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On the Onsager-Yang-Value of the Spontaneous Magnetization

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Abstract. We show that the value of the spontaneous magnetization for the two-dimensional Ising model computed by Onsager is indeed, the appropriate derivative of the free energy with respect to the magnetic field. The argument is based on a simple application of the duality transformation.

1. Introduction

It is well known that Onsager never published his proof that the spontaneous magnetization in the Ising model is given by [1]

$$m_0(\beta) = \left(1 - \frac{1}{(\operatorname{sh} 2\beta J)^4}\right)^{1/8} \qquad \beta \geqq \beta_c \tag{1.1}$$

where β_c, determined by $\operatorname{sh} 2\beta_c J = 1$, is the critical inverse temperature ($\beta_c = 1/T_c$). J is the ferromagnetic coupling constant.

The above value was found by Yang [2] to coincide with

$$m_y(\beta) = \lim_{h \to 0^+} \lim_{N \to \infty} \lim_{M \to \infty} \frac{f_{NM}(\beta, h/N) - f_{NM}(\beta, 0)}{h/N} \tag{1.2}$$

and by Montroll, Potts, Ward [3] (see also footnote on p. 810 of [2]) to coincide with

$$m_A(\beta) = \lim_{|x-y| \to \infty} \lim_{N \to \infty} \lim_{M \to \infty} \sqrt{\langle \sigma_x \sigma_y \rangle_{P,NM}} \tag{1.3}$$

where f_{NM} and $\langle \sigma_x \sigma_y \rangle_{P,NM}$ denote, respectively, the free energy and the two spin correlation function of a rectangular lattice of $N \times M$ sites with periodic boundary conditions. h is the external magnetic field.

Although $m_y(\beta) \equiv m_0(\beta) \equiv m_A(\beta)$ it has never been proven [4] that these values coincide with the "true" spontaneous magnetization

$$m(\beta) = \left. \frac{\partial f(\beta, h)}{\partial \beta h} \right|_{h = 0^+} \tag{1.4}$$

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where f is the infinite volume free energy (which is boundary condition independent).

It is easy to see that $m(\beta) \geqq m_0(\beta)$ [4] and it has been proven that

$$m(\beta) = \lim_{|x-y| \to \infty} \sqrt{\langle \sigma_x \sigma_y \rangle_+} \tag{1.5}$$

where $\langle \sigma_x \sigma_y \rangle_+$ is the infinite volume limit of $\langle \sigma_x \sigma_y \rangle_{+,NM}$ which denotes the two-spin correlation function when all the spins of the boundary are fixed to be $+1$ [5]. We shall often refer to these boundary conditions as to "closed" boundary conditions.

It is of course important to establish that $m_0(\beta) = m(\beta)\ \forall \beta$: in fact if $m_0(\beta) < m(\beta)$ near β_c this would imply that the critical exponents computed from (1.1) need not be the right ones; also, it might happen that $m(\beta) > 0$ for some $\beta < \beta_c$ i.e. that the critical temperature computed by Onsager does not correspond to the "true" critical temperature defined as the infimum of all the β's such that $m(\beta) > 0$.

It has recently been shown that $m(\beta) \equiv 0$ for $\beta < \beta_c$ and this shows that the Onsager value of the critical temperature and the "true" critical temperature coincide [6]; furthermore it has been shown that $m_0(\beta) \equiv m(\beta)$ for $\beta > \bar{\beta}_c > \beta_c$ where $\bar{\beta}_c$ is about 9% different from β_c [7].

It remains to prove that $m(\beta) = m_0(\beta)$ for $\beta_c \leqq \beta \leqq \bar{\beta}_c$; this is the purpose of this paper which relies heavily on the result of [6] quoted above while it does not use the techniques of [7]. The existence and importance of this problem has been, for the first time, clearly stated in the literature in the second paper of [3].

We will reach our goal by proving that for all β's, in the infinite volume limit

$$\langle \sigma_x \sigma_y \rangle_a (\beta) = \langle \sigma_x \sigma_y \rangle_p (\beta) = \langle \sigma_x \sigma_y \rangle_+ (\beta) \tag{1.6}$$

where p means periodic boundary conditions, $+$ means closed boundary conditions and a means open (perfect wall) boundary conditions. This equation when combined with (1.3) and (1.5) provides the desired result $m(\beta) = m_0(\beta)\ \forall\ \beta \geqq \beta_c$. Eq. (1.6) follows from a careful study of the implications of "duality" which is a symmetry property of the two-dimensional square-lattice Ising model with nearest neighbour interactions [8, 9].

