h)/(\operatorname{tr} e^A).$$

Therefore (1.2) implies

$$\operatorname{tr} e^{A+h}/(\operatorname{tr} e^A) \geqq \exp\{\operatorname{tr}(e^A h)/(\operatorname{tr} e^A)\}$$

which is the Peierls-Bogolubov inequality.

Next set $\Psi = e^{A/2}\Omega$ in (1.3). We have

$$\|\Psi(h)\|^2 = \operatorname{tr} e^{A+h},$$
$$\psi(e^h) = \operatorname{tr} e^A e^h.$$

Therefore (1.3) implies

$$\operatorname{tr} e^A e^h \geqq \operatorname{tr} e^{A+h}$$

which is the Golden-Thompson inequality.

§ 2. Proof of Theorem 1

Lemma 1. *Let* $f(x) = \log\{\|\Psi(xh)\|^2\}$ *for a real number* x. *Then*

$$f'(0) = \psi(h)/\psi(1), \tag{2.1}$$

$$f''(0) \geqq 0. \tag{2.2}$$

Proof. $\Psi(xh)$ is an absolutely convergent power series in x by Proposition 4.1 of [3] and $\Psi(xh) \neq 0$ by Corollary 4.4 of [3]. Hence $f(x)$ is a C^∞ function of x. Furthermore

$$\Psi_1 \equiv (\mathrm{d}/\mathrm{d}x)\ \Psi(xh)|_{x=0} = \int_0^{1/2} \Delta_\Psi^t h \Psi\,\mathrm{d}t,$$

$$\Psi_2 \equiv (\mathrm{d}/\mathrm{d}x)^2\ \Psi(xh)|_{x=0} = 2\int_0^{1/2}\mathrm{d}t\int_0^t \mathrm{d}s\,\Delta_\Psi^s h\,\Delta_\Psi^{t-s} h\Psi.$$

Since $\Delta_\Psi^t \Psi = \Psi$, the derivatives of $F(x) \equiv \|\Psi(xh)\|^2$ at $x=0$ are given by

$$F'(0) = (\Psi_1, \Psi) + (\Psi, \Psi_1) = \psi(h), \tag{2.3}$$

$$F''(0) = (\Psi_2, \Psi) + (\Psi, \Psi_2) + 2\|\Psi_1\|^2$$
$$= 2\int_0^1 \|\Delta_\Psi^{u/2} h\Psi\|^2 (1-u)\,du\,.$$

Let E_0 be the projection onto the one-dimensional space spanned by Ψ. Then

$$F''(0) - F'(0)^2/F(0) = 2\int_0^1 \|(1-E_0)\Delta_\Psi^{u/2} h\Psi\|^2 (1-u)\,du \geqq 0\,. \quad (2.4)$$

(2.1) and (2.2) follow from (2.3) and (2.4). Q.E.D.

Lemma 2. $\log\|\Psi(xh)\|$ *is a convex function of* x.

Proof. Let $\Phi \equiv \Psi(xh)$. Then $\Psi((x+y)h) = \Phi(yh)$, by Proposition 4.5 of [3]. If we replace Ψ of Lemma 1 by Φ, we obtain

$$(d/dx)^2 \log\|\Psi(xh)\| = \tfrac{1}{2}(d/dy)^2 \log\{\|\Phi(yh)\|^2\}|_{y=0} \geqq 0\,. \quad \text{Q.E.D.}$$

Remark. Since $\Psi(\lambda h_1 + (1-\lambda)h_2) = [\Psi(h_2)](\lambda(h_1-h_2))$, Lemma 2 implies the convexity of $\log\|\Psi(h)\|$ in h.

Proof of Theorem 1. Set $f(x) = \log\{\|\Psi(xh)\|^2\}$. If $\|\Psi\| = 1$, we have $f(0) = 0$. By Lemma 1, we also have $f'(0) = \psi(h)$. By Lemma 2, $f(x)$ is a C^∞ convex function of x. Hence

$$f(1) \geqq f(0) + f'(0) = \psi(h)\,,$$

which is the inequality (1.2). Q.E.D.

§ 3. A Trotter Product Formula

Lemma 3. *Let H be a self-adjoint operator. If Ψ is in the domain of $e^{\delta H}$ for a $\delta > 0$, then $e^{zH}\Psi$ is holomorphic in z for* $\mathrm{Re}\, z \in (0, \delta)$ *and continuous in z for* $\mathrm{Re}\, z \in [0, \delta]$. *Conversely, if $\Psi(z)$ is holomorphic in a domain D, weakly continuous on its closure $\bar{D}$, which contains an open interval I on the imaginary axis, and $\Psi(z) = e^{zH}\Psi$ for $z \in I$, then Ψ is in the domain of e^{zH} for $z \in \bar{D}$ and $\Psi(z) = e^{zH}\Psi$ for $z \in \bar{D}$.*

The first half is immediate from the spectral theory (cf. Lemma 4 in [1]). The proof of the second half is contained in the proof of Proposition 4.12 in [3].

