Commun. math. Phys. 30, 55—67 (1973)
© by Springer-Verlag 1973

On the Purification Map

S. L. Woronowicz
Department of Mathematical Methods of Physics, University of Warsaw, Poland

Received August 7, 1972

Abstract. The investigation of purifications of factor states has been carried on. It is shown, that any factor state ω of a C^*-algebra admits at most one purification $\tilde{\omega}$, so one can introduce the purification map $\phi : \phi(\omega) = \tilde{\omega}$. It turns out, that the Powers and Størmer inequality is valid in this general situation.

0. Introduction

Let $\mathfrak{A}$ be a C^*-algebra and $\mathfrak{A}^\circ$ be an opposite algebra. It means, that $\mathfrak{A}^\circ$ is a C^*-algebra and that an antilinear, multiplicative, $*$-invariant isometry of $\mathfrak{A}$ onto $\mathfrak{A}^\circ$ is given. The image of an element $a \in \mathfrak{A}$ will be denoted by $\bar{a} \in \mathfrak{A}^\circ$. As in [7] we introduce

$$\tilde{\mathfrak{A}} = \mathfrak{A}^\circ \otimes \mathfrak{A}$$

where the tensor product is taken in the sense of the C^*-algebra theory (it includes a suitable completion such that $\tilde{\mathfrak{A}}$ becomes a C^*-algebra). We shall assume, that $\mathfrak{A}$ contains the unity 1 and shall identify any element $a \in \mathfrak{A}$ with $\bar{1} \otimes a \in \tilde{\mathfrak{A}}$. This way $\mathfrak{A}$ becomes a subalgebra of $\tilde{\mathfrak{A}} : \mathfrak{A} \subset \tilde{\mathfrak{A}}$.

In what follows, we shall consider only such states of C^*-algebras, which give rise (by G.N.S.-construction) to representations in separable Hilbert spaces.

Let us recall (see [7]), that a state $\tilde{\omega}$ of $\tilde{\mathfrak{A}}$ is said to be j-positive iff

$$\tilde{\omega}(\bar{a} \otimes a) \geqq 0, \qquad a \in \mathfrak{A} \tag{0.1}$$

Any such state is j-invariant i.e.:

$$\tilde{\omega}(j(\tilde{a})) = \overline{\tilde{\omega}(\tilde{a})} \tag{0.2}$$

for any $\tilde{a} \in \tilde{\mathfrak{A}}$. In the above equation j denotes the antilinear, multiplicative, $*$-invariant, involutive (i.e. $j^2 = \mathrm{id}$) mapping

$$j : \tilde{\mathfrak{A}} \to \tilde{\mathfrak{A}}$$

introduced by the formula

$$j(\bar{a} \otimes b) = \bar{b} \otimes a .$$

Let $\tilde{\omega}$ be a state of $\tilde{\mathfrak{A}}$ and let $\pi_{\tilde{\omega}}$ denotes the representation of $\tilde{\mathfrak{A}}$ induced by $\tilde{\omega}$. We say, that $\tilde{\omega}$ is an exact state, iff $\{\pi_{\tilde{\omega}}(\bar{a} \otimes 1) : a \in \mathfrak{A}\}$ is weakly dense in the von Neumann algebra of all operators commuting with $\pi_{\tilde{\omega}}(\bar{1} \otimes a)$ for all $a \in \mathfrak{A}$.

We proved in [7], that any factor state ω of $\mathfrak{A}$ can be extended to an exact, j-positive pure state of $\tilde{\mathfrak{A}}$. Any such extension is called a purification of ω. Now we shall prove, that any factor state ω admits at most one purification. Therefore we can introduce purification map ϕ, denoting by $\phi(\omega)$ the only purification of a factor state ω. It turns out, that

$$\|\phi(\omega_1) - \phi(\omega_2)\|^2 \leqq 4\|\omega_1 - \omega_2\|$$

for any two normalized factor states ω_1 and ω_2 of $\mathfrak{A}$. This inequality generalizes the finite dimensional case result of Powers and Størmer (see Lemma 3.1 of [4]).

1. The Main Theorem

All the results announced in the introduction are implied by the following theorem.

Theorem 1.1. *Let ω_1 and ω_2 be two normalized (i.e. $\omega_1(1) = 1 = \omega_2(1)$) factor states of a C^*-algebra $\mathfrak{A}$ and let $\tilde{\omega}_1$ and $\tilde{\omega}_2$ denote their purifications. Then*

$$\|\tilde{\omega}_1 - \tilde{\omega}_2\|^2 \leqq 4\|\omega_1 - \omega_2\| . \tag{1.1}$$

Before the proof, we have to analyse the structure of representations of $\tilde{\mathfrak{A}}$ induced by exact, j-positive, pure states. Let $\tilde{\omega}$ be such a state, π be the representation induced by $\tilde{\omega}$, H be the carrier Hilbert space of π. Von Neumann algebras generated by $\{\pi(\bar{1} \otimes a) : a \in \mathfrak{A}\}$ and $\{\pi(\bar{a} \otimes 1) : a \in \mathfrak{A}\}$ will be denoted by $\mathscr{A}$ and $\mathscr{A}'$ respectively. Note, that, due to the assumed exactness of $\tilde{\omega}$, $\mathscr{A}'$ coincides with the commutant of $\mathscr{A}$, so our notation is justified.