Our use of duality in the present context is close in spirit to that of the third paper of Ref. [10].

2. Duality and Boundary Conditions

It is well known that duality (see appendix) implies a relationship between the high and low temperature properties of the two-dimensional Ising model in zero external field. The center of the symmetry is just the point β_c (i.e. the Onsager critical temperature) where $m_0(\beta)$ vanishes.

Roughly speaking the duality transformation allows to express the high temperature free energy and correlation functions in terms of the low temperature ones and vice-versa [10]. This statement requires further explanations. In fact one has to remark that, in presence of phase transitions, the correlation functions will depend upon the boundary conditions; for example, at least in principle

$$\langle \sigma_x \sigma_y \sigma_z \ldots \rangle_+ \neq \langle \sigma_x \sigma_y \sigma_z \ldots \rangle_- \neq \langle \sigma_x \sigma_y \sigma_z \ldots \rangle_p \neq \langle \sigma_x \sigma_y \sigma_z \ldots \rangle_a \tag{2.1}$$

where the subscripts denote as before the type of boundary conditions used.

The question then arises of determining what happens to a given boundary condition under the duality transformation. An answer to this question is, because of (2.1), clearly preliminary in order to establish the transformation law of correlation functions under duality. It turns out that in general a given boundary condition corresponds to some other "dual" boundary condition. In particular, for example, periodic boundary conditions are dual to some awkward boundary conditions, while closed boundary conditions are dual to open boundary conditions.

Let us express quantitatively the above statements.

Let Λ be an $N \times M$ rectangular box centered at the origin. Define for $X \equiv (x_1, x_2, \ldots, x_{2n})$ $x_i \in \Lambda$

$$\sigma^X = \prod_{i=1}^{2n} \sigma_{x_i} \qquad n = 1, 2, \ldots .$$

We then go over to the dual lattice Λ^* which is obtained (see appendix and Fig. 2) by considering the segments orthogonal at the middle point to the sides of the original lattice Λ. By defining the dual temperature

$$e^{-2\beta^* J} = \operatorname{th} \beta J \tag{2.2}$$

according to the results of the appendix, one can establish the equations valid also in the thermodynamic limit

$$\langle \sigma^X \rangle_a(\beta) = \left\langle \prod_{b^* \in \Gamma^*} [\operatorname{ch}(2\beta^* J) - \tilde{\sigma}_{b^*} \operatorname{sh}(2\beta^* J)] \right\rangle_+ (\beta^*) \tag{2.3}$$

$$\langle \sigma^X \rangle_+(\beta) = \left\langle \prod_{b^* \in \Gamma^*} [\operatorname{ch}(2\beta^* J) - \tilde{\sigma}_{b^*} \operatorname{sh}(2\beta^* J)] \right\rangle_a (\beta^*) \tag{2.4}$$

where $\Gamma^* \in \Lambda^*$ is constructed as follows:

a) first divide X into n pairs of spins and associate with each pair a path contained in Λ joining the two spins.

b) To each link of this path associate the corresponding orthogonal link b^* on the dual lattice Λ^*. The set of these dual links will be Γ^*. $\tilde{\sigma}_{b^*}$ is the product of the spins situated at the end points of the link b^*.

3. The Main Result

In this section we consider the two spin correlation functions and show that

$$\langle \sigma_x \sigma_y \rangle_a = \langle \sigma_x \sigma_y \rangle_p = \langle \sigma_x \sigma_y \rangle_+ \, . \tag{3.1}$$

We shall use a number of known results some of which have already been mentioned in Section 1.

a) The following limits exist and are translationally invariant ($\forall \beta$)

$$\begin{aligned} \lim_{N,M\to\infty} \langle \sigma^X \rangle_{a,NM} &= \langle \sigma^X \rangle_a \\ \lim_{N,M\to\infty} \langle \sigma^X \rangle_{+,NM} &= \langle \sigma^X \rangle_+ \end{aligned} \qquad X = (x_1, x_2, \ldots, x_n) \tag{3.2}$$

where the limit has to be taken when $N\to\infty$, $M\to\infty$ in any order or simultaneously and the result is independent on the way we take the limit. (3.2) follows from Griffiths inequalities.

b) If x, y are two sites on the same row

$$\langle \sigma_x \sigma_y \rangle_p = \lim_{N\to\infty} \lim_{M\to\infty} \langle \sigma_x \sigma_y \rangle_{P,NM} \tag{3.3}$$

exists and

$$m_0^2(\beta) = \lim_{|x-y|\to\infty} \langle \sigma_x \sigma_y \rangle_p \, . \tag{3.4}$$

