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On Clustering States

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Abstract. Clustering properties of states over the algebra of local observables are discussed under the weak form of "asymptotic abelianness".

1. Introduction

In the framework of the algebraic approach to the quantum field theory and statistical mechanics, we have various notions about locality called "asymptotic abelianness" ([4]–[11]). All of them are the constraints imposed upon the algebra of local observables, under which clustering properties of states over the algebra are discussed.

These "algebraic" conditions seem to be too restrictive for characterizing "states". Besides, though the conditions are reasonable when the automorphism group consists of space translations, we may not expect algebraic "asymptotic abelianness" in the case of time translations. So in this note, we require "asymptotic abelianness" as restrictions over states. Our conditions are much weaker than algebraic ones and we can still discuss clustering properties of states under them.

2. Strongly Clustering State

Let $\mathfrak{A}$ be a C^*-algebra, τ a mapping from an infinite set G into automorphisms of $\mathfrak{A}$ and S the set of states on $\mathfrak{A}$. For any ϕ in S there correspond a Hilbert space H_ϕ, a representation π_ϕ of $\mathfrak{A}$ on H_ϕ and a unit cyclic vector Ω_ϕ such that

$$\phi(a) = (\Omega_\phi, \pi_\phi(a)\,\Omega_\phi), \qquad a \in \mathfrak{A}.$$

Let E_Ω be the projection on vector Ω_ϕ.

Definition 1. A state $\phi \in S$ is called

i) *strongly G-central iff for any $a, b \in \mathfrak{A}$*

$$\lim_{g \to \infty} \phi([\tau_g(a), b]) = 0$$

and

ii) *strongly clustering iff for any $a, b \in \mathfrak{A}$*

$$\lim_{g \to \infty} |\phi(\tau_g(a)\,b) - \phi(\tau_g(a))\,\phi(b)| = 0.$$

Theorem 1. *If $\phi \in S$ is pure and strongly G-central, ϕ is strongly clustering.*

We prove this theorem in a more general form:

Theorem 1′. *Let ϕ be a pure state and $\{a_n\}$ a bounded sequence in $\mathfrak{A}$ such that for any $b \in \mathfrak{A}$*

$$\lim_{n\to\infty} \phi([a_n, b]) = 0\,,$$

then

$$\lim_{n\to\infty} |\phi(a_n b) - \phi(a_n)\,\phi(b)| = 0\,.$$

For the proof of this theorem, we use the following lemma.

Lemma 1. *Let $\mathfrak{A}, \mathfrak{B}$ be *-algebras of bounded operators on a Hilbert space H and $\mathfrak{A}$ weakly dense in $\mathfrak{B}$, then for any $\varepsilon > 0$, $B \in \mathfrak{B}$ and $\Omega \in H$ there exists $A \in \mathfrak{A}$ such that*

$$\|(A-B)\,\Omega\| < \varepsilon,\ \|(A-B)^*\,\Omega\| < \varepsilon\,.$$

Proof. Let $\mathfrak{A}^s$ (resp. $\mathfrak{B}^s$) be the set of all self-adjoint elements of $\mathfrak{A}$ (resp. $\mathfrak{B}$). Then $\mathfrak{A}^s$ is weakly dense in $\mathfrak{B}^s$. Since $\mathfrak{A}^s$ is a convex set, it is strongly dense in $\mathfrak{B}^s$ ([2] density theorem).

Since $B_1 = (B+B^*)/2$ and $B_2 = (B-B^*)/2$ are in $\mathfrak{B}^s$, we can choose A_1 and A_2 in $\mathfrak{A}^s$ such that

$$\|(A_1 - B_1)\,\Omega_\phi\| < \varepsilon/2, \qquad \|(A_2 - B_2)\,\Omega_\phi\| < \varepsilon/2\,.$$

Then $A = A_1 + i A_2$ satisfies the required condition.

