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Spacetime b-Boundaries

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Abstract. It is shown that Schmidt's b-boundary for a spacetime can be analyzed using a submanifold of the tangent bundle, rather than the principal bundle or the bundle of orthogonal frames.

1. Introduction

Schmidt [1] has shown that every spacetime can be assigned a boundary, called the b-boundary. Roughly speaking, the boundary points are ideal endpoints for those inextendible curves which do not escape to infinity. Though useful in general arguments, such as those in Hawking and Ellis [2], the b-boundary is hard to construct in specific examples. The purpose of this paper is to point out that the construction can be carried out using only a submanifold of the tangent bundle. Section 2 states the results, Section 3 supplies the proofs, and Section 4 gives 2 examples.

In discussing differential geometry, the notation and terminology of Bishop and Goldberg [3] will usually be used. Hu [4] will be taken as the standard topology reference. Throughout the paper (M, g, D) will denote a *spacetime*: a real, 4-dimensional, connected, Hausdorff, oriented, time-oriented, C^∞ Lorentzian manifold (M, g) together with the Levi-Civita connection D of g. TM denotes the tangent bundle, with projection $\pi : TM \to M$. The main idea is the following. Suppose $\alpha : E \to M$ is an inextendible C^∞ curve. α may be lightlike and need not be geodesic so, in general, neither arc length nor an affine parameter supplies an adequate criterion for when α fails to escape to infinity. But suppose we had a unit timelike vector field $P : M \to TM$ available. Then we could use arc length with respect to the positive definite metric $g + 2g(P, \cdot) \otimes g(P, \cdot)$. The game is to introduce P and then amputate it back out.

2. The Unit Future

The *unit future* UM of M is the following C^∞ submanifold of the tangent bundle: $UM = \{(x, P) \in TM \mid g(P, P) = -1, P \text{ is future-pointing}\}$. Thus U, defined by $U = \pi|_{UM}$, is a C^∞ onto map $U : UM \to M$. As in

Bishop and Goldberg [3] we can regard the identity map of UM onto itself as a C^∞ vector field $P: UM \to TM$ over the map U. For example, suppose $x \in M$ and $y, z \in U^{-1}\{x\}$. Then Py, $Pz \in M_{Uy} = M_x$ and $g(Py, Pz) \leqq -1$, where equality holds iff $y = z$. As pointed out in Bishop and Goldberg [3], U^*D is a C^∞ connection over ("on") the map U. For example, suppose $y \in UM$ and $Y \in (UM)_y$. Then Y is *vertical* iff $U_* Y = 0$, *horizontal* iff $U^* D_Y P = 0$, and zero iff it is both horizontal and vertical.

Proposition 2.1. *There is a unique Riemannian metric G on UM such that for all $(y, Y) \in TUM$, $G(Y, Y) = g(U_* Y, U_* Y) + 2[g(U_* Y, Py)]^2 + g(U^* D_Y P, U^* D_Y P)$.*

Proof. Since g is Lorentzian, $g(Uy) + 2g(Py, .) \otimes g(Py, .)$ is a positive definite quadratic form on M_{Uy}. Thus if Y is horizontal, $G(Y, Y) \geqq 0$, with equality holding iff $Y = 0$. Moreover, for any Y, $g(U^* D_Y P, Py) = \frac{1}{2} U^* D_Y[(g \circ U)(P, P)] = \frac{1}{2} Y[-1] = 0$. Thus $U^* D_Y P \in (Py)^\perp \subset M_{Uy}$ for any Y. But g restricted to $(Py)^\perp$ is positive definite. Thus if Y is vertical $G(Y, Y) \geqq 0$, with equality holding iff $Y = 0$. Thus G is positive definite. The rest is straightforward. $\square$