(3.4) follows from the exact solution of the Ising model [3]. The other results we need are:

c) the Onsager critical temperature β_c coincides with the "true" critical temperature [6].

d) The infinite volume correlation functions for $\beta \leqq \beta_c$ are boundary condition independent. Furthermore for all β's

$$m^2(\beta) = \lim_{|x-y|\to\infty} \langle \sigma_x \sigma_y \rangle_+ \, . \tag{3.5}$$

These statements follow immediately from the results of [5, 6]. It is therefore clear that (3.1) implies $m(\beta) = m_0(\beta)\,\forall \beta$. To prove (3.1) we proceed as follows. Consider the model in a box Λ_{NM} with open boundary conditions: if we introduce additional couplings which couple opposite sites on the boundary $\partial\Lambda$ we obtain a model with periodic boundary conditions. If afterwards we introduce an infinite magnetic field acting on the spins of $\partial\Lambda$ we obtain the $+$, $-$ type boundary conditions on the same box Λ_{NM}.

Hence an application of the second Griffiths inequality yields

$$\langle \sigma_x \sigma_y \rangle_{a,NM} \leqq \langle \sigma_x \sigma_y \rangle_{p,NM} \leqq \langle \sigma_x \sigma_y \rangle_{+,NM} \, . \tag{3.6}$$

Taking the limit as $N \to \infty$, $M \to \infty$ and using the results a) and b) (assume also that x, y are on the same row) we find

$$\langle \sigma_x \sigma_y \rangle_a(\beta) \leqq \langle \sigma_x \sigma_y \rangle_p(\beta) \leqq \langle \sigma_x \sigma_y \rangle_+(\beta) \tag{3.7}$$

where we have explicitely written the β dependence of the correlation functions for further use.

By applying now Eqs. (2.3) and (2.4)

$$\langle \sigma_x \sigma_y \rangle_a(\beta) = \left\langle \prod_{b^* \in \Gamma^*} [\mathrm{ch}(2\beta^* J) - \tilde{\sigma}_{b^*} \mathrm{sh}(2\beta^* J)] \right\rangle_+ (\beta^*) \tag{3.8}$$

$$\langle \sigma_x \sigma_y \rangle_+(\beta) = \left\langle \prod_{b^* \in \Gamma^*} [\mathrm{ch}(2\beta^* J) - \tilde{\sigma}_{b^*} \mathrm{sh}(2\beta^* J)] \right\rangle_a (\beta^*) . \tag{3.9}$$

However if $\beta > \beta_c$ it follows that $\beta^* < \beta_c$ and result d) can be used to guarantee that the right hand side of (3.8) and (3.9) are equal; hence

$$\langle \sigma_x \sigma_y \rangle_a(\beta) = \langle \sigma_x \sigma_y \rangle_+(\beta) \qquad \beta > \beta_c \tag{3.10}$$

which implies together with (3.7) $m_0(\beta) = m(\beta)$.

4. Conclusions

The above results prove that the Onsager value of the spontaneous magnetization is the appropriate derivative of the free energy. An interesting related question remains still open: how many equilibrium pure phases can coexist for $\beta > \beta_c$?

It is known that for β large enough ($\beta > \bar{\beta}_c$, see Section 1) there are only two translational invariant equilibrium states and it has been announced [13] and partially proven [12] that, for large β at least, there are no non-translational invariant equilibrium states (which is a peculiarity of the dimension 2, [13]).

It is however a reasonable conjecture to assume that in two dimensions there are only two translation invariant equilibrium states and no non-translational invariant equilibrium states all the way from $\beta = +\infty$ to $\beta = \beta_c$.

It is possible that the duality symmetry is a key to the proof of the above conjecture: however we have not been able to use this key.

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Appendix. Proof of the Duality Relations [14]

a) High and Low Temperature Expansions of the Partition Function

In this appendix we shall confine ourselves as in the text, to periodic, open and closed boundary conditions.

