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Exact Solution of the Dirac Equation with a Central Potential

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Abstract. The exact solution of the Dirac equation with a central potential, in the semi-relativistic approximation, is derived and formulae for phase shifts and eigenvalue equations are given.

Introduction

The integro-iteration method, introduced in Ref. [1] is applied to the solution of the Dirac's coupled radial equations. The solutions are obtained in a form similar to that of the Schrödinger equation [2], i.e., in simple series which converge strongly when the following restrictions are imposed on the potential $V(r)$:

$$V_{r\to 0}(r) \rightsquigarrow r^{-\beta}\ \beta \leqq 1 \tag{1a}$$

and

$$\int_a^\infty V(r)\,dr < \infty \quad \text{for} \quad 0 < a < \infty\,. \tag{1b}$$

Condition (1 b) excludes the Coulomb potential, but in this case the solutions are already known [3, 4]. On the other hand in cases with a screened or modified Coulomb potential [5] the method is applicable and one can get results to any desired accuracy.

I. Formulation

In semi-relativistic approximation the Dirac equation with central potential, after separation of the angular part, [3], is reduced to a system of two coupled radial equations [5];

$$\begin{aligned} (E+V+m)F_\nu + \frac{dG_\nu}{dr} - \frac{\nu}{r}\, G_\nu &= 0 \\ -(E+V-m)G_\nu + \frac{dF_\nu}{dr} + \frac{\nu+2}{r}\, F_\nu &= 0\,. \end{aligned} \tag{2}$$

Here we use the same notation as in Ref. [5], but for simplicity we have put $\hbar = c = 1$, and $\nu = l$ for $j = l - \frac{1}{2}$ and $\nu = -l - 1$ for $j = l + \frac{1}{2}$.

If we put

$$G_\nu = \frac{g_\nu}{r}, \qquad F_\nu = \frac{f_\nu}{r} \tag{3}$$

we obtain the more symmetrical form:

$$\begin{aligned} (E + V + m) f_\nu + g'_\nu - \frac{\nu + 1}{r} g_\nu &= 0 \\ -(E + V - m) g_\nu + f'_\nu + \frac{\nu + 1}{r} f_\nu &= 0 . \end{aligned} \tag{4}$$

For $V = 0$ the solutions of (4) are readily obtainable and are expressed in terms of Bessel functions if $E^2 - m^2 = k^2 > 0$ and modified Bessel functions if $E^2 - m^2 = -\kappa^2 < 0$.

Let u_1 and u_2 be two independent solutions of (4), with $V = 0$, regular respectively irregular at the origin, corresponding to g_ν^0 and v_1 and v_2 those corresponding to f_ν^0. We normalize them in such a way that

$$\det \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} = 1\,^{1} . \tag{5}$$

Next we look for a solution of (4) in the form:

$$\begin{aligned} g_\nu(r) &= C_1(r)\, u_1(r) + C_2(r)\, u_2(r) \\ f_\nu(r) &= C_1(r)\, v_1(r) + C_2(r)\, v_2(r) \end{aligned} \tag{6}$$

where $C_1(r)$ and $C_2(r)$ are functions to be specified, such that (6) are solutions of Eqs. (4). Using Lagrange's method of undetermined coefficients, and taking into account (5) we find:

$$\begin{aligned} C'_1(r) &= -C_1 V(u_1 u_2 + v_1 v_2) - C_2 V(u_2^2 + v_2^2) \\ C'_2(r) &= +C_2 V(u_1 u_2 + v_1 v_2) + C_1 V(u_1^2 + v_1^2) . \end{aligned} \tag{7}$$