Let $d: UM^2 \to [0, \infty)$ be the topological metric determined by G, as in Helgason [5; Section 1.9]. Let $(\boldsymbol{UM}, \boldsymbol{d})$ be the complete metric space in which (UM, d) is dense. Denote the positive integers by Z^+. Define a relation $R \subset \boldsymbol{UM}^2$ as follows. wRy iff there are Cauchy sequences $w': Z^+ \to UM$ and $y': Z^+ \to UM$ such that: (A) w' converges to w and y' converges to y; (B) the projections coincide, i.e. $U \circ w' = U \circ y'$; and (C) there is a uniform lower bound $A \in (-\infty, -1]$ such that, for all $n \in Z^+$, $g(Pw'n, Py'n) \geqq A$. We now show that R is an equivalence relation and that the decomposition space $\boldsymbol{UM}/R$ is homeomorphic to the union of M with the b-boundary of M.

3. Proofs

To give the proofs and relate $\boldsymbol{UM}/R$ to the space defined by Schmidt we first review a standard definition of the b-boundary. Let OM be the bundle, above M, of those (Lorentzian-) orthonormal frames whose orientation and time-orientation is that determined by (M, g). Let $\theta: OM \to M$ be the projection. Let P_i $(i = 1, \ldots, 4)$ be the four standard vector fields over θ. Thus for each $\delta \in (1, 2, 3)$, $(g \circ \theta)(P_\delta, P_\delta) = 1 = -(g \circ \theta)(P_4, P_4)$, with the other dot products zero. Let $V: OM \to UM$ be the projection onto the unit future. Then $\theta = U \circ V$ and $P_4 = P \circ V$. Define a C^∞, $(0, 2)$ tensor field H on OM by $H(Q, Q) = \sum_{\delta=1}^{3} \{g(\theta^* D_Q P_\delta, \theta^* D_Q P_\delta) + [g(\theta^* D_Q P_\delta, P_4 q)]^2\}$ for all $(q, Q) \in TOM$. By an argument similar to that of Section 2, $G_0 = V^* G + H$ is a Riemannian metric on

OM. Let d_0 be the topological metric determined by G_0, $(\mathbf{OM}, \mathbf{d}_0)$ be the complete metric space in which (OM, d_0) is dense.

One can extend θ to $\mathbf{OM}$ by using the structure group L of OM. The elements of L are real 4×4 matrices and L is isomorphic to that component of the Lorentz group which contains the identity; here and throughout L is assigned its standard topology. The action of $l \in L$ on OM will be denoted by $R_l : OM \to OM$. Thus if $q, r \in OM$ then $\theta q = \theta r$ iff there is an $l \in L$ such that $R_l q = r$. Each R_l has a uniformly continuous extension $\mathbf{R}_l : \mathbf{OM} \to \mathbf{OM}$. For $q, r \in \mathbf{OM}$, $\mathbf{R}_l q = r$ iff there are Cauchy sequences $q' : Z^+ \to OM$ converging to q and $r' : Z^+ \to OM$ converging to r with $R_l \circ q'$ converging to r. There is an equivalence relation $\sim$ on $\mathbf{OM}$, defined as follows: $q \sim r$ iff there is an $l \in L$ such that $\mathbf{R}_l q = r$. The decomposition space $\mathbf{M} = \mathbf{OM}/\sim$ is called the *spacetime with b-boundary* $\mathbf{M}$. The *b-boundary* of (M, g, D) is the topological space $\mathbf{M} - \boldsymbol{\theta}(OM)$ $= \mathbf{M} - \theta(OM) = \mathbf{M} - M$, where $\boldsymbol{\theta}$ is the projection.