Lemma 4. *Let self-adjoint operators H and h, a positive number T and a vector Ψ in the domain of $e^{TH}h^n$ for all integers $n \geqq 0$ be given. Assume that h is bounded and there exist $\alpha > 0$ and $K > 0$ satisfying*

$$\|e^{TH}h^n\Psi\| \leqq \alpha K^n \quad (3.1)$$

for all integers $n \geqq 0$. *Then* Ψ *is in the domain of* $e^{t(H+h)}$ *for all complex* t *satisfying* $\mathrm{Re}\, t \in [0, T]$ *and*

$$e^{t(H+h)}\Psi = \sum_{n=0}^{\infty} t^n \int_0^1 \mathrm{d}x_1 \int_0^{x_1} \mathrm{d}x_2 \ldots \int_0^{x_{n-1}} \mathrm{d}x_n \quad e^{tx_nH} h e^{t(x_{n-1}-x_n)H} h \ldots h e^{t(1-x_1)H}\Psi \tag{3.2}$$

where the right hand side is absolutely and uniformly convergent.

Proof. By the proof of Theorem 3.1 in [3], the assumption (3.1) implies that Ψ is in the domain of

$$A_n(z) \equiv e^{z_1H} h \ldots e^{z_{n-1}H} h e^{z_nH} \tag{3.3}$$

for all integers $n > 0$ and for $z = (z_1, \ldots, z_n)$ satisfying

$$\mathrm{Re}\, z \in \bar{I}_n^T \equiv \{s; s_1 \geqq 0, \ldots, s_n \geqq 0, (s_1 + \cdots + s_n) \leqq T\}, \tag{3.4}$$

that $A_n(z)\Psi$ is weakly continuous and bounded by

$$\|A_n(z)\Psi\| \leqq \alpha\{\max(\|h\|, K)\}^{n-1} \tag{3.5}$$

for $\mathrm{Re}\, z \in \bar{I}_n^T$, and that $A_n(z)\Psi$ is holomorphic in z for $\mathrm{Re}\, z \in I_n^T$ (the interior of $\bar{I}_n^T$). Hence the integrals on the right hand side of (3.2) exist, in strong topology for $\mathrm{Re}\, t \in [0, T)$ and in weak topology for $\mathrm{Re}\, t = T$, and the series converges absolutely and uniformly for $\mathrm{Re}\, t \in [0, T]$. If $t = i\tau$ is pure imaginary, then by Proposition 16 of [2] we have

$$e^{i\tau(H+h)} = \mathrm{Exp}_r\left(\int_0^1 ; e^{ix\tau H} i\tau h e^{-ix\tau H}\, \mathrm{d}x\right) e^{i\tau H}$$

and hence (3.2) holds. By the previous Lemma, Ψ is in the domain of $e^{t(H+h)}$ for $\mathrm{Re}\, t \in [0, T]$ and (3.2) holds for all such t. Q.E.D.

Corollary. *Ψ is in the domain of* $e^{(H+h)/2}$ *and*

$$\Psi(h) = e^{(H+h)/2}\Psi \tag{3.6}$$

where $H = \log \Delta_\Psi$ *and* $h \in \mathfrak{M}$.

This follows from Lemma 4 and $e^{xH}\Psi = \Psi$.

Lemma 5. *Under the assumption of Lemma 4,*

$$\lim_{n\to\infty} (e^{tH/n}(1 + n^{-1}th))^n \Psi = \lim_{n\to\infty} ((1 + n^{-1}th) e^{tH/n})^n \Psi = e^{t(H+h)}\Psi \tag{3.7}$$

where $\mathrm{Re}\, t \in [0, T]$, *the limit is in the strong topology for* $\mathrm{Re}\, t \in [0, T)$ *and in the weak topology for* $\mathrm{Re}\, t = T$.

Proof. Let $t \in [0, T]$, $\Psi_{0,n} \equiv e^{tH}\Psi$ and

$$\Psi_{k,n} \equiv \sum_{0 < j_1 < \cdots < j_k \leqq n} e^{j_1 tH/n} h e^{(j_2 - j_1)tH/n} h \ldots \ldots e^{(j_k - j_{k-1})tH/n} h e^{(n-j_k)tH/n}\Psi \tag{3.8}$$

for $1 \leqq k \leqq n$. By (3.5), we have

$$\|\Psi_{m,n}\| \leqq \binom{n}{m} \alpha \bar{K}^m \,, \quad \bar{K} = \max(\|h\|, K)$$

where $\binom{n}{m}$ is the number of terms in the summation. Since

$$\Psi_n \equiv \{e^{tH/n}(1+n^{-1}th)\}^n \Psi = \sum_{m=0}^{n} (t/n)^m \Psi_{m,n} \,,$$

we have

$$\left\| \Psi_n - \sum_{k=0}^{N} (t/n)^k \Psi_{k,n} \right\| \leqq \alpha \sum_{k=N+1}^{n} \binom{n}{k} (\bar{K}t/n)^k$$

$$= \alpha \left\{ (1+n^{-1}\bar{K}t)^n - \sum_{m=0}^{N} m!^{-1} (\bar{K}t)^m \left(1-\frac{1}{n}\right) \cdots \left(1-\frac{m-1}{n}\right) \right\},$$

for $N \leqq n$ where the right hand side tends to $\alpha\left(e^{\bar{K}t} - \sum_{m=0}^{N} (\bar{K}t)^m/m!\right)$ as $n \to \infty$. Hence we have $\lim_{n\to\infty} \Psi_n = e^{t(H+h)}\Psi$ by Lemma 4 if we prove

$$\lim_{n\to\infty} n^{-k} \Psi_{k,n} = \int_0^1 dx_1 \int_0^{x_1} dx_2 \ldots \int_0^{x_{k-1}} dx_k F(x_1, \ldots, x_k) \,, \tag{3.9}$$

$$F(x_1, \ldots, x_k) = A_{k+1}(tx_k, t(x_{k-1}-x_k), \ldots, t(x_1-x_2), t(1-x_1))\Psi \,.$$

