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# Ising Model and Bernoulli Schemes in One Dimension

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**Abstract.** We prove that the one-dimensional random fields with finite first moment are isomorphic to Bernoulli schemes.

## § 1. Introduction and Notations

We consider stationary processes defined on a space I of states with only two elements: $I=\{0,1\}$. A process will thus be a regular probability measure $\mu$ defined on the compact space $K_Z=\prod_{i\in Z} I$, where Z is the set of the integers and $K_Z$ is considered with the topology product of the discrete topologies on the factors I.

The elements of $K_Z$ will be identified with the subset $X\subset Z$.

If $\Lambda\subset Z$ is a finite set and $K_{Z/\Lambda}$ is defined in analogy with $K_Z$, ($Z/\Lambda$ is the complement of $\Lambda$), the process defines a natural measure $\mu_\Lambda$ on $K_{Z/\Lambda}$ and a natural probability distribution $f_\Lambda$ on the set of subsets of $\Lambda$:

$$\mu_\Lambda(E)=\sum_{X\subset\Lambda}\mu(X\cup E)\qquad \forall\, E\subset K_{Z/\Lambda}\,, \tag{1.1}$$

$$f_\Lambda(X)=\mu(\{Y/Y\in K_Z, Y\cap\Lambda=X\})\qquad \forall\, X\subset\Lambda\,. \tag{1.2}$$

Notice that $\{Y/Y\in K_Z, Y\cap\Lambda=X\}$ can be thought as an atom $A(\Lambda,X)$ of the partition $\bigvee_{i\in\Lambda} T^i P$ where T is the shift operator (rightwards) on K and $P=(P_0,P_1)$ is the two set (generating) partition of $K_Z$ consisting in the sets:

$$\begin{aligned} A(\{0\},\emptyset)&=\{Y/Y\in K_Z, Y\cap\{0\}=\emptyset\}\\ A(\{0\},\{0\})&=\{Y/Y\in K_Z, Y\cap\{0\}=\{0\}\}\,. \end{aligned} \tag{1.3}$$

Stationarity of the process means that $f_\Lambda(X)=f_{\Lambda+s}(X+s)$ where $X+s=(x_1+s, x_2+s,\ldots)$ if $X=(x_1,x_2,\ldots)$ and $s\in Z$.

If $\mu$ is a process we can define the conditional probabilities $f_\Lambda(X/Y)$, for $X\subset\Lambda$, $\Lambda$ finite, $Y\subset Z/\Lambda$, as the conditional probability for finding

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in $K_Z$ an element Q such that $Q \cap \Lambda = X$, knowing that $Q \cap (Z/\Lambda) = Y$: these conditional probabilities are defined for all $X \subset \Lambda$ and $\mu_\Lambda$-almost everywhere in $Y \in Z/\Lambda$ [1].

We shall consider only processes such that $f_\Lambda(X/Y) > 0$ $\mu_\Lambda$-almost everywhere.

A Markov field with memory r will be a process of the above type such that:

$$f_\Lambda(X/Y) = f_\Lambda(X/Y \cap \partial_r \Lambda) \tag{1.4}$$

where $\partial_r \Lambda = \{s/s \in Z, s \notin \Lambda, \text{distance of } s \text{ from } \partial\Lambda \leq r\}$. Notice that the requirement $f_\Lambda(X/Y) > 0$ implies that the process is a mixing Markov chain.

It can be shown that a Markov field with memory r is uniquely determined by its conditional distributions [2] (if, as everywhere in this paper, the time is 1-dimensional).

Furthermore, the $f_\Lambda(X/Y)$ can be uniquely written [3], in terms of a shift invariant function $\Phi$, defined on the non empty subsets of Z, such that $\Phi(S) = 0$ if $\operatorname{diam} S > r$, as

$$f_\Lambda(X/Y) = \frac{\exp \sum\limits_{\substack{S \subset X \\ \emptyset \neq T \subset Y}} \Phi(S \cup T)}{(\text{normalization})} \tag{1.5}$$

$\mu_\Lambda$-almost everywhere.

The r.h.s. of (1.5) makes sense for more general $\Phi$'s: e.g. if $\Phi$ is a shift invariant function such that:

$$\|\Phi\| = \sum_{S \ni \{0\}} |\Phi(S)| < +\infty . \tag{1.6}$$

A stationary process with conditional probabilities given by the r.h.s. of (1.5) with $\Phi$ verifying (1.6) is called a Gibbs process with potential $\Phi$ [2].