Applying the integro-iteration method [1] we find the general solution of (7)[2]:

$$\begin{aligned} C_1(r) &= \lambda_1 e^{-f(r)} \Phi_1\begin{pmatrix} r \\ a, a \end{pmatrix} - \lambda_2 e^{-f(r)} \int_a^r A_{22} e^{2f(r')} \Phi_2\begin{pmatrix} r' \\ a, a \end{pmatrix} dr' \\ C_2(r) &= \lambda_2 e^{f(r)} \Phi_2\begin{pmatrix} r \\ a, a \end{pmatrix} + \lambda_1 e^{f(r)} \int_a^r A_{11} e^{-2f(r')} \Phi_1\begin{pmatrix} r' \\ a, a \end{pmatrix} dr' \end{aligned} \tag{8}$$

[1] For explicit expressions of u_j, v_j $(j = 1, 2)$ for every case see Appendix.

[2] We use the same notation as in Ref. [1].

where λ_1, λ_2 are arbitrary constants,

$$\begin{aligned} A_{11} &= V[u_1^2 + v_1^2] \\ A_{22} &= V[u_2^2 + v_2^2] \\ A_{12} &= V[u_1 u_2 + v_1 v_2] \\ \not{f}(r) &= \int_a^r A_{12}\, dr' \end{aligned} \tag{9}$$

and

$$\Phi_1\begin{pmatrix} r \\ a, a \end{pmatrix} = 1 - \int_a^r A_{22} e^{2\not{f}}\, dr' \int_a^{r'} A_{11} e^{-2\not{f}} \Phi_1\begin{pmatrix} r'' \\ a, a \end{pmatrix} dr'', \tag{10a}$$

$$\Phi_2\begin{pmatrix} r \\ a, a \end{pmatrix} = 1 - \int_a^r A_{11} e^{-2\not{f}} dr' \int_a^{r'} A_{22} e^{2\not{f}}\, \Phi_2\begin{pmatrix} r'' \\ a, a \end{pmatrix} dr''. \tag{10b}$$

The regular solution at $r=0$ is obtained from (8), if we put $\lambda_2 = 0$ and $a = 0$, i.e.:

$$\begin{aligned} C_1(r) &= e^{-\not{f}(r)} \Phi_1\begin{pmatrix} r \\ 0, 0 \end{pmatrix} \\ C_2(r) &= e^{\not{f}(r)} \int_0^r A_{11} e^{-2\not{f}} \Phi_1\begin{pmatrix} r' \\ 0, 0 \end{pmatrix} dr'. \end{aligned} \tag{11}$$

Finally we get:

$$\begin{aligned} g_v &= u_1 e^{-\not{f}} \Phi_1\begin{pmatrix} r \\ 0, 0 \end{pmatrix} + u_2 e^{\not{f}} \int_0^r A_{11} e^{-2\not{f}}\, \Phi_1\begin{pmatrix} r' \\ 0, 0 \end{pmatrix} dr' \\ f_v &= v_1 e^{-\not{f}} \Phi_1\begin{pmatrix} r \\ 0, 0 \end{pmatrix} + v_2 e^{\not{f}} \int_0^r A_{11} e^{-2\not{f}}\, \Phi_1\begin{pmatrix} r' \\ 0, 0 \end{pmatrix} dr'. \end{aligned} \tag{12}$$

For the existence of the solution (11), or (12), we have only to consider the convergence of the central function $\Phi_1\begin{pmatrix} r \\ 0, 0 \end{pmatrix}$. The last is guaranteed by the condition [1]:

$$\bar{q} = \int_0^\infty |A_{22}\, e^{2\not{f}}|\, dr' \int_0^{r'} |A_{11} e^{-2\not{f}}|\, dr'' < \infty .$$

If the potential $V(r)$ fullfils the conditions (1) then the function

$$\not{f}(r) = \int_0^r A_{12}\, dr'$$

is bounded for any $0 \leqq r \leqq \infty$. Let be $|f| < \mu$, then we have

$$\bar{q} \leqq e^{4\mu} \int_0^\infty |A_{22}|\, dr \int_0^r |A_{11}|\, dr'$$

$$\leqq e^{4\mu} \int_0^\infty |V| \{|u_2^2| + |v_2^2|\}\, dr \int_0^r |V| \{|u_1^2| + |v_1^2|\}\, dr' .$$

The r.h.s. consists of four terms. If we apply for each of them the argument used in [2] § III we prove that all of them are bounded, provided that the potential $V(r)$ obeys conditions (1).