Since $F(x_1, \ldots, x_k)$ is continuous in $x = (x_1, \ldots, x_k)$, strongly for $\mathrm{Re}\, t \in [0, T)$ and weakly for $\mathrm{Re}\, t = T$, we obtain (3.8) (as a strong or weak limit according to cases) from the uniform bound $\|F(x_1, \ldots, x_k)\| \leqq \alpha\{\max(\|h\|, K)\}^k$ and the following observation:

$$\Psi_{k,n} = \sum_{0 < j_1 < \cdots < j_k \leqq n} F(j_k/n, j_{k-1}/n, \ldots, j_1/n) \,.$$

The second equality of (3.7) is proved in exactly the same manner where the only difference is that the summation in (3.8) is now for $0 \leqq j_1 < \cdots < j_k < n$. Q.E.D.

Remark 1. If $H = \log \Delta_\Psi$ and $h \in \mathfrak{M}$, then the bound (3.1) holds with $\alpha = \|\Psi\|$, $K = \|h\|$ and $T = 1/2$. Furthermore, $F(x_1, \ldots, x_k)$ is strongly continuous for $\mathrm{Re}\, t \in [0, 1/2]$ by Theorem 3.1 in [3]. Therefore (3.7) holds in strong topology for $\mathrm{Re}\, t \in [0, 1/2]$.

Remark 2. Under the assumption of Lemma 4, it is also possible to prove the usual form of the Trotter product formula:

$$\lim_{n\to\infty} (e^{tH/n} e^{th/n})^n \Psi = \lim_{n\to\infty} (e^{th/n} e^{tH/n})^n \Psi = e^{t(H+h)} \Psi \,. \tag{3.10}$$

For this, we first note the absolute convergence of

$$\sum_{k_1=0}^{\infty} \dots \sum_{k_n=0}^{\infty} e^{tH/n}\{(th/n)^{k_1}/k_1!\} \dots e^{tH/n}\{(th/n)^{k_n}/k_n!\}\,\Psi\,,$$

which is equal to $(e^{tH/n}e^{th/n})^n\Psi$. It is then easy to find out that the difference

$$(e^{tH/n}e^{th/n})^n\,\Psi - (e^{tH/n}(1+n^{-1}th))^n\,\Psi$$

is of the order $(1/n)$ in norm and we obtain (3.10) from (3.7).

§ 4. Some Formulas for Modular Operators

The weak closure of $\Delta^{1/4}\mathfrak{M}^+\,\Psi$ is denoted by V_Ψ as in [1]. We denote $j(x)=J_\Psi x J_\Psi$. The following Lemma is a slightly modified version of Lemma 7 in [1].

Lemma 6. *If* $A\in\mathfrak{M}$, $A^{-1}\in\mathfrak{M}$ *and* $A\Psi\in V_\Psi$, *then*

$$\Delta_{A\Psi}^{1/2} = A\,\Delta_\Psi^{1/2}\,j(A^{-1})\,. \tag{4.1}$$

Proof. By (5.2) of [1], we have $J_\Psi A\Psi = A\Psi$. Hence

$$j(A^{-1})A\Psi = J_\Psi A^{-1}J_\Psi A\Psi = J_\Psi\Psi = \Psi\,.$$

For $Q\in\mathfrak{M}$, we have

$$\begin{aligned} A\,\Delta_\Psi^{1/2}j(A^{-1})Q\,A\,\Psi &= A\,\Delta_\Psi^{1/2}Q\,\Psi = A\,J_\Psi Q^*\,\Psi = j(Q^*)A\,\Psi \\ &= J_\Psi Q^*\,A\,\Psi = \Delta_{A\Psi}^{1/2}Q\,A\,\Psi \end{aligned}$$

where we have used $J_\Psi = J_{A\Psi}$ (Theorem 4(5) of [1]) for the last equality. Since A has a bounded inverse, $A\,\Delta_\Psi^{1/2}j(A^{-1})$ is closed. Since $\mathfrak{M}A\Psi$ is a core of $\Delta_{A\Psi}^{1/2}$, we have

$$\Delta_{A\Psi}^{1/2}\subset A\,\Delta_\Psi^{1/2}j(A^{-1})\,.$$

Since $V_\Psi = V_{A\Psi}$ (Theorem 4(4) of [1]), we may interchange the role of Ψ and $A\Psi$, replacing A by A^{-1} at the same time. We then obtain

$$\Delta_\Psi^{1/2}\subset A^{-1}\,\Delta_{A\Psi}^{1/2}j(A)\,.$$

Therefore we have (4.1). Q.E.D.