We can see the relation of these definitions to the discussion of Section 2 by filling in the two missing maps, $\mathbf{V}$ and $\mathbf{U}$, in the following diagram.

$$\begin{tikzcd}
OM \arrow[r, "V"] \arrow[rr, bend left, "\theta"] \arrow[d, phantom, "\cap"] & UM \arrow[r, "U"] \arrow[d, phantom, "\cap"] & M \arrow[d, phantom, "\cap"] \\
\mathbf{OM} \arrow[r, "\mathbf{V}"] \arrow[rr, bend right, "\boldsymbol{\theta}"'] & \mathbf{UM} \arrow[r, "\mathbf{U}"] & \mathbf{M}
\end{tikzcd}$$

Proposition 3.1. *For all $q, r \in \mathbf{OM}$ and $y \in \mathbf{UM}$:*

(A) $\mathbf{d}_0(q, r) \geqq \mathbf{d}(\mathbf{V}q, \mathbf{V}r)$;

(B) *there is an $s \in \mathbf{V}^{-1}\{y\}$ such that $\mathbf{d}_0(s, q) = \mathbf{d}(y, \mathbf{V}q)$.*

Proof. The tensor field H defined above is positive semi-definite. Since $G_0 = H + V^* G$, assertion (A) follows. To prove (B) we shall construct an "optimum lift" into OM of each curve into UM. The following notation will be convenient. Let β be a C^∞ curve into OM. Abbreviate $(\theta \circ \beta)^* D_{(d/dt)}(P_4 \circ \beta)$, where t is the curve parameter, by $\dot{P}_4$, etc. Now let $\alpha : [0, a] \to UM$ be a C^∞ curve from Vq to y. Then there is a unique C^∞ curve $\beta : [0, a] \to OM$ such that: (i) $V \circ \beta = \alpha$; (ii) $\beta 0 = q$; and (iii) β obeys the Fermi-Walker transport law in the sense that for all $\delta \in (1, 2, 3)$ $\dot{P}_\delta = [(g \circ \theta \circ \beta)(\dot{P}_4, P_\delta \circ \beta)]\,(P_4 \circ \beta)$. From the form of H, the length of β is the same as the length of α. Moreover $V^{-1}\{y\}$ is compact. (B) above now follows by considering a sequence of curves $\alpha_1, \alpha_2, \ldots$ into UM whose lengths approach $d(y, Vq)$, with each α_i going from Vq to y. $\square$

Theorem 3.2. *There is a unique, uniformly continuous, uniformly open, onto extension $\mathbf{V} : \mathbf{OM} \to \mathbf{UM}$ of $V : OM \to UM$.*

Proof. V is uniformly continuous by 3.1.A. Therefore, as shown in Kelley [6; Chapter 6], V has a unique uniformly continuous extension $\boldsymbol{V}: \boldsymbol{OM} \to \boldsymbol{UM}$. If we can show that 3.1.B extends, the uniform openess of $\boldsymbol{V}$ will follow; compare Kelley [6; Chapter 6]. Suppose that $q \in OM$ and $y \in \boldsymbol{V}(\boldsymbol{OM}) \subset \boldsymbol{UM}$. Let $r' : Z^+ \to OM$ be a Cauchy sequence such that $V \circ r'$ converges to y. For each $n \in Z^+$ we can, by 3.1.B, choose $s'(n) \in V^{-1}\{Vr'n\}$ such that $d_0(s'(n), q) = d(Vr'n, Vq)$. This determines a sequence $s' : Z^+ \to OM$; it also determines a sequence $l' : Z^+ \to L$ by the rule $R_{l'n} s'n = r'n$ for all such n. Now $P_4(s'n) = P(Vs'n) = P(Vr'n) = P_4(r'n)$. Thus the image of l' is contained in a compact subset of L and there is at least one cluster point, say $l \in L$. Then $R_l^{-1} r'$ is a Cauchy sequence; let $s \in \boldsymbol{OM}$ be its limit. Then $s \in \boldsymbol{V}^{-1}\{y\}$ and $\boldsymbol{d}_0(s, q) = \boldsymbol{d}(\boldsymbol{V}s, Vq) = \boldsymbol{d}(y, Vq)$. Thus 3.1.B extends to this case. 3.1.B also extends to the more general case $y \in \boldsymbol{V}(\boldsymbol{OM})$, $q \in \boldsymbol{OM}$; the proof is so similar to that just given it is omitted. Thus $\boldsymbol{V}$ is uniformly open. Kelley [6; Chapter 6] shows that the range of a continuous, uniformly open map of a complete metric space into a Hausdorff uniform space is complete. It follows that $\boldsymbol{V}$ is onto. □