Let

$$Z_B\{\beta J_b\} = \sum_{\sigma^B} e^{\beta \sum_b j_b \tilde{\sigma}_b} = Z_B\{K_b\} = \sum_{\sigma^B} e^{\sum_b K_b \tilde{\sigma}_b} \qquad K_b = \beta J_b$$

be the partition function, where the index B means that the sum over all possible configurations must be restricted according to the particular boundary condition chosen. A simple calculation now gives the so called "high temperature" expansion

$$Z_B\{K_b\} = \prod_b \operatorname{ch} K_b \sum_{\sigma^B} \prod_b (1 + \tilde{\sigma}_b \operatorname{th} K_b) . \tag{A.1}$$

The general term

$$T = \left(\prod_{i=1}^{K} \operatorname{th} K_{b_i} \right) \tilde{\sigma}_{b_1} \tilde{\sigma}_{b_2} \dots \tilde{\sigma}_{b_K} \tag{A.2}$$

contributes only if $\tilde{\sigma}_{b_1} \tilde{\sigma}_{b_2} \dots \tilde{\sigma}_{b_K} \equiv 1$. This means that the set of lattice bonds $b_1, b_2, \dots, b_K$, characterizing T, constitutes a closed multipolygon for open or periodic boundary conditions, and a collection of multipolygons and paths, whose end-points belong to the boundary, for closed ($+$ or $-$) conditions (see Fig. 1).

Let us write $T = T(\gamma)$, where $\gamma = b_1 \cup b_2 \dots \cup b_K$; as a consequence (A.1) can be given the following form

$$Z_B\{K_b\} = 2^L \prod_b \operatorname{ch} K_b \sum_{\gamma^B} T(\gamma) \tag{A.3}$$

where L is the number of free lattice spins and the sum is extended to the γ's which, for the given boundary condition, furnish a non vanishing contribution in (A.1).

The partition function of the model can also be expressed by the so called "low temperature" expansion, whose key is the geometrical description of the spin configurations [11].

Given a spin configuration σ we can draw a unit segment perpendicular to the center of each bond having opposite spins at its extremes. Clearly the union of all these segments, λ, will be a closed multipolygon, for closed or periodic boundary conditions, and a collection of multipolygons and paths, whose terminal segments intersect some links of the boundary, for open conditions (see Fig. 1).

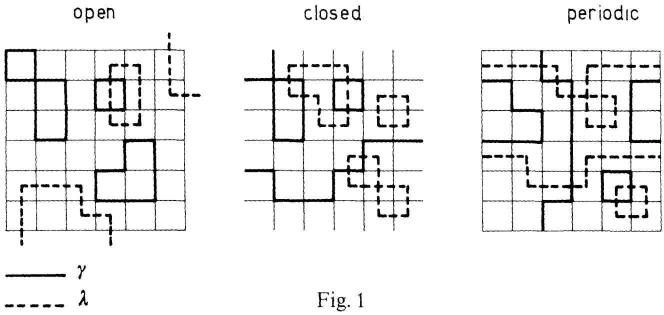

Fig. 1

With open or periodic boundary conditions two configurations with opposite spins will correspond to the same λ; a factor 2 will take into account this degeneracy in expanding the partition function.

The energy of a general configuration is

$$H(\lambda) = -\sum_{b} J_b + 2 \sum_{b \in \lambda} J_b$$

and consequently the partition function can be written

$$Z_B\{K_b\} = 2^{\nu} e^{\sum_b K_b} \sum_{\lambda^B} \left(\prod_{b \in \lambda} e^{-2K_b} \right) \tag{A.4}$$

where ν is equal to 0 for closed and to 1 for open or periodic boundary conditions.

	H. T.	L. T.
open:	multipolygons	multipolygons and paths
periodic:	multipolygons	multipolygons
closed:	multipolygons and paths	multipolygons

b) Duality Relations

Consider a $N \times M$ rectangular lattice Λ with open boundary conditions and a second $(N+1) \times (M+1)$ lattice Λ^*, obtained from Λ by drawing a unit segment b^* perpendicular to each link b of Λ at its middle point (see Fig. 2); for Λ^* we choose closed boundary conditions.

It is immediate to see that the γ's of the high temperature expansion (A.3) for Λ coincide with the λ's of the low temperature expansion (A.4) for Λ^*, and vice versa. So if we choose for each link b^* of Λ^* a coupling constant K_b^* such that

$$\operatorname{th} K_b^* = e^{-2K_b} \tag{A.5}$$

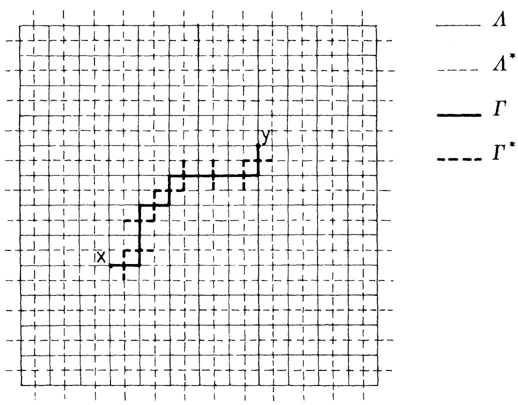

Fig. 2

where b and b^* intersect each other, we obtain immediately

$$\frac{Z_{\Lambda,a}\{K_b\}}{2^L \prod_b \operatorname{ch} K_b} = \frac{Z_{\Lambda^*,+(\text{or}-)}\{K_b^*\}}{e^{\sum_{b^*} K_b^*}} \qquad L = N \times M \tag{A.6}$$