$\mathscr{A}$ is a factor. Indeed, since π is an irreducible representation, $\mathscr{A} \cup \mathscr{A}'$ is an irreducible set of operators and $\mathscr{A} \cap \mathscr{A}' = (\mathscr{A} \cup \mathscr{A}')' = (B(H))' = \{\lambda I\}$.

By using (0.2) one can easily prove, that mapping j is implemented by an antiunitary involutive operator, which will be denoted by J. It means, that $J^2 = I$ and

$$J\pi(\tilde{a})\,J = \pi(j(\tilde{a})) \tag{1.2}$$

for any $\tilde{a} \in \tilde{\mathfrak{A}}$. Setting $\tilde{a} = \bar{1} \otimes a$ we have

$$J\pi(\bar{1} \otimes a)\,J = \pi(\bar{a} \otimes 1) .$$

It shows, that

$$J\mathscr{A}J = \mathscr{A}' .$$

Therefore J is an exchange involution (see [7]) for $(\mathscr{A}, \mathscr{A}')$. Let us recall, that a vector $x' \in H$ is said to be $(\mathscr{A}, J)$-positive iff $Jx' = x'$ and

$$(x' \mid AJAx') \geqq 0 \tag{1.3}$$

for any $A \in \mathscr{A}$.

Lemma 1.1. *Let $x \in H$ be such, that the state*

$$\tilde{\omega}(\tilde{a}) = (x \mid \pi(\tilde{a})\, x) \tag{1.4}$$

is j-positive. Then $x = \lambda x'$, where $\lambda \in \mathbb{C}^1$, $|\lambda| = 1$ and x' is $(\mathscr{A}, J)$-positive.

Proof. Remembering, that π is an irreducible representation and taking into account (1.4), (0.2) and (1.2) we get

$$Jx = \mu x$$

where $\mu \in \mathbb{C}^1$ and $|\mu| = 1$. Let λ be a complex number, such that $\lambda^2 = \bar{\mu}$. Evidently $|\lambda| = 1$ and $\bar{\lambda} = \lambda^{-1}$. We put $x' = \bar{\lambda} x$. Then $x = \lambda x'$ and one can easily check, that $Jx' = x'$.

To end the proof, we have to show, that (1.3) is satisfied by any operator $A \in \mathscr{A}$. Since $\{\pi(\bar{1} \otimes a) : a \in \mathfrak{A}\}$ is dense in $\mathscr{A}$ (with respect to the strong operator topology), we may assume, that $A = \pi(\bar{1} \otimes a)$. We have:

$$\begin{aligned}(x' \mid AJAx') &= (x' \mid AJAJx') = (x \mid AJAJx) \\ &= (x \mid \pi(\bar{1} \otimes a)\, J\pi(\bar{1} \otimes a)\, Jx) \\ &= (x \mid \pi(\bar{1} \otimes a)\, \pi(\bar{a} \otimes 1)\, x) \\ &= (x \mid \pi(\bar{a} \otimes a)\, x) = \tilde{\omega}(\bar{a} \otimes a)\end{aligned}$$

and (1.3) is equivalent to (0.1). Q.E.D.

We shall also need the following

Lemma 1.2. *Let π be a representation of a C^*-algebra $\mathfrak{B}$ acting in a Hilbert space H, $\mathscr{B}$ be a von Neumann algebra generated by $\pi(\mathfrak{B})$, f be a weakly continuous linear functional defined on $\mathscr{B}$. Then $f \circ \pi$ is a linear functional defined on $\mathfrak{B}$ and*

$$\|f \circ \pi\| = \|f\|.$$

Proof. The lemma follows immediately from Corollary 1.8.3 of [1] and the Kaplansky density theorem. Q.E.D.

Proof of the Theorem. We shall consider two cases:

I. States ω_1 and ω_2 are not quasiequivalent. Then (see [2]) $\|\omega_1 - \omega_2\| = 2$, whereas $\|\tilde{\omega}_1 - \tilde{\omega}_2\| \leqq 2$ and (1.1) is satisfied.

II. States ω_1 and ω_2 are quasiequivalent. Then (see [7], Theorem 1.2) purifications $\tilde{\omega}_1$ and $\tilde{\omega}_2$ give rise to the same representation of $\tilde{\mathfrak{A}}$. Let π be this representation. Then

$$\omega_1(\tilde{a}) = (x_1 \mid \pi(\tilde{a})\, x_1)$$

and

$$\omega_2(\tilde{a}) = (x_2 \mid \pi(\tilde{a})\, x_2)$$

where x_1 and x_2 are normalized vectors belonging to H (we use the notation introduced before, in particular H is the carrier Hilbert space of π).