Proof of Theorem 1′. Let K be an upper bound of $\{\|a_n\| \cdot \|b\|\}$. Obviously we have

$$\phi(a_n b) - \phi(a_n)\,\phi(b) = (\Omega_\phi, A_n(1-E_\Omega)\,B\,\Omega_\phi)$$

where $A_n = \pi_\phi(a_n)$ and $B = \pi_\phi(b)$. Since ϕ is a pure state, $\pi_\phi(\mathfrak{A})$ is irreducible and therefore weakly dense in the *-algebra consisting of all bounded operators on H_ϕ. By Lemma 1, for any $\varepsilon > 0$ there exists $B' \in \pi_\phi(\mathfrak{A})$ such that

$$\|\{(1-E_\Omega)\,B - B'\}\,\Omega_\phi\| < \varepsilon/3K, \qquad \|\{(1-E_\Omega)\,B - B'\}^*\,\Omega_\phi\| < \varepsilon/3K\,.$$

By assumption, there exists an integer N such that

$$|(\Omega_\phi, [A_n, B']\,\Omega_\phi)| < \varepsilon/3$$

for all $n > N$. Then we have

$$\begin{aligned}&|(\Omega_\phi, A_n(1-E_\Omega)\,B\Omega_\phi) - (\Omega_\phi, (1-E_\Omega)\,B A_n \Omega_\phi)|\\ &\qquad\qquad < \varepsilon/3 + \varepsilon/3 + |(\Omega_\phi, [A_n, B']\,\Omega_\phi)| < \varepsilon\end{aligned}$$

for all $n > N$. Since $(\Omega_\phi, (1-E_\Omega)\,B A_n \Omega_\phi) = 0$, the theorem is proved.

3. Weakly Clustering State

In addition to the notations defined in the previous section, let G be a locally compact group and τ a continuous representation of G by automorphisms of $\mathfrak{A}$. We denote by S_0 the set of states invariant under $\tau(G)$. If $\phi \in S_0$, there exists a strongly continuous unitary representation U_ϕ of G on H_ϕ such that

$$\pi_\phi(\tau_g(a)) = U_\phi(g)\, \pi_\phi(a)\, U_\phi(g)^{-1},$$
$$U_\phi(G)\, \Omega_\phi = \Omega_\phi$$

for $a \in \mathfrak{A}$, $g \in G$. We denote by E_ϕ the projection on the subspace spanned by all $U_\phi(G)$-invariant vectors and by R_ϕ the W^*-algebra generated by $\pi_\phi(\mathfrak{A}) \cup U_\phi(G)$, namely

$$R_\phi = (\pi_\phi(\mathfrak{A}) \cup U_\phi(G))''.$$

Definition 2. A $\tau(G)$-invariant state ϕ is called
i) *weakly G-central iff for any $a, b \in \mathfrak{A}$,*

$$\mathrm{M}\,\phi([\tau_g(a), b]) = 0,$$

ii) *weakly clustering iff for any $a, b \in \mathfrak{A}$,*

$$\mathrm{M}\,\phi(\tau_g(a)\, b) = \phi(a)\, \phi(b)$$

where M *is the Godement mean over G and*
iii) *G-abelian iff $E_\phi \pi_\phi(\mathfrak{A}) E_\phi$ is abelian.*

Lemma 2. *Let ϕ be a weakly G-central state. Then for $A, B \in R_\phi$*

$$(\Omega_\phi, [E_\phi A E_\phi, E_\phi B E_\phi]\, \Omega_\phi) = 0.$$

Proof. The lemma holds for $A, B \in \pi_\phi(\mathfrak{A})$, since

$$(\Omega_\phi, [E_\phi A E_\phi, E_\phi B E_\phi]\, \Omega_\phi) = \mathrm{M}\,\phi([\tau_g(a), b]) = 0$$

with $A = \pi_\phi(a)$, $B = \pi_\phi(b)$ and $a, b \in \mathfrak{A}$. Then it can be extended to the weak closure $\pi_\phi(\mathfrak{A})''$. Since

$$E_\phi R_\phi E_\phi = E_\phi \pi_\phi(\mathfrak{A})'' E_\phi,$$

the lemma is proved.

The next theorem is implicitly proved in [1].