Lemma 7. *If* A, A^{-1}, B, B^{-1} *are all in* $\mathfrak{M}$ *and satisfy*

$$B\,\Delta_\Psi^{\alpha/2} = \Delta_\Psi^{\alpha/2}B^*\,, \tag{4.2}$$

$$\Delta_\Psi^{-\alpha/4}B\,\Delta_\Psi^{\alpha/4}\geqq 0\,, \tag{4.3}$$

$$A\,\Delta_\Psi^{\alpha} = (B\,\Delta_\Psi^{\alpha/2})^2\,, \tag{4.4}$$

$$\Delta_\Phi^{\alpha} = A\,\Delta_\Psi^{\alpha}J(A^{-1})\,, \tag{4.5}$$

where $\Phi \in V_\Psi$ *and* $\alpha \in [0, 1/2]$, *then*

$$\Delta_\Phi^{\alpha/2} = B\Delta_\Psi^{\alpha/2} j(B^{-1}). \tag{4.6}$$

Proof. Since B has a bounded inverse and Δ_Ψ^α is self-adjoint, we have $(B\Delta_\Psi^{\alpha/2})^* = \Delta_\Psi^{\alpha/2} B^*$. Hence (4.2) implies that $B\Delta_\Psi^{\alpha/2}$ is self-adjoint. (4.4) then implies that $A\Delta_\Psi^\alpha$ is positive self-adjoint.

By Lemma 6 of [1], (4.2) implies that $\sigma_t^\alpha(B)$ has an analytic continuation $\sigma_z^\psi(B) \in \mathfrak{M}$ for $\mathrm{Im}\, z \in [0, \alpha/2]$ satisfying $\sigma_z^\psi(B)^* = \sigma_{i\alpha/2+\bar{z}}^\psi(B)$ and (4.3) implies that $\sigma_{i\alpha/4}^\psi(B) = (\Delta_\Psi^{-\alpha/4} B \Delta_\Psi^{\alpha/4})^-$ is positive, where $\sigma_t^\psi(B)$ for real t denotes the modular automorphism $\Delta_\Psi^{it} B \Delta_\Psi^{-it}$ and $(\ldots)^-$ denotes the closure of $(\ldots)$.

By (4.2), we also have $B^{-1}\Delta_\Psi^{\alpha/2} = \Delta_\Psi^{\alpha/2}(B^{-1})^*$. Hence by the same reason as above, $\sigma_t^\psi(B^{-1})$ has an analytic continuation $\sigma_z^\psi(B^{-1}) \in \mathfrak{M}$ for $\mathrm{Im}\, z \in [0, \alpha/2]$. Since $\sigma_z^\psi(B)\,\sigma_z^\psi(B^{-1}) = \sigma_z^\psi(B^{-1})\,\sigma_z^\psi(B) = 1$ holds for real t, it holds for $\mathrm{Im}\, z \in [0, \alpha/2]$ and hence $\sigma_z^\psi(B^{-1}) = \sigma_z^\psi(B)^{-1}$. Namely $\sigma_z^\psi(B)^{-1} \in \mathfrak{M}$.

From (4.5) and (4.4), we have

$$\begin{aligned}\Delta_\Phi^\alpha &= (B\Delta_\Psi^{\alpha/2})^2 j(A^{-1}) \\ &= \{B\Delta_\Psi^{\alpha/2} j(B^{-1})\}\,\{j(B) B\Delta_\Psi^{\alpha/2} j(A^{-1})\}.\end{aligned}$$

By (4.4), we have

$$\Delta_\Psi^\alpha j(A^{-1}) = j(\Delta_\Psi^{-\alpha} A^{-1}) = j(B\Delta_\Psi^{\alpha/2})^{-2} = (\Delta_\Psi^{\alpha/2} j(B)^{-1})^2.$$

Hence

$$j(B)B\Delta_\Psi^{\alpha/2} j(A^{-1}) = Bj(B)\Delta_\Psi^{-\alpha/2}\Delta_\Psi^\alpha j(A^{-1}) = B\Delta_\Psi^{\alpha/2} j(B)^{-1}.$$

When restricted to the domain of $\Delta_\Psi^\alpha j(A^{-1})$. By (4.5), the domain of Δ_Φ^α is the same as the domain of $\Delta_\Psi^\alpha j(A^{-1})$. Therefore

$$\Delta_\Phi^\alpha \subset \{B\Delta_\Psi^{\alpha/2} j(B^{-1})\}^2.$$

Since B and $j(B^{-1})$ have bounded inverses, we have

$$(B\Delta_\Psi^{\alpha/2} j(B^{-1}))^* = j(B^*)^{-1}\Delta_\Psi^{\alpha/2} B^* = Bj(B^*)^{-1}\Delta_\Psi^{\alpha/2},$$

where we have used (4.2). By (4.2) again, we have

$$j(B^*)^{-1}\Delta_\Psi^{\alpha/2} = j(\Delta_\Psi^{\alpha/2} B^*)^{-1} = j(B\Delta_\Psi^{\alpha/2})^{-1} = \Delta_\Psi^{\alpha/2} j(B^{-1}).$$

Hence $B\Delta_\Psi^{\alpha/2} j(B^{-1})$ is self-adjoint and

$$\Delta_\Phi^\alpha = \{B\Delta_\Psi^{\alpha/2} j(B^{-1})\}^2.$$

Since $B\Delta_\Psi^{\alpha/4} = \Delta_\Psi^{\alpha/4}\sigma_{i\alpha/4}^\psi(B)$ and $\Delta_\Psi^{\alpha/4} j(B^{-1}) = j(\sigma_{i\alpha/4}^\psi(B)^{-1})\Delta_\Psi^{\alpha/4}$ we have

$$B\Delta_\Psi^{\alpha/2} j(B^{-1}) = \Delta_\Psi^{\alpha/4}\sigma_{i\alpha/4}^\psi(B) j(\sigma_{i\alpha/4}^\psi(B)^{-1})\Delta_\Psi^{\alpha/4}.$$