Setting in Lemma 1.2 $\mathfrak{B} = \tilde{\mathfrak{A}}$, $f(A) = (x_1 \mid A x_1) - (x_2 \mid A x_2)$ and remembering that π is an irreducible representation of $\tilde{\mathfrak{A}}$ (so $\mathscr{B} = B(H)$ coincides with the algebra of all bounded operators) we get

$$\|\tilde{\omega}_1 - \tilde{\omega}_2\| = \sup_{\substack{A \in B(H) \\ \|A\| \leqq 1}} |(x_1 \mid A x_1) - (x_2 \mid A x_2)|\,.$$

The right hand side of the above formula is equal to the trace norm of operator $|x_1)(x_1| - |x_2)(x_2|$ and can be easily evaluated. We obtain

$$\|\tilde{\omega}_1 - \tilde{\omega}_2\| = 2\sqrt{1 - |(x_1 \mid x_2)|^2}\,. \tag{1.5}$$

We know, that ω_1 and ω_2 are restrictions of $\tilde{\omega}_1$ and $\tilde{\omega}_2$ to subalgebra $\mathfrak{A} \subset \tilde{\mathfrak{A}}$. Therefore setting in Lemma 1.2 $\mathfrak{B} = \mathfrak{A}$ and $f(A) = (x_1 \mid A x_1) - (x_2 \mid A x_2)$ we get $\mathscr{B} = \mathscr{A}$ and

$$\|\omega_1 - \omega_2\| = \sup_{\substack{A \in \mathscr{A} \\ \|A\| \leqq 1}} |(x_1 \mid A x_1) - (x_2 \mid A x_2)|.$$

According to Lemma 1.1 vectors x_1 and x_2 are $(\mathscr{A}, J)$-positive modulo complex factor of modulus 1. We shall prove in Section 3, that for such vectors, the expression on the right hand side of the above equation is larger than $2(1 - |(x_1 \mid x_2)|)$. Therefore

$$\|\omega_1 - \omega_2\| \geqq 2(1 - |(x_1 \mid x_2)|)\,. \tag{1.6}$$

Taking into account (1.5) and (1.6) we have:

$$\begin{aligned} \|\tilde{\omega}_1 - \tilde{\omega}_2\|^2 &= 4(1 - |(x_1 \mid x_2)|^2) = 4(1 + |(x_1 \mid x_2)|)\,(1 - |(x_1 \mid x_2)|) \\ &\leqq 4 \cdot 2(1 - |(x_1 \mid x_2)|) \leqq 4\, \|\omega_1 - \omega_2\| \qquad \text{Q.E.D.} \end{aligned}$$

2. The Modular Operator

Let $\mathscr{A}$ be a factor, J be a positive exchange involution for $(\mathscr{A}, \mathscr{A}')$ and y be a $(\mathscr{A}, J)$-positive separating and cyclic vector. For reader's convenience we recall the basic facts concerning the so called modular operator Δ assigned to y (see [6] and [7]). This is a selfadjoint, positive, invertibe (i.e. 0 is not an eigenvalue of Δ) operator such that:

1° $\{By : B \in \mathscr{A}\}$ is a core of $\Delta^{\frac{1}{2}}$ and

$$\Delta^{\frac{1}{2}} By = JB^* y, \qquad B \in \mathscr{A}.$$

It is known, that

2° $\{B'y : B' \in \mathscr{A}'\}$ is a core of $\Delta^{-\frac{1}{2}}$ and

$$\Delta^{-\frac{1}{2}} B'y = JB'^* y \qquad B' \in \mathscr{A}'.$$

3° $J\Delta^s J = \Delta^{-s}$ for any $s \in \mathbb{R}^1$.

4° Δ defines an one-parameter group of automorphisms of $\mathscr{A}$:

$$\Delta^{it} \mathscr{A} \Delta^{-it} = \mathscr{A}.$$

We shall frequently use these properties of the modular operator without any special reference.

Let $A \in \mathscr{A}$. It turns out, that $\{By : B \in \mathscr{A}\}$ is a core of $\Delta^{\frac{1}{2}} A^*$. Now we are going to prove this statement, which will play an important role in the next section. We start with the following lemma, which is a slightly improved version of a Schwartz lemma (see [5], Chapter II, Section 2, Lemma 4).

Lemma 2.1. *Let y be a cyclic and separating vector of a von Neumann algebra $\mathscr{A} \subset B(H)$ (the latter denotes the algebra of all bounded operators acting in a Hilbert space H) and $x \in H$. Then there exist a bounded operator C and a selfadjoint operator K such that:*

1° $C \in \mathscr{A}$, K is affiliated to $\mathscr{A}$.

2° $y \in D(K)$ and $x = CKy$.

3° $\{A'y : A' \in \mathscr{A}'\}$ is a core of K.

Let us recall, that a selfadjoint operator K is said to be affiliated to $\mathscr{A}$ iff

$$W'KW'^{-1} = K \tag{2.1}$$

for any unitary operator $W' \in \mathscr{A}'$. If this is the case, then all bounded functions of K belong to $\mathscr{A}$.

Proof. Since y is a cyclic vector one can find operators $A_n \in \mathscr{A}$ such that

$$\|A_n y - x\| \leqq \frac{1}{n^2}.$$

For any $z \in H$ we put

$$|||z|||^2 = \|z\|^2 + \sum_{n=1}^{\infty} n^2 \|(A_{n+1} - A_n) z\|^2.$$

Let $D = \{z \in H : |||z||| < \infty\}$. It is seen, that D endowed with norm $|||\cdot|||$ is a Hilbert space and that $\{A'y : A' \in \mathscr{A}'\} \subset D$. By D_0 we shall denote the

$|||\cdot|||$-closure of $\{A'y: A' \in \mathscr{A}'\}$. Evidently D_0 is dense in H. Therefore (see for example [3], Chapter VI, Theorem 2.33, p. 331) there exists a positive selfadjoint operator K such that

$$D(K) = D_0, \tag{2.2}$$

$$\|Kz\|^2 = |||z|||^2, \quad z \in D_0. \tag{2.3}$$

It is seen, that the graph topology of D_0 induced by K coincides with the topology given by norm $|||\cdot|||$. Therefore $\{A'y: A' \in \mathscr{A}'\}$ is a core of K.

