Commun. math. Phys. 27, 137—145 (1972)


# Asymptotic Completeness for Multi-Particle Schroedinger Hamiltonians with Weak Potentials

RAFAEL JOSE IORIO, JR. and MICHAEL O'CARROLL
Departamento de Matemática, Pontifícia Universidade Católica, Rio de Janeiro, Brasil

Received March 15, 1972

**Abstract.** We show that the non-relativistic quantum mechanical $n$-body Hamiltonians $T(k) = T + kV$ and $T$, the free particle Hamiltonian, are unitarily equivalent in the center of mass system, i.e., $T(k) = W_{\pm}(k)\, T W_{\pm}(k)^{-1}$ for $k$ sufficiently small and real. $V = \sum_i V_i$, a sum of $\frac{n(n-1)}{2}$ real pair potentials, $V_i$, depending on the relative coordinate $x_i \in R^3$ of the pair $i$, where $V_i$ is required to behave like $|x_i|^{-2-\varepsilon}$ as $|x_i| \to \infty$ and like $|x_i|^{-2+\varepsilon}$ as $|x_i| \to 0$. $T(k)$ is the self-adjoint operator associated with the form sum $T + kV$. There are no smoothness requirements imposed on the $V_i$. Furthermore $W_{\pm}(k) = s\text{-}\lim_{t \to \pm\infty} e^{iT(k)t} e^{-iTt}$ are the wave operators of time dependent scattering theory and are unitary. This result gives a quantitative form of the intuitive argument based on the Heisenberg uncertainty principle that a certain minimum potential well depth and range is needed before a bound state can be formed. This is the best possible long range behavior in the sense that if $kV_i \leqq C_i |x_i|^{-b}$, $0 < b \leqq 2$ for $|x_i| > R_i (0 < R_i < \infty)$ and all $C_i$ are negative then $T(k)$ has discrete eigenvalues and $W_{\pm}(k)$ are not unitary.

## 0. Introduction

In this article we treat the scattering and spectral problem for an $n$-body system in non-relativistic quantum mechanics with weak potentials. We show that the method of Kato [1] used to show asymptotic completeness and unitarity of the wave operators for weak potentials in the two-body case can be applied to obtain similar results in the $n$-body case. More precisely we show that in the center of mass system Hilbert space $H = L^2(R^{3n-3})$ the self-adjoint operators $T(k) = T + kV$ (the self-adjoint operator associated with a form sum) and $T$ (the free particle Hamiltonian) are unitarily equivalent for sufficiently small, real $k$. The potential $V = \sum_i V_i$ is a sum of pair potentials, $V_i$, which are real-valued measurable functions depending on the relative coordinates $x_i \in R^3$ of the pair $i$. Writing

$$A_i = |V_i|^{1/2}, \qquad B_i = (\operatorname{sign} V_i)\, A_i\,,$$

the result follows from the crucial fact that the operators $A_i(T - z)^{-1} B_j^*$ admit bounded analytic extensions for $\operatorname{Im} z \neq 0$, the bound being in-dependent of $z$ for $\operatorname{Im} z \neq 0$ and moreover that this is sufficient, i.e. compactness is not necessary.

The equivalence of $T$ and $T(k)$ is implemented by the unitary operators

$$W_{\pm}(k), \text{ i.e.,} \qquad T(k) = W_{\pm}(k)\, T\, W_{\pm}(k)^{-1}$$

where $W_{\pm}(k)^{-1} = W_{\pm}(k)^*$. The operators $W_{\pm}(k)$ admit an absolutely convergent power series expansion in $k$. Pair potentials behaving like $1/|x_i|^{2-\varepsilon}$ as $|x_i| \to 0$ and like $1/|x_i|^{2+\varepsilon}$ as $|x_i| \to \infty$ are allowed ($\varepsilon > 0$) (see Theorem 1.1 for the precise conditions). There are no conditions of smoothness imposed on the potential functions. We also show that the operators

$$W_{\pm}(k) = \underset{t \to \pm\infty}{s\text{-lim}}\, e^{iT(k)t} e^{-iTt}$$

$$W_{\pm}(k)^{-1} = \underset{t \to \pm\infty}{s\text{-lim}}\, e^{iTt} e^{-iT(k)t} = W_{\pm}(k)^*,$$