The above mentioned results mean that a memory r Markov field is the same thing as a "finite range" Gibbs' process.

In view of the well known theorem showing that Markov chains are isomorphic to Bernoulli schemes [5], it is natural to ask wether a general Gibbs' process also shares this property.

In this paper we show that a Gibbs process with a potential such that:

$$\|\Phi\|_1 = \sum_{S \supset \{0\}} |\Phi(S)| (\operatorname{diam} S) < +\infty \tag{1.7}$$

is isomorphic to a Bernoulli scheme [6]. It is known that, if (1.7) is verified, there is one and only one Gibbs' process with conditional probabilities given by (1.6), [4], and, furthermore, such a system is K-system [4].

We shall actually prove that the process is a weak-Bernoulli shift and then apply the Friedman-Ornstein isomorphism theorem [5]. The weak Bernoulli character of our processes will be proved by showing that the partitions $\bigvee_{i\in\Lambda_1} T^i P$ and $\bigvee_{i\in(\Lambda_2+n)} T^i P$ with $\Lambda_1=(1,2,\ldots,m_1)$ and $\Lambda_2=(1,2,\ldots,m_2)$ are $\varepsilon$-independent for all $m_1, m_2$ provided $n-m_1$ is large enough. In terms of the notations (1.2), (1.3) and the remarks in between, this means that there is $N_\varepsilon>0$ such that:

$$\sum_{\substack{X_1\subset\Lambda_1\\ X_2\subset(\Lambda_2+n)}} |f_{\Lambda_1\cup(\Lambda_2+n)}(X_1\cup X_2)-f_{\Lambda_1}(X_1)\,f_{\Lambda_2+n}(X_2)|<\varepsilon \tag{1.8}$$

for $n-m_1\geqq N_\varepsilon$.

The proof is a generalization of the original proof of Friedman and Ornstein for Markov shifts and is technically based on the results of Ruelle [4]. In the next section we give a brief survey of Ruelle's theorem. In Section 3 we use the theorem of Section 2 to write (1.8) in a different form. Section 4 contains the technical part of the paper. In Section 5 we discuss some open problems.

## § 2. Ruelle's Theorem

Let $\Phi$ be a potential verifying (1.7) which is fixed once for all.

Let $Z_+$ be the positive integers and let $K_{Z_+}=\prod_{i\in Z_+} I$: as in Section 1 the space $K_{Z_+}$ will be regarded as the family of subsets of $Z_+$ and considered with the product topology.

Let $\mathscr{C}(K_{Z_+})$ be the set of the continuous functions on $K_{Z_+}$ and let $U(X/\tau_1 Y)=\sum_{\substack{S\subset X\\ \emptyset\neq T\subset Y}} \Phi(S\cup\tau_1 T)$ where $\tau_s T$ is a short hand notation for $T+s$. Define on $\mathscr{C}(K_{Z_+})$ the transfer operator:

$$\mathscr{L}f(Y)=\sum_{X\subset\{1\}} e^{-U(X|\tau_1 Y)} f(X\cup\tau_1 Y) \tag{2.1}$$

(where X can obviously be either $\emptyset$ or $\{1\}$).

If $\mu$ denotes the Gibbs process associated with $\Phi$ we have:

**Theorem 1.** *There exist, and are unique, a number* $\lambda>0$, *a function* $h\in\mathscr{C}(K_{Z_+})$ *and a measure* $v$ *on* $K_{Z_+}$ *such that:*

i) $\|\lambda^{-n}\mathscr{L}^n\|\leqq C$ *where* C *is a suitable* n*-independent constant.*

ii) $\mathscr{L}h=\lambda h, \quad \mathscr{L}^* v=\lambda v, \quad v(h)\equiv\int h(Y)\,v(dY)=1.$

*Furthermore:*

iii) *if* $\Lambda\subset Z_+$ *and* $\mathscr{C}_\Lambda$ *denotes the subspace of* $\mathscr{C}(K_{Z_+})$ *consisting in the* $f\in\mathscr{C}(K_{Z_+})$ *such that* $f(Y)=f(Y\cap\Lambda), \forall\, Y\subset Z_+$, *then if* $\Lambda=(1,2,\ldots,m)$:

$$v(|\lambda^{-n}\mathscr{L}^n f|)\leqq(1-e^{-2\|\Phi\|_1})\,v(|f|)$$

*for all* $f\in\mathscr{C}_\Lambda$ *such that* $v(f)=\int f(Y)\,v(dY)=0$ *and for all* $n\geqq m$.