We have to show, that K is affiliated to $\mathscr{A}$. To this end, we note, that K is the only positive selfadjoint operator satisfying (2.2) and (2.3) and that D_0 and $|||\cdot|||$ are invariant under all unitary operators $W' \in \mathscr{A}'$. Therefore (2.1) follows.

K^{-1} is a bounded operator (indeed $\|Kz\| = |||z||| \geqq \|z\|$) and evidently $K^{-1} \in \mathscr{A}$. For any $u \in H$ we have $K^{-1}u \in D_0$ and

$$\sum_{n=0}^{\infty} \|(A_{n+1} - A_n) K^{-1} u\| \leqq \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} n^2 \|(A_{n+1} - A_n) K^{-1} u\|^2\right)^{\frac{1}{2}}$$
$$\leqq \frac{\pi}{\sqrt{6}} |||K^{-1}u||| = \frac{\pi}{\sqrt{6}} \|u\|$$

It shows, that sequence $A_n K^{-1}$ is strongly convergent. Let $C = s\text{-}\lim A_n K^{-1}$. Then $C \in \mathscr{A}$ and

$$x = \lim_{n\to\infty} A_n y = \lim_{n\to\infty} A_n K^{-1} K y = CKy \qquad \text{Q.E.D.}$$

Now we can prove the main result of this section.

Lemma 2.2. *Let $\mathscr{A}$ be a factor, J be an exchange involution for $(\mathscr{A}, \mathscr{A}')$ and Δ be the modular operator assigned to a separating and cyclic $(\mathscr{A}, J)$-positive vector y. Then for any $A \in \mathscr{A}: (By: B \in \mathscr{A}\}$ is a core of $\Delta^{\frac{1}{2}} A^*$.*

Proof. Let $x \in D(\Delta^{\frac{1}{2}} A^*)$. We have to find a sequence $(B_n)_{n=1,2,\ldots}$ of elements of $\mathscr{A}$ such that

$$x = \lim B_n y, \tag{2.4}$$

$$\Delta^{\frac{1}{2}} A^* x = \lim \Delta^{\frac{1}{2}} A^* B_n y. \tag{2.5}$$

For any $B' \in \mathscr{A}'$ we have:

$$(B'y \mid J\Delta^{\frac{1}{2}} A^* x) = (A^* x \mid \Delta^{\frac{1}{2}} J B' y)$$
$$= (A^* x \mid B'^* y) = (B' x \mid Ay).$$

We may assume, that $x = CKy$, where C and K are such as in Lemma 2.1. Then

$$(B'y \mid J\Delta^{\frac{1}{2}} A^* x) = (B'CKy \mid Ay) = (KB'y \mid C^* Ay).$$

Remembering, that $\{B'y : B' \in \mathscr{A}'\}$ is a core of selfadjoint operator K, we get $C^*Ay \in D(K)$ and

$$KC^*Ay = J\Delta^{\frac{1}{2}}A^*x. \tag{2.6}$$

Let $K_n = f_n(K)$, where $f_n(\lambda) = \min\{\lambda, n\}$. Then $K_n \in \mathscr{A}$ and for any $u \in D(K)$ we have

$$Ku = \lim K_n u.$$

We shall prove, that (2.4) and (2.5) are satisfied by sequence $B_n = CK_n$. Indeed

$$x = CKy = \lim CK_n y$$

and taking into account (2.6)

$$\begin{aligned}\Delta^{\frac{1}{2}}A^*x &= JKC^*Ay = \lim JK_nC^*Ay \\ &= \lim \Delta^{\frac{1}{2}}(K_nC^*A)^*\, y = \lim \Delta^{\frac{1}{2}}A^*CK_n y \qquad \text{Q.E.D.}\end{aligned}$$

3. $(\mathscr{A}, J)$-Positive Vectors

In this section we investigate properties of $(\mathscr{A}, J)$-positive vectors. We shall find, on what terms a vector of the form Ay (where $A \in \mathscr{A}$ and y is a separating and cyclic $(\mathscr{A}, J)$-positive vector) is J-invariant (Lemma 3.1) and is $(\mathscr{A}, J)$-positive (Lemma 3.3).

Any vector defines a state of $\mathscr{A}$. The main problem solved in the section (see Lemma 3.5 and Theorem 3.1) is following: Can the difference (or the sum) of two $(\mathscr{A}, J)$-positive vectors be estimated in terms of the difference of the corresponding states of $\mathscr{A}$. We shall find an interesting estimate. We saw in Section 1, how this estimate entered the proof of the Theorem 1.1.

In the following lemmas $\mathscr{A}$ is a factor, J is an exchange involution for $(\mathscr{A}, \mathscr{A}')$, y is a separating and cyclic $(\mathscr{A}, J)$-positive vector and Δ is the modular operator assigned to y.