i.e., are the wave operators of time-dependent scattering theory. Thus the wave operators exist and their ranges are all $H$. This result of course also gives information about the spectrum of $T(k)$, in the center of mass system; namely, it is absolutely continuous, the singular continuous spectrum being absent. Thus there can be no point spectrum and we have a quantitative form of the heuristic argument using the Heisenberg uncertainty principle on the potential shape and well depth which will exclude bound states. Furthermore the long-range behavior of our result can not be improved. What is meant by this is that for pair potentials which fall off like $1/|x_i|^b$, $0 < b \leqq 2$, for $|x_i| \to \infty$, no matter how small $k > 0$ is there is an infinite point spectrum if all $V_i$ are negative. For a proof of this fact see Simon [2]. Thus the $W_{\pm}(k)$ can not be unitary since the ranges of $W_{\pm}(k)$ are orthogonal to the point spectrum of $T(k)$.

In Section I we introduce necessary notation and state our results. In Section II we give the proofs of the results.

For other results on $n$-body completeness in the general case see Hepp [3] and for repulsive potentials see Lavine [4]. For $n$-body spectral results see Simon [2], Balslev and Combes [5] and Albeverio [6].

## 1. Results

Throughout this paper we will maintain the notation of Kato [1]. We consider the $n$-particle self-adjoint Schroedinger Hamiltonian operator $T(k)$ which formally equals $T + kV$ where $T$ is the free particle self-adjoint center of mass kinetic energy operator and $V = \sum_i V_i$, the potential, is a sum of $n(n-1)/2$ pair potentials $V_i$, such as $i = (lm)$,

$l, m = 1, 2, \ldots, n (l < m)$. Each $V_i$ is a real-valued, measurable function depending only on the difference coordinates, $x_i \in R^3$, of the pair $i$ and $k$ is a real parameter. We write $V_i = B_i^* A_i$, where

$$A_i = |V_i|^{1/2}, \qquad B_i = (\operatorname{sign} V_i)\,|V_i|^{1/2}\,.$$

The operators $T$ and $A_i$, $B_i$ are defined as maximum multiplication operators on the Fourier transform of $H = L^2(R^{3n-3})$ and $H$, respectively. Thus $T$ is self-adjoint and $A_i, B_i$ are closed.

In order that the theorem of Kato [1] be directly applicable we introduce the Hilbert space $H' = H \oplus H \oplus \cdots \oplus H (n(n-1)/2$ summands) and let $V = B^* A$. The operators $A$ and $B$ are defined by

$$Xu = (X_1 u, X_2 u, \ldots, X_{\frac{n(n-1)}{2}} u) \text{ for } u \in D(X) = \bigcap_i D(X_i) \subset H, \quad X = A(B),$$

$X_i = A_i(B_i)$ and are closed linear operators as mappings from $D(X) \subset H$ to $H'$. We point out however that it is not necessary to introduce $H'$ and the results will still be valid but the Kato proof will then require modification. The resolvent of the self-adjoint operator $T$ will be denoted by $R(z)$.

We not state the crucial lemma from which our results will follow.

**Lemma 1.1.** *The norm of the closure, $[C_i R(z) D_j]$, of the operator $C_i R(z) D_j$ for $\operatorname{Im} z \neq 0$ where $C_i, D_i = A_i, A_i^*, B_i, B_i^*$ has the majorization*

$$\begin{aligned} \|[C_i R(z) D_j]\| \leqq {} & (2 m_j c_{ij})/(4\pi) \cdot (a^{-1} + b^{-1}) \\ & \cdot [\|V_i\|_{L^{p/2}(R^3)} \|V_j\|_{L^{p/2}(R^3)}]^{b 2^{-1}(a+b)^{-1}} \\ & \cdot [\|V_i\|_{L^{q/2}(R^3)} \|V_j\|_{L^{q/2}(R^3)}]^{a 2^{-1}(a+b)^{-1}}, \end{aligned} \tag{1.1}$$

*where $i, j$ are the same pair or not completely disjoint pairs. In (1.1) $a = -3p^{-1} + 1 > 0$, $b = 3q^{-1} - 1 > 0$, $1 \leqq q < 3 < p \leqq \infty$ and $m_j^{-1} = m_k^{-1} + m_l^{-1}$ with $j = (kl)$, $k < l$. The constant*

$$c_{ij} = 1 \quad \text{if} \quad i = j \quad \text{and} \quad c_{ij} = (m_k + m_l) m_k^{-1} \quad \text{for} \quad k < l,$$