Hence for any f in $D(B\Delta_\Psi^{\alpha/2}j(B^{-1}))\subset D(\Delta_\Psi^{\alpha/4})$, we have

$$(f, B\Delta_\Psi^{\alpha/2}j(B^{-1})f)=(\Delta_\Psi^{\alpha/4}f, \sigma_{i\alpha/4}^\psi(B)j(\sigma_{i\alpha/4}^\psi(B)^{-1})\Delta_\Psi^{\alpha/4}f).$$

Since $\sigma_{i\alpha/4}^\psi(B)\in\mathfrak{M}$ and $j(\sigma_{i\alpha/4}^\psi(B)^{-1})\in\mathfrak{M}'$ are both positive and commute, we have

$$B\Delta_\Psi^{\alpha/2}j(B^{-1})\geqq 0.$$

Hence we have (4.6). Q.E.D.

Lemma 8. *Assume that* $a\in\mathfrak{M}^+$, $\sigma_t^\psi(a)$ *has an analytic continuation* $\sigma_z^\psi(a)\in\mathfrak{M}$ *for* $\operatorname{Im} z\in[-\frac{1}{2},\frac{1}{2}]$ *and* $\sigma_z^\psi(a)^{-1}\in\mathfrak{M}$ *for all such z. Let*

$$\Phi=(\Delta_\Psi^{2^{-(m+1)}}a\Delta_\Psi^{2^{-(m+1)}})^{2(m-1)}\Psi. \tag{4.7}$$

Then

$$\Delta_\Phi^{2^{-m}}=b\Delta_\Psi^{2^{-m}}j(b^{-1}),\quad b=\sigma_{-i\delta}^\psi(a),\ \delta=2^{-(m+1)}. \tag{4.8}$$

Proof. Let

$$Q(l)=\sigma_{-in(1)}^\psi(a)\dots\sigma_{-in(l)}^\psi(a),\quad n(j)=(j-\tfrac{1}{2})2^{-m}. \tag{4.9}$$

Then

$$\Phi=Q(2^{m-1})\Psi.$$

Since $\sigma_t^\psi(a)^*=\sigma_t^\psi(a)$ for real t, we have $\sigma_z^\psi(a)^*=\sigma_{\bar z}^\psi(a)$ and hence

$$Q(l)^*=\sigma_{i(n(l)+n(1))}^\psi Q(l),\quad n(l)+n(1)=l2^{-m}. \tag{4.10}$$

We also have

$$\begin{aligned}Q(2l)&=Q(l)\sigma_{-i(n(l+1)-n(1))}^\psi(Q(l))\\&=Q(l)\sigma_{-2i(n(l)+n(1))}^\psi(Q(l)^*).\end{aligned} \tag{4.11}$$

Hence

$$\sigma_{i(n(l)+n(1))}^\psi(Q(2l))\geqq 0. \tag{4.12}$$

Due to $a\geqq 0$, (4.12) holds also for $l=\frac{1}{2}$ $(n(l)=0)$.

For $l=2^{m-2}$, we have $n(l)+n(1)=\frac{1}{4}$ and hence $\sigma_{i/4}^\psi Q(2^{m-1})\geqq 0$. By Theorem 3(7) of [1], this implies $\Phi\in V_\Psi$. By Lemma 6 with $A=Q(2^{m-1})$, we have

$$\Delta_\Phi^{1/2}=Q(2^{m-1})\Delta_\Psi^{1/2}j(Q(2^{m-1})^{-1}). \tag{4.13}$$

If we set $\alpha=2^{-k}$, $A=Q(2^{m-k})$ and $B=Q(2^{m-k-1})$ in Lemma 7, then (i) (4.10) with $l=2^{m-k-1}$ implies (4.2), (ii) (4.12) with $l=2^{m-k-2}$ implies (4.3) and (iii) (4.11) with $l=2^{m-k-1}$ implies (4.4), where $1\leqq k\leqq m-1$. Since (4.5) is satisfied for $k=1$, Lemma 7 implies recursively (4.6) for $k=1,\dots,m-1$. The case $k=m-1$ yields (4.8) due to $Q(2^{m-k-1})=Q(1)=b$ for $k=m-1$. Q.E.D.

§ 5. Proof of Theorem 2

Lemma 9. *If* $h \in \mathfrak{M}^+$ *and* $n = 0, 1, 2, \ldots$, *then*

$$\|(\Delta_\Psi^{2^{-(n+2)}} h \Delta_\Psi^{2^{-(n+2)}})^{2^n} \Psi\| \leqq \|(\Delta_\Psi^{2^{-(n+1)}} h^2 \Delta_\Psi^{2^{-(n+1)}})^{2^{n-1}} \Psi\| . \quad (5.1)$$

Proof. We give proof for the following 3 cases in that order: (i) $h^{-1} \in \mathfrak{M}$ and $\sigma_t^\psi(h)$ as well as $\sigma_t^\psi(h^{-1})$ have analytic continuations to M-valued entire functions, (ii) $h^{-1} \in \mathfrak{M}$, (iii) general h.