Lemma 3.1. *Let $A \in \mathscr{A}$. Then the following conditions are equivalent:*
(1) *Vector Ay is J-invariant i.e. $JAy = Ay$.*
(2) *Operator $A\Delta^{\frac{1}{2}}$ is symmetric.*
(3) *Operator $A\Delta^{\frac{1}{2}}$ is ess selfadjoint.*

Proof. (1)⇒(3). For any $B \in \mathscr{A}$ we have

$$A\Delta^{\frac{1}{2}}By = AJB^*y = AJB^*Jy = JB^*JAy.$$

Assume, that Ay is J-invariant. Then

$$A\Delta^{\frac{1}{2}}By = JB^*Ay = \Delta^{\frac{1}{2}}A^*By.$$

Remembering, that $\{By : B \in \mathscr{A}\}$ is a core of $\Delta^{\frac{1}{2}}$ and $\Delta^{\frac{1}{2}} A^*$ (Lemma 2.2) and that $\Delta^{\frac{1}{2}} A^*$ is a closed operator, we have:

$$\overline{A\Delta^{\frac{1}{2}}} = \Delta^{\frac{1}{2}} A^* = (A\Delta^{\frac{1}{2}})^* .$$

(3) ⇒ (2). It is obvious.

(2) ⇒ (1). Assume, that $A\Delta^{\frac{1}{2}}$ is a symmetric operator. Then $A\Delta^{\frac{1}{2}} \subset (A\Delta^{\frac{1}{2}})^* = \Delta^{\frac{1}{2}} A^*$ and

$$JAy = JA\Delta^{\frac{1}{2}} y = J\Delta^{\frac{1}{2}} A^* y = Ay \qquad \text{Q.E.D.}$$

Lemma 3.2. *Let $A \in \mathscr{A}$ be such that $A\Delta^{\frac{1}{2}}$ is a symmetric operator. Then there exists a selfadjoint operator $\tilde{A} \in \mathscr{A}$ such that*

$$A\Delta^{\frac{1}{4}} \subset \Delta^{\frac{1}{4}} \tilde{A} . \tag{3.1}$$

Proof. For any $u \in D(\Delta^{-\frac{1}{2}})$ and $v \in D(\Delta^{\frac{1}{2}})$ we have:

$$|(u|Av)| \leqq \|A\| \, \|u\| \, \|v\|$$

$$|(\Delta^{-\frac{1}{2}} u| \, A\Delta^{\frac{1}{2}} v)| = |(\Delta^{-\frac{1}{2}} u | \Delta^{\frac{1}{2}} A^* v)|$$

$$= |(u|A^* v)| \leqq \|A\| \, \|u\| \, \|v\|$$

Setting $\Delta^{i\lambda} u$ and $\Delta^{i\lambda} v$ instead of u and v respectively, we get

$$|(\Delta^{i\lambda} u | A\Delta^{i\lambda} v)| \leqq \|A\| \, \|u\| \, \|v\| ,$$

$$|(\Delta^{-\frac{1}{2}+i\lambda} u | A\Delta^{\frac{1}{2}+i\lambda} v)| \leqq \|A\| \, \|u\| \, \|v\| .$$

The maximum principle of the holomorphic function theory shows immediately, that

$$|(\Delta^{-\bar{s}} u | A\Delta^{s} v)| \leqq \|A\| \, \|u\| \, \|v\|$$

for all complex s such that $0 \leqq \mathrm{Re}\, s \leqq \frac{1}{2}$. It means that for any such s, there exists a bounded operator A_s such that

$$(\Delta^{-\bar{s}} u | A\Delta^{s} v) = (u | A_s v) \tag{3.2}$$

for all $u \in D(\Delta^{-\frac{1}{2}})$ and $v \in D(\Delta^{\frac{1}{2}})$. In fact (3.2) holds for all vectors $u \in D(\Delta^{-\bar{s}})$ and $v \in D(\Delta^{s})$, since $D(\Delta^{-\frac{1}{2}})$ and $D(\Delta^{\frac{1}{2}})$ are cores of $\Delta^{-\bar{s}}$ and Δ^{s} respectively. Evidently $s \to A_s$ is a weakly holomorphic mapping and for $s = i\lambda$ (where $\lambda \in \mathbb{R}^1$) we have: $A_{i\lambda} = \Delta^{-i\lambda} A \Delta^{i\lambda} \in \mathscr{A}$. Therefore $A_s \in \mathscr{A}$ for all s. We put

$$\tilde{A} = A_{\frac{1}{4}} .$$

Let $v \in D(\Delta^{\frac{1}{4}})$. Setting $s = \frac{1}{4}$ in (3.2) we have:

$$(\Delta^{-\frac{1}{4}} u | A\Delta^{\frac{1}{4}} v) = (u | \tilde{A} v) \tag{3.3}$$

for all $u \in D(\Delta^{-\frac{1}{4}})$. It means, that $A\Delta^{\frac{1}{4}} v \in D(\Delta^{-\frac{1}{4}})$ and

$$\Delta^{-\frac{1}{4}} A\Delta^{\frac{1}{4}} v = \tilde{A} v .$$

Hence $\tilde{A}v \in D(\Delta^{\frac{1}{4}})$, $v \in D(\Delta^{\frac{1}{4}}\tilde{A})$ and

$$A\Delta^{\frac{1}{4}}v = \Delta^{\frac{1}{4}}\tilde{A}v\,.$$

So, inclusion (3.1) is proved.