Since $\boldsymbol{V}$ is open and continuous it is an identification. Moreover, note that any Cauchy sequence $y' : Z^+ \to UM$ can be lifted to a Cauchy sequence $r' : Z^+ \to OM$, with $V \circ r' = y'$. For let $s' : Z^+ \to OM$ be a Cauchy sequence such that $V \circ s'$ converges to the limit $y \in UM$ of y'. For each $n \in Z^+$, choose $r'n$ such that $d_0(r'n, s'n) = d(y'n, Vs'n)$ and $r'n \in V^{-1}\{y'n\}$. Then r' is Cauchy. Having extended V we can now extend U. Suppose $y \in \boldsymbol{UM}$ and $q, r \in \boldsymbol{V}^{-1}\{y\}$.

Proposition 3.3. $\boldsymbol{\theta} q = \boldsymbol{\theta} r$.

Proof. Suppose y' is a Cauchy sequence which converges to y. Lift y' to a Cauchy sequence q' which converges to q, using the method just discussed; also lift y' to a Cauchy sequence r' which converges to r. Define a sequence $l' : Z^+ \to L$ by $R_{l'n} q'n = r'n$. As in the theorem, there is a cluster point $l \in L$. $\boldsymbol{R}_l q = r$ so $\boldsymbol{\theta} q = \boldsymbol{\theta} r$. □

Thus we can define $\boldsymbol{U} : \boldsymbol{UM} \to \boldsymbol{M}$ by $\boldsymbol{U} y = \boldsymbol{\theta}(\boldsymbol{V}^{-1}\{y\})$ for all $y \in \boldsymbol{UM}$. Since $\boldsymbol{\theta}$ and $\boldsymbol{V}$ are identifications, $\boldsymbol{U}$ is an identification. The last step is to describe $\boldsymbol{U}$ wholly in terms of structures defined on UM. Suppose $w, y \in \boldsymbol{UM}$; let R be as in 2.

Proposition 3.4. $\boldsymbol{U} w = \boldsymbol{U} y$ *iff* wRy.

Proof. Suppose $\boldsymbol{U} w = \boldsymbol{U} y$. Thus if q' is a Cauchy sequence which converges to $q \in \boldsymbol{V}^{-1}\{w\}$ and r' is a Cauchy sequence which converges to $r \in \boldsymbol{V}^{-1}\{y\}$ then there is an $l \in L$ such that $R_l \circ q'$ converges to r. Form a Cauchy sequence which converges to w by alternating terms from $V \circ q'$ and $V \circ R_l^{-1} \circ r'$ and a Cauchy sequence which converges to y by alternating terms from $V \circ R_l \circ q'$ and $V \circ r'$. The projections of these two Cauchy sequences into M coincide and the existence of a uniform lower bound is implied by the fact that l is fixed. Thus wRy. Conversely, suppose wRy. Suppose w' converges to w and y' converges to y, with the projections of w' and y' identical. Lift w' to a Cauchy sequence q' into OM, y' to a Cauchy sequence r' into OM. Define $l': Z^+ \to L$ by $R_{l'n} q'n = r'n$. The existence of a uniform lower bound on $g(Pw'n, Pr'n)$ implies that l' has at least one cluster point $l \in L$. Then $R_l \circ q'$ converges to the same point as r' so $\boldsymbol{U}w = \boldsymbol{U}y$. $\square$

As corollaries we have that R is an equivalence relation and that $\boldsymbol{U}\boldsymbol{M}/R = \boldsymbol{M}$, as claimed in Section 2.