Case (i). By Hölder inequality,

$$\|\Delta^\beta \Phi\| \leqq \|\Delta^\alpha \Phi\|^\lambda \|\Phi\|^{1-\lambda} \leqq \max\{\|\Delta^\alpha \Phi\|, \|\Phi\|\}, \qquad \Phi \in D(\Delta^\alpha),$$

where $\beta \leqq \alpha$, $\lambda = \beta/\alpha$. Hence it is enough to prove the following 2 inequalities:

$$\|(h \Delta_\Psi^{2^{-(n+1)}})^{2^n} \Psi\| \leqq \|(\Delta_\Psi^{2^{-(n+1)}} h^2 \Delta_\Psi^{2^{-(n+1)}})^{2(n-1)} \Psi\| , \quad (5.2)$$

$$\|(\Delta_\Psi^{2^{-(n+1)}} h)^{2^n} \Psi\| \leqq \|(\Delta_\Psi^{2^{-(n+1)}} h^2 \Delta_\Psi^{2^{-(n+1)}})^{2(n-1)} \Psi\| . \quad (5.3)$$

Let $h\Delta_\Psi^\delta = u|h\Delta_\Psi^\delta|$ be a polar decomposition, where $\delta = 2^{-(n+1)}$. By Lemma 4 of [4], $u \in \mathfrak{M}$. Since h^{-1} is bounded, $h\Delta_\Psi^\delta$ is closed. Since h and Δ_Ψ are strictly positive, $h\Delta_\Psi^\delta$ has 0 kernel and dense range. Hence u must be unitary. Let $q = u^* h \in \mathfrak{M}$. Then $q^{-1} = h^{-1} u \in \mathfrak{M}$. We have $|h\Delta_\Psi^\delta| = q\Delta_\Psi^\delta$.

Let

$$\Phi = |h\Delta_\Psi^\delta|^{2^n} \Psi = (\Delta_\Psi^\delta h^2 \Delta_\Psi^\delta)^{2^{(n-1)}} \Psi , \quad (5.4)$$

where Ψ is in the domain of $(\Delta_\Psi^\delta h^2 \Delta_\Psi^\delta)^{2^{(n-1)}}$ due to Theorem 3.1 of [3].

By assumption, both $\sigma_t^\psi(h^2) = \sigma_t^\psi(h)^2$ and $\sigma_t^\psi(h^{-2}) = \sigma_t^\psi(h^{-1})^2$ have analytic continuations to M-valued entire functions $\sigma_z^\psi(h^2)$ and $\sigma_z^\psi(h^{-2})$. Since $\sigma_z^\psi(h^2)\sigma_z^\psi(h^{-2}) = \sigma_z^\psi(h^{-2})\sigma_z^\psi(h^2) = 1$ for real z, the same equality holds for all z. Hence $\sigma_z^\psi(h^2)^{-1} \in \mathfrak{M}$ for all z. By Lemma 8, we have

$$\Delta_\Phi^{2\delta} = b\Delta_\Psi^{2\delta} j(b^{-1}), \qquad b = \sigma_{-i\delta}^\psi(h^2) . \quad (5.5)$$

Since q^{-1} is bounded, $(q\Delta_\Psi^\delta)^* = \Delta_\Psi^\delta q^*$. Since $q\Delta_\Psi^\delta = |h\Delta_\Psi^\delta|$ is self-adjoint, (4.2) is satisfied for $B = q$ and $\alpha = 2\delta = 2^{-n}$. It then implies, by Lemma 6 of [1], that $\sigma_t^\psi(q)$ has an analytic continuation $\sigma_z^\psi(q) \in \mathfrak{M}$ for $\mathrm{Im}\, z \in [0, \alpha/2]$. If x is in the domain of $\Delta_\Psi^{-\alpha/4}$ as well as in the domain of $\Delta_\Psi^{\alpha/4}$, then

$$(x, \sigma_{i\alpha/4}^\psi(q)x) = (\Delta_\Psi^{-\alpha/4} x, (q\Delta_\Psi^{\alpha/2})\Delta_\Psi^{-\alpha/4} x) \geqq 0$$

due to $q\Delta_\Psi^\delta = |h\Delta_\Psi^\delta| \geqq 0$. Hence (4.3) is satisfied. Since

$$(q\Delta_\Psi^{\alpha/2})^2 = |h\Delta_\Psi^\delta|^2 = \Delta_\Psi^\delta h^2 \Delta_\Psi^\delta = b\Delta_\Psi^{2\delta} = b\Delta_\Psi^\alpha , \quad (5.6)$$

(4.4) is satisfied for $A = b$. (5.5) is then the same as (4.5). By Lemma 9 of [4], $\Phi \in V\Psi$. Therefore Lemma 7 is applicable and

$$\Delta_\Phi^\delta = q\Delta_\Psi^\delta j(q^{-1}) . \quad (5.7)$$

We now have

$$(h\Delta_{\Psi}^{\delta})^{2^n}\Psi=(u|h\Delta_{\Psi}^{\delta}|)^{2^n}\Psi=(u\Delta_{\Phi}^{\delta}j(q))^{2^n}(\Delta_{\Phi}^{\delta}j(q))^{-2^n}\Phi\,.$$

As we shall see immediately below, $\sigma_t^{\phi}(u)$ for $\phi=\omega_{\Phi}$ has an analytic continuation to an M-valued entire function $\sigma_z^{\phi}(u)$. Hence

$$(\Delta_{\Phi}^{\delta}j(q))^k u=\sigma_{-ik\delta}^{\phi}(u)(\Delta_{\Phi}^{\delta}j(q))^k\,.$$