II. Results and Discussion

(i) The application of the integro-iteration method leads, also in the present case, to the explicit expressions of the radial wave functions in a very simple way.

(ii) For $E^2 - m^2 = k^2 > 0$ (scattering problems) we find for the phase shifts η_ν:

$$\tan \eta_\nu = + \frac{e^{f(\infty)} \int_0^\infty A_{11} e^{-2f} \Phi_1 \begin{pmatrix} r \\ 0, 0 \end{pmatrix} dr}{e^{-f(\infty)} \Phi_1 \begin{pmatrix} \infty \\ 0, 0 \end{pmatrix}} \tag{13}$$

where $\nu = l$ or $-l-1$. It is understood that for every case we have to employ, for the calculations of f, A_{11} and A_{22}, the corresponding expressions of u_j, v_j $(j = 1, 2)$ given in the Appendix.

(iii) On the other hand if $E^2 - m^2 = -\kappa^2 < 0$ (bound states) we find the eigenvalue equation:

$$\Phi_1 \begin{pmatrix} \infty \\ 0, 0 \end{pmatrix} = 0 . \tag{14}$$

(iv) The phase function [6] also is explicitly obtained:

$$S(r) = + \frac{e^{f(r)} \int_0^r A_{11} e^{-2f} \Phi_1 \begin{pmatrix} r' \\ 0, 0 \end{pmatrix} dr'}{e^{-f(r)} \Phi_1 \begin{pmatrix} r \\ 0, 0 \end{pmatrix}} . \tag{15}$$

It is easy to verify that the phase function (15) is the solution of the Riccati equation:

$$S' = [A_{11} + 2A_{12} S + A_{22} S^2]$$

or,

$$S' = V[(u_1 + u_2 S)^2 + (v_1 + v_2 S)^2] \tag{16}$$

with

$$S(0) = 0 .$$

This expression is found in [6]. The difference is due to the different "normalization" of u_i, v_i which we adopted in order to have

$$\det\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} = 1\,.$$

(v) From (12) we find:

$$\int_0^\infty V(g_\nu u_1 + f_\nu v_1)\,dr = \int_0^\infty A_{11} e^{-f} \Phi_1\begin{pmatrix} r \\ 0,0 \end{pmatrix} dr$$

$$+ \int_0^\infty A_{12}\, e^{f}\, dr \int_0^r A_{11} e^{-2f}\, \Phi_1\begin{pmatrix} r' \\ 0,0 \end{pmatrix} dr'\,.$$

Integrating by parts the second term in the r.h.s. we find:

$$\int_0^\infty V(g_\nu u_1 + f_\nu v_1)\,dr = e^{f(\infty)} \int_0^\infty A_{11} e^{-2f}\, \Phi_1\begin{pmatrix} r \\ 0,0 \end{pmatrix} dr\,. \tag{17}$$

In a similar way we find:

$$\int_0^\infty V(g_\nu u_2 + f_\nu v_2)\,dr = -e^{-f(\infty)} \Phi_1\begin{pmatrix} \infty \\ 0,0 \end{pmatrix} + 1\,. \tag{18}$$

From (17) and (18) we have:

$$\tan\eta_\nu = \frac{\int_0^\infty V(g_\nu u_1 + f_\nu v_1)\,dr}{1 - \int_0^\infty V(g_\nu u_2 + f_\nu v_2)\,dr} \tag{19}$$

with $\nu = l$ or $-l-1$.

The expression (19) is analogeous to that given by Parzen [7], Eq. (71).