iv) $C^{-1} \leqq \dfrac{h(X)}{h(Y)} \leqq C \quad \forall X, Y \in K_{Z_+}$.

v) $\lim_{n\to\infty} \|\lambda^{-1}\mathscr{L}^n 1 - h\|_\infty = 0$, *where* $1$ *denotes the function which is identically one.*

vi) *if* $f \in \mathscr{C}(K_Z)$ *and* $f(Y) = f(Y \cap Z_+)$, $\forall Y \in K_Z$, *then (with an obvious meaning for the symbols):*

$$\mu(f) \equiv \int_{K_Z} f(Y)\,\mu(dY) = \int_{K_{Z_+}} f(Y)\,h(Y)\,\nu(dY) \equiv \nu(fh). \tag{2.2}$$

An immediate consequence of vi) and of definition (1.2) is the following: if $\chi_{\Lambda,X}$ is the characteristic function of the atom $A(\Lambda, X)$ (i.e. if $\chi_{\Lambda,X}(Y) = 1$ if $Y \cap \Lambda = X$ and $\chi_{\Lambda,X}(Y) = 0$ otherwise) and if $\Lambda \subset Z_+$ we can use vi) to write $f_\Lambda(X)$ as:

$$f_\Lambda(X) = \int_{K_Z} \chi_{\Lambda,X}(Y)\,\mu(dY) = \int_{K_{Z_+}} \chi_{\Lambda,X}(Y)\,h(Y)\,\nu(dY) \tag{2.3}$$

a formula that will be useful later.

## § 3. A Restatement of (1.8)

Assume that $\Lambda_1 = (1, 2, \ldots, m_1)$ and $\Lambda_2 = (1, 2, \ldots, m_2)$: using (2.3) we find:

$$f_{\Lambda_1 \cup (\Lambda_2 + n)}(X_1 \cup X_2) = \int \chi_{\Lambda_1, X_1}(Y)\,\chi_{(\Lambda_2+n), X_2}(Y)\,h(Y)\,\nu(dY) \quad \forall X_i \subset \Lambda_i. \tag{3.1}$$

This formula can be written in a more convenient form in terms of the transfer operator. Notice first that the definition (2.1) implies:

$$(\mathscr{L}^n f)(Y) = \sum_{Q \subset (1,2,\ldots,n)} e^{-U(Q/\tau_n Y)} f(Q \cup \tau_n Y) \qquad Y \in K_{Z_+} \tag{3.2}$$

and similarly:

$$\int_{K_{Z_+}} f(Y)\,(\mathscr{L}^{*n}\varrho)(dY) = \sum_{Q \subset (1,2,\ldots,n)} \int f(Q \cup \tau_n Y)\, e^{-U(Q/\tau_n Y)}\varrho(dY) \tag{3.2}$$

for all measures $\varrho \in \mathscr{C}(K_{Z_+})$.

Therefore, using (3.2) and $\mathscr{L}^*\nu = \lambda\nu$, it is easy to see that:

$$\chi_{(\Lambda_2+n), X_2}(Y) \cdot \nu(dY) = [\lambda^{-n}\mathscr{L}^{*n}(\chi_{\Lambda_2, (X_2-n)} \cdot \nu)]\,(dY) \tag{3.3}$$

hence:

$$\begin{aligned} f_{\Lambda_1 \cup (\Lambda_2+n)}(X_1 \cup X_2) &= \int \chi_{\Lambda_1, X_1}\,\chi_{(\Lambda_2+n), X_2}\,h\,d\nu \\ &= \int \chi_{\Lambda_1, X_1}(Y)\,h(Y)\,(\lambda^{-n}\mathscr{L}^{*n}\chi_{\Lambda_2, X_2-n} \cdot \nu)\,(dY) \\ &= \int \chi_{\Lambda_2, X_2-n}(Y)\,(\lambda^{-n}\mathscr{L}^n(\chi_{\Lambda_1, X_1} \cdot h))\,(Y)\,\nu(dY). \end{aligned} \tag{3.4}$$