For any $v \in D(\Delta^{\frac{1}{2}})$ we have $\Delta^{\frac{1}{4}}v \in D(\Delta^{\frac{1}{4}}) \cap D(\Delta^{-\frac{1}{4}})$ and (3.3) shows, that

$$(\Delta^{\frac{1}{4}}v \,|\, \tilde{A}\Delta^{\frac{1}{4}}v) = (v \,|\, A\Delta^{\frac{1}{2}}v)\,. \tag{3.4}$$

According to this formula, $\tilde{A}$ is selfadjoint iff $A\Delta^{\frac{1}{2}}$ is symmetric. Q.E.D.

Let $A_1, A_2 \in \mathscr{A}$ be such that $A_1\Delta^{\frac{1}{2}} \geqq 0$ and $A_2\Delta^{\frac{1}{2}} \geqq 0$. Then, in virtue of (3.4): $\tilde{A}_1 \geqq 0$ and $\tilde{A}_2 \geqq 0$. We have:

$$\begin{aligned}(A_1 y \,|\, A_2 y) &= (A_1\Delta^{\frac{1}{4}}y \,|\, A_2\Delta^{\frac{1}{4}}y) = (\Delta^{\frac{1}{4}}\tilde{A}_1 y \,|\, \Delta^{\frac{1}{4}}\tilde{A}_2 y)\\ &= (y \,|\, \tilde{A}_1\Delta^{\frac{1}{2}}\tilde{A}_2 y) = (y \,|\, \tilde{A}_1 J\tilde{A}_2 y) = (y \,|\, \tilde{A}_1 J\tilde{A}_2 J y)\,.\end{aligned}$$

Since the product of two commuting positive bounded operators is positive, we get

$$(A_1 y \,|\, A_2 y) \geqq 0\,. \tag{3.5}$$

An operator A is said to be definite iff $A \geqq 0$ or $A \leqq 0$. Let A and A' be selfadjoint elements of a factor and its commutant, respectively. One can check, that AA' is a definite operator iff both A and A' are definite.

Lemma 3.3. *Let $A \in \mathscr{A}$. Then Ay is $(\mathscr{A}, J)$-positive if and only if $A\Delta^{\frac{1}{2}}$ is definite.*

Proof. We may assume, that $A\Delta^{\frac{1}{2}}$ is symmetric. For any $B \in \mathscr{A}$ we have:

$$JBAy = JBJAy = AJBy = A\Delta^{\frac{1}{2}}B^*y = \Delta^{\frac{1}{4}}\tilde{A}\Delta^{\frac{1}{4}}B^*y\,.$$

Setting in this equation B^* instead of B and making use of relation $\Delta^{\frac{1}{4}}By = J\Delta^{\frac{1}{4}}B^*y$ we get

$$JB^*Ay = \Delta^{\frac{1}{4}}\tilde{A}\Delta^{\frac{1}{4}}By$$

and

$$B^*Ay = \Delta^{-\frac{1}{4}}J\tilde{A}J\Delta^{\frac{1}{4}}B^*y\,.$$

Therefore

$$\begin{aligned}(Ay \,|\, BJBAy) &= (B^*Ay \,|\, JBAy)\\ &= (\Delta^{-\frac{1}{4}}J\tilde{A}J\Delta^{\frac{1}{4}}B^*y \,|\, \Delta^{\frac{1}{4}}\tilde{A}\Delta^{\frac{1}{4}}B^*y) \qquad (3.6)\\ &= (\Delta^{\frac{1}{4}}B^*y \,|\, J\tilde{A}J\tilde{A}\Delta^{\frac{1}{4}}B^*y)\,.\end{aligned}$$

Assume, that $A\Delta^{\frac{1}{2}}$ is definite. Equation (3.4) shows, that $\tilde{A}$ is also definite. Therefore either $\tilde{A} \geqq 0$ and $J\tilde{A}J \geqq 0$ or $\tilde{A} \leqq 0$ and $J\tilde{A}J \leqq 0$. In both cases $J\tilde{A}J\tilde{A} \geqq 0$ and (3.6) shows, that Ay is $(\mathscr{A}, J)$-positive.

Conversely assume, that Ay is $(\mathscr{A}, J)$-positive. Then equation (3.6) shows, that $J\tilde{A}J\tilde{A} \geqq 0$ and $\tilde{A}$ must be definite (note, that $\tilde{A}$ and $J\tilde{A}J$

belong to factor $\mathscr{A}$ and its commutant $\mathscr{A}'$ respectively). Equation (3.4) shows now, that $A\Delta^{\frac{1}{2}}$ is definite. Q.E.D.

In what follows, $R(X)$ denotes the range of an operator X. Let us note, that for any bounded operator $A: R(A\Delta^{\frac{1}{2}})$ is dense in $R(A)$.