Therefore

$$\begin{aligned}(h\Delta_{\Psi}^{\delta})^{2^n}\Psi&=u\sigma_{-i\delta}^{\phi}(u)\,\sigma_{-2i\delta}^{\phi}(u)\ldots\sigma_{-(i/2)+i\delta}^{\phi}(u)\Phi\\&=(u\Delta_{\Phi}^{\delta})^{2^n}\Phi\,.\end{aligned}\tag{5.8}$$

Similarly, we have $\Delta_{\Psi}^{\delta}h=|h\Delta_{\Psi}^{\delta}|u^*$ and hence

$$(\Delta_{\Psi}^{\delta}h)^{2^n}\Psi=(\Delta_{\Phi}^{\delta}u^*)^{2^n}\Phi\,.\tag{5.9}$$

By Theorem 3.1 of [3], we have

$$\|(u\Delta_{\Phi}^{\delta})^{2^n}\Phi\|\leqq\|u\|^{2^n}\|\Phi\|=\|\Phi\|\,,$$

$$\|(\Delta_{\Phi}^{\delta}u^*)^{2^n}\Phi\|\leqq\|u^*\|^{2^n}\|\Phi\|=\|\Phi\|\,.$$

This proves (5.2) and (5.3), hence (5.1) for this case.

To prove that $\sigma_t^{\psi}(u)$ (and hence $\sigma_t^{\phi}(u^*)=\sigma_{\bar{t}}^{\phi}(u)^*$) has an analytic continuation to an entire function, we first remember that $\sigma_t^{\psi}(q)$ has an analytic continuation $\sigma_z^{\psi}(q)$ for $\operatorname{Im}z\in[0,\delta]$. By (5.6), we have $\sigma_{i\delta}(q)q=\sigma_{i\delta}^{\psi}(b)=h^2$. We then have

$$\Delta_{\Psi}^{-\delta}q=\sigma_{i\delta}^{\psi}(q)\Delta_{\Psi}^{-\delta}=h^2q^{-1}\Delta_{\Psi}^{-\delta}$$

and hence

$$qh^{-2}\Delta_{\Psi}^{-\delta}=\Delta_{\Psi}^{-\delta}q^{-1}\,.$$

Again by Lemma 6 of [1], we obtain an analytic continuation $\sigma_z^{\psi}(q^{-1})$ for $\operatorname{Im}z\in[0,\delta]$ and $\sigma_{i\delta}^{\psi}(q^{-1})=qh^{-2}$. By repeated use of relations

$$\sigma_{i\delta}^{\psi}(q)=h^2q^{-1}\,,\qquad\sigma_{i\delta}^{\psi}(q^{-1})=qh^{-2}\,,\tag{5.10}$$

we obtain analytic continuations for $\operatorname{Im}z\in[k\delta,k\delta+\delta]$:

$$\sigma_z^{\psi}(q)=\begin{cases}\sigma_{z-ik\delta}^{\psi}[\sigma_{ik\delta-i\delta}^{\psi}(h^2)\sigma_{ik\delta-3i\delta}^{\psi}(h^2)\ldots h^2q^{-1}\sigma_{i\delta}^{\psi}(h^{-2})\\\qquad\ldots\sigma_{ik\delta-2i\delta}^{\psi}(h^{-2})]\quad\text{if } k \text{ is odd}\\\sigma_{z-ik\delta}^{\psi}[\sigma_{ik\delta-i\delta}^{\psi}(h^2)\ldots\sigma_{i\delta}^{\psi}(h^2)qh^{-2}\\\qquad\ldots\sigma_{ik\delta-2i\delta}^{\psi}(h^{-2})]\quad\text{if } k \text{ is even}\,,\end{cases}$$

and a similar equation for $\sigma_z^{\psi}(q^{-1})$. Reading (5.10) backwards as

$$q=\sigma_{i\delta}^{\psi}(q^{-1})h^2\,,\qquad q^{-1}=h^{-2}\sigma_{i\delta}^{\psi}(q)\,,$$

we also obtain $\sigma_z^\psi(q)$ and $\sigma_z^\psi(q^{-1})$ for $\operatorname{Im} z<0$. Thus $\sigma_t^\psi(q)$ and $\sigma_t^\psi(q^{-1})$ have analytic continuations to Q-valued entire functions.

Since $u=hq^{-1}$ and $u^*=u^{-1}=qh^{-1}$, $\sigma_t^\psi(u)$ and $\sigma_t^\psi(u^*)$ also have analytic continuations to all z:

$$\sigma_z^\psi(u)=\sigma_z^\psi(h)\,\sigma_z^\psi(q^{-1}), \qquad \sigma_z^\psi(u^*)=\sigma_z^\psi(q)\,\sigma_z^\psi(h^{-1}).$$

By (5.7), we have

$$\Delta_\Phi^{k\delta}u=u^{(k)}\Delta_\Phi^{k\delta},$$

where

$$u^{(k)}=q\sigma_{-i\delta}^\psi(u^{(k-1)})q^{-1} \qquad (k>0),$$

$$u^{(k)}=\sigma_{i\delta}^\psi(q^{-1}u^{(k+1)}q) \qquad (k<0),$$

$$u^{(0)}=u.$$

By Lemma 6 of [1], $\sigma_t^\phi(u)$ has an analytic continuation $\sigma_z^\phi(u)$ for all z. Similar conclusion holds for u^*.