Combining (3.4) with (2.3) and using $\lambda^{-1}\mathscr{L}h=h$:

$$\sum_{X_2\subset\Lambda_2+n} |f_{\Lambda_1\cup(\Lambda_2+n)}(X_1\cup X_2)-f_{\Lambda_1}(X_1)\,f_{\Lambda_2+n}(X_2)|$$
$$\leqq \sum_{X_2\subset\Lambda_2+n} \int \chi_{\Lambda_2,(X_2-n)}(Y)\,|[\lambda^{-n}\mathscr{L}^n(\chi_{\Lambda_1,X_1}h-f_{\Lambda_1}(X_1)\,h)]\,(Y)|\,\nu(dY)$$
$$=\int|\lambda^{-n}\mathscr{L}^n(\chi_{\Lambda_1,X_2}-f_{\Lambda_1}(X_1))\,h|\,d\nu\equiv\int|\lambda^{-n}\mathscr{L}^n F|\,d\nu\,. \tag{3.5}$$

Where $F(Y)=(\chi_{\Lambda_1,X_1}(Y)-f_{\Lambda_1}(X_1))\,h(Y)$.

Our main result will follow from the inequality:

$$\int|\lambda^{-n-m_1}\mathscr{L}^{n+m_1}F|\,d\nu\leqq f_{\Lambda_1}(X_1)\,\varepsilon(n) \tag{3.6}$$

where $\varepsilon(n)\xrightarrow[n\to\infty]{}0$. Formula (3.6) is proven in the next section.

## § 4. Proof of (3.6)

The proof is based on the following estimates:

**Lemma 1.** *The function* h *has the property:*

$$|h(Y)-h(Y\cap(1,2,\ldots,n))|\leqq e^{\|\Phi\|_1}h(Y)\,\eta(n) \tag{4.1}$$

*where, here and below,* $\eta(n)=\sum_{\substack{S\supset\{0\}\\ \operatorname{diam}S\geqq n}}(\operatorname{diam}S)\,|\Phi(S)|$.

*Proof.* Part v) of Theorem 1 allows to write, setting $Y_n=Y\cap(1,2,\ldots,n)$

$$|h(Y)-h(Y_n)|=\lim_{s\to\infty}|(\lambda^{-s}\mathscr{L}^s 1)(Y)-(\lambda^{-s}\mathscr{L}^s 1)(Y_n)|$$
$$\leqq\lim_{s\to\infty}\lambda^{-s}\sum_{Q\subset(1,\ldots,s)}|e^{-U(Q/\tau_sY)}-e^{-U(Q/\tau_sY_n)}|$$
$$\leqq\lim_{s\to\infty}\lambda^{-s}\sum_{Q\subset(1,\ldots,s)}e^{-U(Q/\tau_sY)}\left|1-e^{-\sum_{\substack{\emptyset\neq S\subset Q\\ \emptyset\neq T\subset Y/Y_n}}\Phi(S\cup\tau_sT)}\right| \tag{4.2}$$
$$\leqq\lim_{s\to\infty}\lambda^{-s}\sum_{Q\subset(1,\ldots,s)}e^{-U(Q/\tau_sY)}\,e^{\|\Phi\|_1}\,\eta(n)$$
$$=h(Y)\,e^{\|\Phi\|_1}\,\eta(n)$$

and the lemma is proved.

**Lemma 2.** *Let* $n_0>0$. *For all* $n\geqq n_0$ *there is a function* $F_n\in\mathscr{C}_{(1,2,\ldots,n_0)}$ *(see* iii) *Theorem 1 for the symbol* $\mathscr{C}_{(1,2,\ldots,n_0)}$*) such that:*

a) $\nu(F_n)=0$. (4.3)

b) *There is a* $C'>0$ *such that, if* $\delta_n=(\lambda^{-n-m_1}\mathscr{L}^{n+m_1}F)-F_n$, *then:*

$$\nu(|\delta_n|)\leqq C'f_{\Lambda_1}(X_1)\,\eta(n_0)\,. \tag{4.4}$$

*Proof.* Notice first that the definition of F [after (3.5)] together with ii) in Theorem 1 implies $\nu(F)=0$.