Lemma 3.4. *Let $A \in \mathscr{A}$ be such that $A\Delta^{\frac{1}{2}}$ is symmetric. Then there exist operators A_+ and A_- belonging to $\mathscr{A}$ such that*

$$A = A_+ - A_- , \tag{3.7}$$

$$A_+\Delta^{\frac{1}{2}} \geqq 0 \quad \text{and} \quad A_-\Delta^{\frac{1}{2}} \geqq 0 , \tag{3.8}$$

$$R(A_+) \perp R(A_-) . \tag{3.9}$$

Proof. In virtue of Lemma 3.1, $A\Delta^{\frac{1}{2}}$ is ess. selfadjoint. Let

$$\overline{A\Delta^{\frac{1}{2}}} = \int_{-\infty}^{+\infty} \lambda dE(\lambda) \tag{3.10}$$

be the spectral resolution of $\overline{A\Delta^{\frac{1}{2}}}$. Then

$$(\Delta^{\frac{1}{2}} A^* \overline{A\Delta^{\frac{1}{2}}})^{\frac{1}{2}} = \int_{-\infty}^{+\infty} |\lambda|\, dE(\lambda) . \tag{3.11}$$

It can be proved (see [7], Section 3), that the operator standing on the left hand side of (3.11) is of the form $\overline{B\Delta^{\frac{1}{2}}}$, where $B \in \mathscr{A}$. Let

$$A_+ = \tfrac{1}{2}(B + A) ,$$
$$A_- = \tfrac{1}{2}(B - A) .$$

Then (3.7) is fulfiled. By using (3.10) and (3.11) we have:

$$A_+\Delta^{\frac{1}{2}} \subset \int_0^{\infty} |\lambda|\, dE(\lambda) ,$$

$$A_-\Delta^{\frac{1}{2}} \subset \int_{-\infty}^{0} |\lambda|\, dE(\lambda) ,$$

hence (3.8) is proven. Moreover the above equations show, that $R(A_+\Delta^{\frac{1}{2}}) \perp R(A_-\Delta^{\frac{1}{2}})$ and (3.9) follows. Q.E.D.

The following lemma may be compared with Lemma 4.1 of [4].

Lemma 3.5. *Let $A_1, A_2 \in \mathscr{A}$. Assume that $A_1\Delta^{\frac{1}{2}} \geqq 0$ and $A_2\Delta^{\frac{1}{2}} \geqq 0$. Then*

$$\sup_{\substack{W \in \mathscr{A} \\ \|W\| \leqq 1}} |(A_1 y | W A_1 y) - (A_2 y | W A_2 y)| \geqq \|A_1 y - A_2 y\|^2$$

Proof. Let

$$\omega(W) = (A_1 y | W A_1 y) - (A_2 y | W A_2 y) .$$

We have to find an operator $W \in \mathscr{A}$ such that $\|W\| \leqq 1$ and

$$\omega(W) \geqq \|(A_1 - A_2)\, y\|^2 . \tag{3.12}$$

Let $A = A_1 - A_2$ and let A_+, A_- be operators introduced in Lemma 3.4. We shall prove, that (3.12) is satisfied by operator

$$W = E_+ - E_- \tag{3.13}$$

where E_+ and E_- are projection operators projecting onto $\overline{R(A_+)}$ and $\overline{R(A_-)}$ respectively.

One can check, that for selfadjoint W:

$$\omega(W) = ((A_1 + A_2)\, y \,|\, W(A_1 - A_2)\, y)\,.$$

Assume, that W is given by (3.13). Then

$$W(A_1 - A_2) = (E_+ - E_-)(A_+ - A_-) = A_+ + A_-$$

and therefore

$$\omega(W) = ((A_1 + A_2)\, y \,|\, (A_+ + A_-)\, y)\,.$$

It is seen, that

$$\begin{aligned}((A_1 + A_2)\, y \,|\, (A_+ + A_-)\, y) - 2(A_2 y \,|\, A_+ y) - 2(A_1 y \,|\, A_- y) \\ = ((A_1 - A_2)\, y \,|\, A_+ y) - ((A_1 - A_2)\, y \,|\, A_- y) \\ = ((A_1 - A_2)\, y \,|\, (A_+ - A_-)\, y) = \|(A_1 - A_2)\, y\|^2\,.\end{aligned}$$

In virtue of (3.5): $(A_2 y \,|\, A_+ y) \geqq 0$, $(A_1 y \,|\, A_- y) \geqq 0$ and (3.12) follows. Q.E.D.

Now, we can prove the following, very important

Theorem 3.1. *Let $\mathscr{A}$ be a factor, J be an exchange involution for $(\mathscr{A}, \mathscr{A}')$ and x_1, x_2 be $(\mathscr{A}, J)$-positive vectors. Then either*

$$\sup_{\substack{W \in \mathscr{A} \\ \|W\| \leqq 1}} |(x_1 \,|\, W x_1) - (x_2 \,|\, W x_2)| \geqq \|x_1 - x_2\|^2 \tag{3.14}$$

or

$$\sup_{\substack{W \in \mathscr{A} \\ \|W\| \leqq 1}} |(x_1 \,|\, W x_1) - (x_2 \,|\, W x_2)| \geqq \|x_1 + x_2\|^2\,. \tag{3.15}$$

Proof. Let ω be a faithful normal state of $\mathscr{A}$ such that

$$\omega(A^* A) \geqq \|A x_1\|^2 \tag{3.16}$$

and

$$\omega(A^* A) \geqq \|A x_2\|^2 \tag{3.17}$$

for any $A \in \mathscr{A}$. For example one can choose an orthonormal basis (e_n) of the Hilbert space and put

$$\omega(A) = (x_1 \,|\, A x_1) + (x_2 \,|\, A x_2) + \sum_{n=1}^{\infty} \frac{1}{n^2}\, (e_n \,|\, A e_n)\,.$$