or in a more symmetric form

$$\frac{Z_{\Lambda,a}\{K_b\}}{2^{L/2} \prod_b (\operatorname{ch} 2K_b)^{1/2}} = \frac{2^{1/2} Z_{\Lambda^*,+(\text{or}-)}\{K_b^*\}}{2^{L^*/2} \prod_{b^*} (\operatorname{ch} 2K_b^*)^{1/2}} \quad L^* = (N-1) \times (M-1). \tag{A.7}$$

The same relation can of course be obtained by using for Λ the low and for Λ^* the high temperature expansion.

The situation is not that simple for periodic boundary conditions because of the existence of additional multipolygons which exploit the connectivity properties of the torus. Moreover, these extra configurations give a different contribution to the high and low temperature expansions.

We may notice that if $J_b = J_{b'} = J$ for all b, b' equation (A.5) defines uniquely a dual temperature β^*, if J_{b^*} is taken to be equal to J_b. It is well known that the value β_c for which $\beta^* = \beta = \beta_c$ is the Onsager critical temperature. For $\beta > \beta_c$, we have $\beta^* < \beta_c$ and vice versa.

c) Path Definition for the Two-spin Correlation Function

Consider the two-spin correlation function $\langle \sigma_x \sigma_y \rangle_B$, where B stays in this paragraph for open or closed boundary conditions. If Γ is an arbitrary path connecting the two sites x and y, constituted by the lattice bonds $b_1, b_2, \ldots, b_n$, it can be established the path independent relation

$$\langle \sigma_x \sigma_y \rangle_B = \frac{Z_B\{K'_b\} \prod_b (\mathrm{ch}\, 2K'_b)^{-1/2}}{Z_B\{K_b\} \prod_b (\mathrm{ch}\, 2K_b)^{-1/2}} \tag{A.8}$$

where $\{K'_b\}$ is the new set of coupling constants obtained from $\{K_b\}$ by substituting

$$K_{b_i} \to K'_{b_i} = K_{b_i} + i\frac{\pi}{2} \tag{A.9}$$

for the b's belonging to Γ. (A.8) is trivially verified.

If we denote by Γ^* the configuration in Λ^* obtained by considering all the bonds b^* orthogonal to the b's $\in \Gamma$ (see Fig. 2), the duality relation (A.7), gives, noting that

$$(K'_b)^* = \left(K_b + i\frac{\pi}{2}\right)^* = -K^*_b \quad \text{for} \quad b^* \in \Gamma^* \quad \text{and putting} \quad J_b = J_{b'} = J$$

$$\begin{aligned}\langle \sigma_x \sigma_y \rangle_B(\beta) &= \left\langle \prod_{b^* \in \Gamma^*} e^{-2\beta^* \tilde{\sigma}_{b^*}} \right\rangle_{B^*} (\beta^*) \\ &= \left\langle \prod_{b^* \in \Gamma^*} (\mathrm{ch}\, 2\beta^* J - \tilde{\sigma}_{b^*} \,\mathrm{sh}\, 2\beta^* J) \right\rangle_{B^*} (\beta^*)\end{aligned} \tag{A.10}$$

where, for $B =$ open (closed), $B^* =$ closed (open).

During the previous proof the volume of the lattices was kept finite. However, (A.10) holds also in the thermodynamic limit.

The above argument can be generalized to any correlation function $\langle \sigma^X \rangle_B$ where X contains an even number of sites, $2n$, by dividing them into n pairs and by considering paths connecting the two spins of each pair.

Note added in Proof. A similar scheme for the proof in this paper has been independently proposed by D. Ruelle (private communication).

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14. In this appendix for convenience of the reader we give a treatment of duality suitable for the purpose of the present paper. It should be mentioned however, that it contains certain remarks on the role of boundary conditions which do not appear in full detail elsewhere in the literature. For recent formulations of duality in the case of more general models see Wegner, F. J.: J. Math. Phys. **12**, 2259 (1971); – Merlini, D., Gruber, C.: preprint (1972).

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