4. Examples and Comments

The first example shows that the condition of a uniform lower bound in the definition of R, Section 2, cannot be dropped. Let $\alpha: (-\infty, 0) \to M$ be a lightlike geodesic with the following property. There is an $x \in M$ such that, for all $n \in Z^+, \alpha(-1/2^n) = x$ and $\alpha_*(-1/2^n) = 2^n \alpha_*(-1) \in M_x$. Thus the image of α is $\alpha[-1, -\frac{1}{2}) \subset M$ and α winds around an infinite number of times as the affine parameter, say t, approaches zero from below. Hawking and Ellis [2] show this rather peculiar behavoir can in fact occur. Now let $X_0 \in M_x$ be unit, timelike, and future-pointing. The constant sequence $y': Z^+ \to UM$ given by $y'n = (x, X_0)$ has $y = (x, X_0)$ as limit and $\boldsymbol{U}y = Uy = x$. Next define a vector field over α, $X: (-\infty, 0) \to TM$, as follows: X is parallel, i.e. $\alpha^* D_{d/dt}(X \circ \alpha) = 0$; and $X(-1) = X_0$. Then the sequence w' defined by $w'n = (x, X(-1/2^n))$ is also Cauchy. For the only contribution to the arc length of the curve $\beta = (\alpha, X \circ \alpha)$ $:(-\infty, 0) \to UM$ comes from the term $2[(g \circ \alpha)(\alpha_*, X)]^2$, which is constant; let w be the limit of w'. w' has the same projection into M as y', but $\boldsymbol{U}w \neq \boldsymbol{U}y$, as discussed in Hawking and Ellis [2]. The catch is that $g(Pw'n, Py'n) = g(X(-1/2^n), X_0)$ is not bounded from below.

The second example shows that, at least in one artificially constructed case, working with UM rather than OM gives a major simplification. Let N be R^3 with the origin $(0,0,0)$ deleted. Let h be a C^0 Riemannian metric on R^3 which is C^∞ on N. Let $M = N \times (-\infty, \infty)$, with projections $S: M \to N$ and $T: M \to (-\infty, \infty)$. Define g on M by $g = S^*h - dT \otimes dT$. Supply (M, g) with the natural orientation, natural time-orientation, and the Levi-Civita connection D. Then (M, g, D) is a spacetime. The claim is that $\boldsymbol{M}$ is homeomorphic to R^4; roughly speaking, the b-boundary consists simply of the "missing points" $(0,0,0) \times (-\infty, \infty)$. Only an outline of the rather tedious proof will be given.

Let $\alpha: [0, a] \to UM$ be a C^∞ curve. Then $(G \circ \alpha)(\alpha_*, \alpha_*) \geqq (g \circ U \circ \alpha)$ $(\dot{P}, \dot{P})$, with $\dot{P}$ essentially as in 3.1. Define the vector field $X: M \to TM$

by $T_* X = d/ds$, $S_* X = 0$. X is unit, timelike, future pointing, and parallel ("covariant constant"). Let $f\colon [0, a] \to [0, \infty)$ be the function defined by $\cosh f = -(g \circ U \circ \alpha)(P \circ \alpha, X \circ U \circ \alpha)$. Using the inequality mentioned above and the fact that X is parallel one finds that the arc length of α is at least $|f0 - fa|$. Now let $w' : Z^+ \to UM$ be a Cauchy sequence with limit w. The above estimate shows there is a uniform lower bound on $g(Pw'n, XUw'n)$. Next one can work with the "horizontal part" $g(U_* Y, U_* Y) + 2[g(U_* Y, Py)]^2$ of G, rather than the "vertical part" $g(U^* D_Y P, U^* D_Y P)$ used above. One finds that the sequence y', defined by $y'n = (Uw'n, XUw'n) \in UM$ is also Cauchy and that its limit y obeys $\boldsymbol{U} y = \boldsymbol{U} w$. Thus one can confine attention to sequences y' with the property $P \circ y' = X \circ U \circ y'$. The rest is straightforward and gives the result already mentioned.

If one tries to work directly with OM in this second example a terrible mess results. Unfortunately, in more realistic cases even using UM still leads to quite difficult computations. Whether one can develop effective techniques for computing the b-boundaries of the various physically interesting spacetimes remains to be seen. If not, the physical relevance of b-boundary techniques may remain rather obscure.

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