*$i = (lm)$, $j = (kl)$; $c_{ij} = (m_k + m_l) m_l^{-1}$ for $k < l$, $i = (km)$, $j = (kl)$. For $i$, $j$ completely disjoint pairs we have*

$$\begin{aligned} \|[C_i R(z) D_j]\| & \leqq 2 \left(\sup_z \|[A_i R(z) A_i]\|\right)^{1/2} \left(\sup_z \|[A_j R(z) A_j]\|\right)^{1/2} \\ & \leqq 2 \max_{i,j} \left(\sup_z \|[A_i R(z) A_j]\|\right) \end{aligned} \tag{1.2}$$

*where the* sup *is taken over all $z$, $\operatorname{Im} z \neq 0$.*

By specializing theorems 1.5, 3.9 and 4.1 of Kato [1] to our case and using the same notation as in Lemma 1.1 we have

**Theorem 1.1.** *For some constant $N<\infty$ let $V$ be such that*

$$\max_{ij} 2(n(n-1)/2)\left[\left(\frac{2m_j c_{ij}}{4\pi}\right)(a^{-1}+b^{-1})\right] \cdot [\|V_i\|_{L^{p/2}(R^3)}\|V_j\|_{L^{p/2}(R^3)}]^{b2^{-1}(a+b)^{-1}} \cdot [\|V_i\|_{L^{q/2}(R^3)}\|V_j\|_{L^{q/2}(R^3)}]^{a2^{-1}(a+b)^{-1}} < N \tag{1.3}$$

*(the factor 2 occurring after* max *in (1.3) can be dropped for $n=2$ or 3) then for any real $k$ with $|k|<1/N$*

*a) $T(k)$ is the unique self-adjoint operator determined from the resolvent*

$$R(z,k)=R(z)-k[R(z)B^*](1+kQ(z))^{-1}AR(z) \tag{1.4}$$

*where $Q(z)=[AR(z)B^*]$ is a bounded operator from $H'$ to $H'$ and $\|Q(z)\|<N$ for* $\operatorname{Im} z \neq 0$.

*b) The operators $T$ and $T(k)$ are unitarily equivalent, i.e.*

$$T(k)=W_\pm(k)\,TW_\pm(k)^*$$

*where $W_\pm(k)$ are unitary and are defined by*

$$(W_\pm u,v)=(u,v)\mp(k/2\pi i)\int_{-\infty}^{+\infty}(AR(\lambda\pm i0)u, BR(\lambda\mp i0,k)^*v)\,d\lambda \tag{1.5}$$

*with $u,v\in H$. $W_\pm(k)$, $W_\pm(k)^*$ admit absolutely convergent (in operator norm) series expansions in $k$.*

*c) The operators $W_\pm(k)$, $W_\pm(k)^*$ of part b are the wave operators of scattering theory, i.e. on all $H$ we have*

$$W_\pm(k)=\operatorname*{s-lim}_{t\to\pm\infty} e^{iT(k)t}e^{-iTt}$$

$$W_\pm(k)^*=\operatorname*{s-lim}_{t\to\pm\infty} e^{iTt}e^{-iT(k)t}$$

*and the scattering operator $S=W_+(k)^*\,W_-(k)$ is unitary.*

*Remarks*

1. $T(k)$ defined through (1.4) agrees with the definition of $T(k)$ defined by the quadratic forms method of Simon [7] as each $V_i\in L^{3/2}(R^3)$ by (1.3) and $L^{3/2}(R^3)\subset R$(the Rollnik class).

2. In (1.5) $AR(\lambda\pm i0)u$ are $H'$ valued functions of $\lambda$ and exist a.e. for $-\infty<\lambda<\infty$ as boundary values of the analytic vector $AR(z)u$. Similar remarks apply to $BR(\lambda\mp i0,k)^*v$.

3. For the case $n=2$ Kato [1] (see Theorem 6.1) showed that $V\in R$ is sufficient for the conclusions of Theorem 1.1 to hold for small $k$.

## 2. Proof of Lemma and Theorem

*Proof of Lemma 1.1.* We will make estimates to establish sufficient conditions on the $V_i$ so that the operators

$$C_i R(z) D_j \qquad C_i, D_j = A_i, A_i^*, B_i, B_i^* \tag{2.1}$$

admit norm bounded analytic extensions, $[C_i R(z) D_j]$, where the bound is uniform for $\operatorname{Im} z \neq 0$.