The state ω can be represented (see Appendix) by a separating and cyclic $(\mathscr{A}, J)$-positive vector y:

$$\omega(A) = (y \,|\, A y)\,.$$

Relation (3.16) means, that $\|Ax_1\| \leqq \|Ay\|$. Setting in this inequality $JA'J$ instead of A we get $\|A'x_1\| \leqq \|A'y\|$ for all $A' \in \mathscr{A}'$. Therefore there exists a bounded operator A_1 such that $A_1 A' y = A' x_1$. One can easily prove, that $A_1 \in \mathscr{A}$. Moreover setting $A' = I$ we get

$$x_1 = A_1 y .$$

In the same way, starting from (3.17) we can prove, that

$$x_2 = A_2 y$$

where A_2 is an element of $\mathscr{A}$.

Let Δ be the modular operator assigned to y. In virtue of Lemma 3.3 operators $A_1 \Delta^{\frac{1}{2}}$ and $A_2 \Delta^{\frac{1}{2}}$ are definite.

Assume for an instant, that these operators are positive. Then Lemma 3.5 leads immediately to relation (3.14). In the general case we have $\varepsilon_1 A_1 \Delta^{\frac{1}{2}} \geqq 0$ and $\varepsilon_2 A_2 \Delta^{\frac{1}{2}} \geqq 0$ (where $\varepsilon_1, \varepsilon_2 = \pm 1$ are suitably chosen). By using Lemma 3.5 we get either (3.14) or (3.15) depending on the value of $\varepsilon_1 \varepsilon_2$. Q.E.D.

Assume now, that $\|x_1\| = 1 = \|x_2\|$. Then

$$\|x_1 - x_2\|^2 \geqq 2(1 - |(x_1 | x_2)|)$$
$$\|x_1 + x_2\|^2 \geqq 2(1 - |(x_1 | x_2)|)$$

and we immediately get

$$\sup_{\substack{W \in \mathscr{A} \\ \|W\| \leqq 1}} |(x_1 | W x_1) - (x_2 | W x_2)| \geqq 2(1 - |(x_1 | x_2)|) . \tag{3.18}$$

Let us note, that both sides of the last relation remain unchanged, when x_1 and x_2 are multiplied by complex number of modulus 1.

Corollary. *Assume, that $x_1 = \lambda_1 x_1'$ and $x_2 = \lambda_2 x_2'$, where x_1', x_2' are normalized $(\mathscr{A}, J)$-positive vectors and λ_1, λ_2 are complex number of modulus* 1. *Then relation* (3.18) *is fulfiled.*

Appendix

In [7] we proved (see [7], Theorem 2.2), that any normal state of a standart von Neumann algebra $\mathscr{A}$ can be represented by a $(\mathscr{A}, J)$-positive vector, where J is a suitably chosen exchange involution for $(\mathscr{A}, \mathscr{A}')$. Now, we shall slightly improve this result.

Theorem A.1. *Let $\mathscr{A} \subset B(H)$ be a von Neumann algebra and let J be a positive exchange involution for $(\mathscr{A}, \mathscr{A}')$. Then any normal state ω of $\mathscr{A}$ can be represented by a $(\mathscr{A}, J)$-positive vector $y \in H$:*

$$\omega(A) = (y | A y) , \qquad A \in \mathscr{A} .$$

Proof. At first we use Theorem 2.2 of [7]. It says, that there exist a positive exchange involution J' and a $(\mathscr{A}, J')$-positive vector y' such that

$$\omega(A) = (y' | A y')$$

for any $A \in \mathscr{A}$. We know (see [7], Theorem 2.1), that $J' = VJV^{-1}$ where V is an unitary element of $\mathscr{A}'$. Let $y = V^{-1} y'$. One can easily check, that y is $(\mathscr{A}, J)$-positive. Moreover

$$\omega(A) = (y' | A y') = (Vy | AVy) = (y | Ay)$$

for any $A \in \mathscr{A}$. Q.E.D.

If ω is a faithful state, then evidently y is a separating vector. Since J maps separating vectors onto cyclic vectors, y is a cyclic vector.

Acknowledgment. The author is very indebted to Mr. W. Pusz for helpful discussions. Special thanks are due to Professor K. Maurin for his interest in this work.

References

1. Dixmier, J.: Les C^*-algebre et leurs representations. Paris: Gauthier-Villars 1964.
2. Glimm, J., Kadison, R. V.: Unitary operators in C^*-algebras. Pacific J. Math. **10**, 547—556 (1960).
3. Kato, T.: Perturbation theory for linear operators. Berlin-Heidelberg-New York: Springer 1966.
4. Powers, R. T., Størmer, E.: Free states of the canonical anticommutation relations. Commun. math. Phys. **16**, 1—33 (1970).
5. Schwartz, J. T.: W^*-algebras. New York: Gordon and Breach 1967.
6. Takesaki, M.: Tomita's theory of modular Hilbert algebras and its applications. Lecture Notes in Mathematics, Vol. 128. Berlin-Heidelberg-New York: Springer 1970.
7. Woronowicz, S. L.: On the purification of factor states. Commun. math. Phys. **28**, 221—235 (1972).

S. L. Woronowicz
Department of Mathematical
Methods of Physics
University of Warsaw
Warsaw Hoza 74
Warsaw, Poland