(vi) Finally we mention that the method can be applied with the same easiness to the scattering by a modified Coulomb field and it could be useful for the determination of the nuclear charge density $\varrho(r)$ and the corresponding formfactors [5].

Appendix

If we put in (4) $V = 0$ we obtain:

$$\begin{aligned} (E+m) f_\nu^0 + g_\nu^{0\prime} - \frac{\nu+1}{r} g_\nu^0 &= 0 \\ -(E-m)\, g_\nu^0 + f_\nu^{0\prime} + \frac{\nu+1}{r} f_\nu^0 &= 0\,. \end{aligned} \tag{A.1}$$

The system can be reduced to two uncoupled Bessel differential equations. If u_1, u_2 correspond to g_ν^0 and v_1, v_2 to f_ν^0 we make the following choice of the solutions:

(i) $\nu = l,\ E^2 - m^2 = k^2 > 0$

$$u_1 = (E+m)^{\frac{1}{2}} \sqrt{\frac{\pi r}{2}}\, J_{l+\frac{1}{2}}(kr), \qquad v_1 = (E-m)^{\frac{1}{2}} \sqrt{\frac{\pi r}{2}}\, J_{l+\frac{3}{2}}(kr),$$

$$u_2 = -(E+m)^{\frac{1}{2}} \sqrt{\frac{\pi r}{2}}\, Y_{l+\frac{1}{2}}(kr), \qquad v_2 = -(E-m)^{\frac{1}{2}} \sqrt{\frac{\pi r}{2}}\, Y_{l+\frac{3}{2}}(kr).$$

(ii) $\nu = -l-1,\ E^2 - m^2 = k^2 > 0$

$$u_1 = (E+m)^{\frac{1}{2}} \sqrt{\frac{\pi r}{2}}\, J_{l+\frac{1}{2}}(kr), \qquad v_1 = -(E-m)^{\frac{1}{2}} \sqrt{\frac{\pi r}{2}}\, J_{l-\frac{1}{2}}(kr),$$

$$u_2 = -(E+m)^{\frac{1}{2}} \sqrt{\frac{\pi r}{2}}\, Y_{l+\frac{1}{2}}(kr), \qquad v_2 = (E-m)^{\frac{1}{2}} \sqrt{\frac{\pi r}{2}}\, Y_{l-\frac{1}{2}}(kr).$$

(iii) $\nu = l,\ E^2 - m^2 = -\kappa^2 < 0$

$$u_1 = (m+E)^{\frac{1}{2}}\, r^{\frac{1}{2}}\, I_{l+\frac{1}{2}}(\kappa r), \qquad v_1 = -(m-E)^{\frac{1}{2}}\, r^{\frac{1}{2}}\, I_{l+\frac{3}{2}}(\kappa r),$$

$$u_2 = (m+E)^{\frac{1}{2}}\, r^{\frac{1}{2}}\, K_{l+\frac{1}{2}}(\kappa r), \qquad v_2 = (m-E)^{\frac{1}{2}}\, r^{\frac{1}{2}}\, K_{l+\frac{3}{2}}(\kappa r).$$

(iv) $\nu = -l-1,\ E^2 - m^2 = -\kappa^2 < 0$

$$u_1 = (m+E)^{\frac{1}{2}}\, r^{\frac{1}{2}}\, I_{l+\frac{1}{2}}(\kappa r), \qquad v_1 = -(m-E)^{\frac{1}{2}}\, r^{\frac{1}{2}}\, I_{l-\frac{1}{2}}(\kappa r),$$

$$u_2 = (m+E)^{\frac{1}{2}}\, r^{\frac{1}{2}}\, K_{l+\frac{1}{2}}(\kappa r), \qquad v_2 = (m-E)^{\frac{1}{2}}\, r^{\frac{1}{2}}\, K_{l-\frac{1}{2}}(\kappa r).$$

With this choice the couples (u_i, v_i) satisfy Eqs. (A.1) and, [8]

$$\det{}^{-} \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} = 1.$$

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