Put $Y_{n_0} = Y \cap (1, 2, \ldots, n_0)$, then adding and substracting appropriate terms we can write the following straightforward sequence of equalities:

$$
\begin{aligned}
&((\lambda^{-1}\mathscr{L})^{n+m_1} F)(Y) \\
&= \lambda^{-(n+m_1)} \sum_{Q \subset (1,2,\ldots,n+m_1)} e^{-U(Q/\tau_{n+m_1} Y)} (\chi_{A_1, X_1}(Q) - f_A(X_1)) h(Q \cup \tau_{n+m_1} Y) \\
&= \lambda^{-(n+m_1)} \sum_{Q \subset (1,2,\ldots,n+m_1)} e^{-U(Q/\tau_{n+m_1} Y_{n_0})} (\chi_{A_1, X_1}(Q) - f_{A_1}(X_1)) \\
&\quad \cdot h(Q \cup \tau_{n+m_1} Y_{n_0}) \\
&\quad + \lambda^{-(n+m_1)} \sum_{Q \subset (1,\ldots,n+m_1)} e^{-U(Q/\tau_{n+m_1} Y)} (\chi_{A_1, X_1}(Q) - f_{A_1}(X_1)) \\
&\quad \cdot (h(Q \cup \tau_{n+m_1} Y) - h(Q \cup \tau_{n+m_1} Y_{n_0})) \\
&\quad + \lambda^{-(n+m_1)} \sum_{Q \subset (1,\ldots,n+m_1)} (e^{-U(Q/\tau_{n+m_1} Y)} - e^{-U(Q/\tau_{n+m_1} Y_{n_0})}) \\
&\quad \cdot (\chi_{A_1, X_1}(Q) - f_{A_1}(X_1))\, h(Q \cup \tau_{n+m_1} Y_{n_0}).
\end{aligned}
$$

The first sum in the r.h.s. will be called $\tilde{F}_n$ and the sum of the other two terms will be $\tilde{\delta}_n$.

Using Lemma 1 to bound the second sum and iv) Theorem 1 together with the inequality [used in (4.2)]:

$$|e^{-U(Q/\tau_{n+m_1} Y)} - e^{-U(Q/\tau_{n+m_1} Y_{n_0})}| \leqq e^{-U(Q/\tau_{n+m_1} Y)}\, e^{\|\Phi\|_1}\, \eta(n_0),$$

to bound the third sum, one finds:

$$
\begin{aligned}
|\tilde{\delta}_n(Y)| \leqq e^{\|\Phi\|_1}(1 + C)\, \eta(n_0)\, \lambda^{-(n+m_1)} \sum_{Q \subset (1,\ldots,n+m_1)} e^{-U(Q/\tau_{n+m_1} Y)} \\
\cdot |\chi_{A_1, X_1}(Q) - f_{A_1}(X_1)|\, h(Q \cup \tau_{n+m_1} Y)
\end{aligned}
$$

hence, using $\lambda^{-1}\mathscr{L}^* \nu = \nu$ and (2.3), we have:

$$\int \nu(dY)\, |\tilde{\delta}_n(Y)| \leqq 2(1 + C)\, e^{\|\Phi\|_1}\, \eta(n_0)\, f_{A_1}(X_1)$$

therefore, since, as remarked at the beginning of the proof, $\nu(\tilde{F}_n + \tilde{\delta}_n) = \int (\tilde{F}_n + \tilde{\delta}_n)\, d\nu = 0$ we can modify $\tilde{\delta}_n$ into $\delta_n = \tilde{\delta}_n - \nu(\tilde{\delta}_n)$ and, correspondingly $\tilde{F}_n$ into $F_n = \tilde{F}_n + \nu(\tilde{\delta}_n)$ and obtain a couple $F_n, \delta_n$ verifying the lemma.

We can now prove the main theorem:

**Theorem 2.** *Given* $\varepsilon > 0$, $\exists\, n_\varepsilon$ *such that*

$$\int |(\lambda^{-1}\mathscr{L})^{n+m_1} F|\, d\nu < \varepsilon f_{A_1}(X_1) \qquad n \geqq n_\varepsilon \tag{4.5}$$

*and* $n_\varepsilon$ *is* $m_1$*-independent.*

*Proof.* Let $n(N)$ be such that [see (4.4)]: $\eta(n(N))\, C' < 1/N$.

Let k be an arbitrary integer and let:

$$n > 2 \sum_{i=1}^{k} n(N + i). \tag{4.6}$$

Then $n - n(N+k) > n(N+k)$; therefore Lemma 2 applies to the function $(\lambda^{-1}\mathscr{L})^{n-n(N+k)+m_1}F$ and gives a function $F' \in \mathscr{C}_{(1,2,\ldots,n(N+k))}$ and a rest $\delta'$ such that (4.4) holds. Then apply iii) Theorem 1 to the function $(\lambda^{-1}\mathscr{L})^{n(N+k)}$ and obtain:

$$\nu(|\lambda^{-n-m_1}\mathscr{L}^{n+m_1}F|) \equiv \nu(|\lambda^{-n(N+k)}\mathscr{L}^{n(N+k)}\cdot\lambda^{-(n-n(N+k)+m_1)}\mathscr{L}^{n-n(N+k)+m_1}F|)$$
$$\leqq (1-e^{-2\|\Phi\|_1})\,\nu(|F'|) + \|\lambda^{-n(N+k)}\mathscr{L}^{n(N+k)}\|\,\nu(|\delta'|)$$
$$\leqq (1-e^{-2\|\Phi\|_1})\,\nu(|\lambda^{-(n-n(N+k)+m_1)}\mathscr{L}^{n-n(N+k)+m_1}F|)$$
$$+\left((1-e^{-2\|\Phi\|_1}) + \|\lambda^{-n(N+k)}\mathscr{L}^{n(N+k)}\|\right)\nu(|\delta'|)$$

and, using i) Theorem 1 and b) Lemma 2:

$$\nu(|\lambda^{-n-m_1}\mathscr{L}^{n-m_1}F|) \leqq \frac{C'(C+1-e^{-2\|\Phi\|_1})}{N+k} f_{\Lambda_1}(X_1)$$
$$+(1-e^{-2\|\Phi\|_1})\,\nu(|\lambda^{n-n(N+k)+m_1}\mathscr{L}^{n-n(N+k)+m_1}F|)$$

and, iterating the procedure we find (for a suitable $C'' > 0$):

$$\nu(|\lambda^{-n-m_1}\mathscr{L}^{n+m_1}F|) \leqq C'' f_{\Lambda_1}(X_1)\sum_{i=0}^{k-1}\frac{(1-e^{-2\|\Phi\|_1})^i}{N+k-i}$$
$$+(1-e^{-2\|\Phi\|_1})^k\, C\,\nu(|F|)$$
$$\leqq \left(\frac{C''e^{2\|\Phi\|_1}}{N} + 2(1-e^{-2\|\Phi\|_1})^k\right) f_{\Lambda_1}(X_1)$$

and our theorem follows from the arbitrariness of N and k.

## § 5. Conclusion

The reader, familiar with Ref. [4], will have noticed that, technically, the results follow from the improvement of part of the proof of Proposition 5 in [4] which is given in our Lemma 2 via Lemma 1 which is adapted from [7] (Lemma 1).

From a conceptual point of view the proof is nothing else than a generalization of the original method of Friedman and Ornstein for proving the Bernoullicity of Markov processes; the key to this generalization is furnished by Ruelle's theory of the transfer matrix and its use for proving that Gibbs fields are K-systems if their potential verifies (1.7).

If (1.7) is violated the proof of Theorem 1 fails and, actually, one can construct counterexamples to it [8].

Let us consider only the case in which $\Phi$ has only "one" and "two" body components which do not vanish (i.e. assume that $\Phi(S) \equiv 0$ if the number of points in S is larger than 2). Then $\Phi$ is given (since it is a shift invariant function) by a constant $\Phi(s)$ and a function $\Phi(s,t) =$ of the form $\varphi(|s-t|)$. If $\varphi(|s-t|) \leqq 0$, $2\Phi(s) = -\sum_{t \neq s}\varphi(|t-s|)$ and if $\varphi(|s-t|) \sim \dfrac{C}{|s-t|^{1+\varepsilon}}$

as $|s-t| \to \infty$ (so that (1.7) is violated) then it can be shown that the operator $\mathscr{L}$ has a degenerate eigenvalue $\lambda$, [7].

It would be of great interest to know wether the systems just described are Bernoulli schemes. We mention just two possibilities:

1) i) for $2\Phi(s) \neq \sum_{t \neq s} \varphi(|s-t|)$ the systems are Bernoulli schemes.

ii) for $2\Phi(s) = \sum_{t \neq s} \varphi(|s-t|)$ the systems are not always Bernoulli schemes (and many possibilities arise if one varies $\Phi(s)$).

2) the systems are never Bernoulli schemes.

Of course there are other possibilities beyond 1) and 2) above; 1) is most appealing since it would establish a link between the "phase transitions" and the isomorphism of the equilibrium state with a B-system (at least in one dimension). We notice here that under the assumptions of case 1) *i)* above it is known that the Gibbs field is unique and, also, a K-system [9].

A final remark is that the restriction on the space of states I to contain two elements, set at the beginning of this paper, is clearly unnecessary: the same results would be true with any finite I. Things are, however, unclear in the case of more general I's.

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