Case (ii). If $h^{-1}\in\mathfrak{M}$, we can write $h=e^Q$ where $Q=Q^*\in\mathfrak{M}$. ($Q=\log h$.) Let $f\in \mathrm{D}(R)$, $f^*=f$ and consider $h_f=e^{Q(f)}$. Then $h_f^{-1}\in M$ and $\sigma_t^\psi(h_f)=\exp Q(f_t)$ as well as $\sigma_t^\psi(h_f^{-1})=\exp -Q(f_t)$ have analytic continuations. Hence, by case (i), we have (5.1) for h_f. Let f_j be a sequence such that $Q(f_j)$ is uniformly bounded and converges to Q strongly. We can complete the proof of this case if we show that both sides of (5.1) with h replaced by h_{f_j} converges to the same expressions with h. This follows from the following general results:

Let $h_j^*=h_j>0$, $h_j\in\mathfrak{M}$, $\|h_j\|$ uniformly bounded and $h=\lim h_j$ (in strong topology). Then

$$\|\Delta_\Psi^{\alpha_1}h_j\Delta_\Psi^{\alpha_2}h_j\ldots\Delta_\Psi^{\alpha_n}h_j\Psi-\Delta_\Psi^{\alpha_1}h\Delta_\Psi^{\alpha_2}h\ldots\Delta_\Psi^{\alpha_n}h\Psi\|$$

converges to 0 for fixed $\alpha_1\geqq 0\ldots\alpha_n\geqq 0$ satisfying $\alpha_1+\cdots+\alpha_n<1/2$. [In the present application, the strict inequality $\alpha_1+\cdots+\alpha_n<1/2$ can be obtained just by absorbing last Δ_Ψ factor into Ψ on both sides of (5.1).] The proof of this general result is achieved by considering $h_j(f_\beta^G)$ and is given in the proof of Proposition 4.1 of [3].

Case (iii). For any given $h\in\mathfrak{M}^+$, we can find a sequence $h_j\in\mathfrak{M}^+$ such that $h_j^{-1}\in\mathfrak{M}^+$, h_j is uniformly bounded (by $\|h\|$) and h_j tends to h strongly. For h_j, we have (5.1) by case (ii). By the same reason as Case (ii), we obtain (5.1) for the given h from (5.1) for h_n by taking the limit $n\to\infty$.

Q.E.D.

Corollary. *For* $h\in\mathfrak{M}^+$, $n=0,1,\ldots$ *and* $\alpha\in[0,2^{-(n+1)}]$,

$$\|\Delta_\Psi^\alpha(h\Delta_\Psi^{2^{-(n+1)}})^{2^n}\Psi\|\leqq\|(\Delta_\Psi^{2^{-(n+1)}}h^2\Delta_\Psi^{2^{-(n+1)}})^{2^{(n-1)}}\Psi\|, \tag{5.11}$$

$$\|\Delta_\Psi^\alpha(h\Delta_\Psi^{2^{-(n+1)}})^{2^n}\Psi\|\leqq\|h^{2^n}\Psi\|. \tag{5.12}$$

Proof. For the case (i) above, this follows from (5.2) and (5.3) by the Hölder inequality. If $0 \leqq \alpha < 2^{-(n+1)}$, then the continuity argument in the proof of Lemma 9 for cases (ii) and (iii) works and (5.11) is proved for general h. The case $\alpha = 2^{-(n+1)}$ is obtained from the case $\alpha < 2^{-(n+1)}$ by the continuity in α.

By repeated use of (5.1), we obtain

$$\|(\Delta_\Psi^{2^{-(n+2)}} h \Delta_\Psi^{2^{-(n+2)}})^{2^n} \Psi\| \leqq \|\Delta_\Psi^{1/4} h^n \Psi\| \leqq \|h^{2^n} \Psi\|$$

where the last inequality is due to Hölder inequality with $\beta = 1/4$, $\alpha = 1/2$ and due to $\|\Delta_\Psi^{1/2} h^{2^n} \Psi\| = \|J_\Psi h^{2^n} \Psi\| = \|h^{2^n} \Psi\|$. By using this inequality on the right hand side of (5.11), we obtain (5.12). Q.E.D.

Proof of Theorem 2. For any real constant c, we have

$$\Psi(h+c) = [\Psi(h)](c) = e^{c/2} \Psi(h)$$

where the first equality is due to Proposition 4.5 of [3]. Hence

$$\|\Psi(h+c)\|^2 = e^c \|\Psi(h)\|^2, \qquad \psi(e^{h+c}) = e^c \psi(e^h).$$

Therefore, by taking $c = \|h\|$, we may restrict our attention to the case $h \geqq 0$. By (5.12) with $\alpha = 0$, we obtain

$$\|\{(1 + 2^{-(n+1)} h) \Delta_\Psi^{2^{-(n+1)}}\}^{2^n} \Psi\| \leqq \|(1 + 2^{-(n+1)} h)^{2^n} \psi\|. \tag{5.13}$$

By taking the limit $n \to \infty$ and using (3.7), we obtain

$$\|e^{(H+h)/2} \Psi\| \leqq \|e^{h/2} \Psi\|.$$

By Corollary to Lemma 4, this is the same as (1.3). Q.E.D.

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