Theorem 2. *For an invariant state ϕ, the following conditions are equivalent.*
i) *ϕ is weakly clustering.*
ii) *ϕ is weakly G-central and is an extremal invariant state.*

Proof. i)$\to$ii) As is well known, a weakly clustering state is an extremal invariant state ([9]) and is obviously G-central.

ii)$\to$i) We have for any $a, b \in \mathfrak{A}$

$$\begin{aligned}\mathrm{M}\,\phi(\tau_g(a)\,b) - \phi(a)\,\phi(b) &= (\Omega_\phi, A(E_\phi - E_\Omega)\,B\,\Omega_\phi)\\ &= (\Omega_\phi, [E_\phi A E_\phi, E_\phi(1-E_\Omega)\,B E_\phi]\,\Omega_\phi)\end{aligned}$$

where $A = \pi_\phi(a)$, $B = \pi_\phi(b)$. Since ϕ is an extremal invariant state, $\pi_\phi(\mathfrak{A}) \cup U_\phi(G)$ is irreducible. Therefore R_ϕ contains $(1-E_\Omega)B$. This proves the theorem by Lemma 2.

For the decomposition of an invariant state into weakly clustering states, weak G-central property is not sufficient in general. Namely we have:

Theorem 3. *Let $\mathfrak{A}$ be separable. For an invariant state ϕ the following are equivalent.*

i) *ϕ is an integral of weakly clustering invariant states.*

ii) *ϕ is G-abelian.*

Proof. i)$\to$ii). Let ϕ_ξ be a measurable family of weakly clustering states on $\mathfrak{A}$ and

$$\begin{aligned}\phi(a) &= \int \phi_\xi(a)\,d\xi\,,\\ \pi(a) &= \int^{\oplus} \pi_\xi(a)\,d\xi\,,\\ H &= \int^{\oplus} H_\xi\,d\xi\,,\\ \Omega &= \int^{\oplus} \Omega_\xi\,d\xi\,,\\ U(g) &= \int^{\oplus} U_\xi(g)\,d\xi\,,\\ E &= \int^{\oplus} E_\xi\,d\xi\end{aligned}$$

for all $a \in \mathfrak{A}$ and $g \in G$, where E is the projection on $U(G)$-invariant vectors in H and π_ξ is the abbreviation of π_{ϕ_ξ} and so on. The representation π_ϕ and U_ϕ associated with ϕ can be identified with π and U restricted to the closure of $\pi(\mathfrak{A})\,\Omega$, since

$$\phi(a) = (\Omega, \pi(a)\,\Omega), \qquad a \in \mathfrak{A}\,.$$

Then $E\pi(\mathfrak{A})\,E$ hence $E_\phi \pi_\phi(\mathfrak{A})\,E_\phi$ is abelian since $E_\xi \pi_\xi(\mathfrak{A})\,E_\xi$ is abelian. (E_ξ is one-dimensional since ϕ_ξ is weakly clustering.)

ii)$\to$i) If we regard the expressions in the previous paragraph as the central decomposition ([3]) with respect to $R_\phi \cap R'_\phi$. Then $E_\xi \pi_\xi(\mathfrak{A})\,E_\xi$ must be abelian since $E\pi(\mathfrak{A})\,E$ is abelian.

We will prove the remaining part in the following proposition.

Proposition. *If ϕ is a G-abelian state such that R_ϕ is a factor, ϕ is weakly clustering.*

Proof. We abbreviate subscript ϕ in E_ϕ etc. Since R is a factor, we have

$$\{\lambda E\} = E(R \cap R')E = (ERE) \cap (R'E).$$

Since $ERE = E\pi(\mathfrak{A})''E$ is abelian, we have

$$ERE \subset (ERE)'E = R'E.$$

Hence

$$ERE = \{\lambda E\}.$$

Since Ω is a cyclic vector, E is one-dimensional. Namely ϕ is weakly clustering.

4. Discussions

We have discussed clustering properties on the basis of some "asymptotically abelian properties" of states such as strong and weak G-central properties. When algebraic "asymptotically abelian properties" are assumed, the following results are known:

1. If $\mathfrak{A}$ is weakly asymptotically abelian and $\pi_\phi(\mathfrak{A})''$ is a factor, ϕ is strongly clustering ([7, 10]).
2. If $\mathfrak{A}$ is G-abelian and $(\pi_\phi(\mathfrak{A}) \cup U_\phi(G))''$ is a factor, ϕ is weakly clustering.

It is interesting to note that if we replace assumptions of algebraic asymptotic abelian and G-abelian properties in the above by assumptions of G-central properties of states, and the word "a factor" by a word "irreducible", then we obtain our Theorem 1 and 2.

In this replacement, we can not keep the word "factor" because of the following example:

Let G be trivial and ϕ be the trace over the algebra of all 2×2 matrices, then ϕ is a factor state but not a character.

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