More precisely, we will show

$$\begin{aligned}
&\sup \| [C_i R(z) D_j] \| \\
&< \left[ \left( \frac{4\pi}{2m_j} \right)^{-3p^{-1}} a_{ij}^{-3p^{-1}+1} (-3p^{-1}+1)^{-1} \| V_i \|_{L^{p/2}(R^3)}^{1/2} \| V_j \|_{L^{p/2}(R^3)}^{1/2} (c_{ij})^{3p^{-1}} \right. \\
&\left. + \left( \frac{4\pi}{2m_j} \right)^{-3q^{-1}} a_{ij}^{-3q^{-1}+1} (3q^{-1}-1)^{-1} \| V_i \|_{L^{q/2}(R^3)}^{1/2} \| V_j \|_{L^{q/2}(R^3)}^{1/2} (c_{ij})^{3q^{-1}} \right]
\end{aligned} \tag{2.2}$$

where $a_{ij} > 0$, $1 \leqq q < 3 < p \leqq \infty$ and $m_j^{-1} = m_k^{-1} + m_l^{-1}$ for $j = (kl)$, $k < l$. The constant $c_{ij} = 1$ if $i = j$ and $c_{ij} = (m_k + m_l) m_k^{-1}$ for $k < l$, $i = (lm)$, $j = (kl)$; $c_{ij} = (m_k + m_l) m_l^{-1}$ for $k < l$, $i = (km)$, $j = (kl)$. The expression (2.2) holds for $i, j$ the same pair or not completely disjoint pairs. Minimizing the right hand side of (2.2) with respect to the $a_{ij}$ we have

$$\begin{aligned}
\sup_z \| [C_i R(z) D_j] \| \leqq\ & (2m_j c_{ij}) (4\pi)^{-1} \cdot (a^{-1} + b^{-1}) \\
& \cdot [\| V_i \|_{L^{p/2}(R^3)} \| V_j \|_{L^{p/2}(R^3)}]^{b\,2^{-1}(a+b)^{-1}} \\
& \cdot [\| V_i \|_{L^{q/2}(R^3)} \| V_j \|_{L^{q/2}(R^3)}]^{a\,2^{-1}(a+b)^{-1}}
\end{aligned} \tag{2.2'}$$

where $a = -3p^{-1} + 1 > 0$ and $b = 3q^{-1} - 1 > 0$.

For $i, j$ completely disjoint pairs we have

$$\begin{aligned}
\sup_z \| [C_i R(z) D_j] \| &\leqq 2 \left( \sup_z \| [C_i R(z) C_i] \| \right)^{1/2} \left( \sup_z \| [D_j R(z) D_j] \| \right)^{1/2} \\
&\leqq 2 \max_{i,j} \left( \sup_z \| [C_i R(z) D_j] \| \right).
\end{aligned} \tag{2.3}$$

We now make the estimates required to establish (2.2).

Since $A_i = A_i^* \geqq 0$, $B_i = A_i U_i = U_i A_i$ where $U_i$ is a partial isometry it is sufficient to bound the norms $\| A_i R(z) A_j u \|$ where

$$u \in D(A_j) = D(A_j^*) = D(B_j) = D(B_j^*).$$

For the case of $i, j$ the same and $i, j$ different but not disjoint pairs we reduce the calculation to two body considerations and follow a method of Kato [1]. For $i, j$ disjoint pairs we reduce the problem to the case where the pairs are the same.

*Case 1.* $i=j$, i.e., $i=(kl)$. We have

$$\|A_i e^{-iTt} A_i u\|_2 = \|A_i e^{-iT_i t} A_i u\|_2 \tag{2.4}$$

where $T_i$ is the relative kinetic energy of the pair $i$. We show that

$$\|A_i e^{-iT_i t} A_i u\|_2 \leqq (4\pi t/2m_i)^{-3p^{-1}} \|A_i\|^2_{L^p(R^3)} \|u\|_2, \quad u \in D(A_i),\ t>0 \tag{2.5}$$

where $m_i^{-1} = m_k^{-1} + m_l^{-1}$ so that, using the representation

$$R(z)u = i\int_0^\infty e^{izt} e^{-iTt} u\,dt, \qquad \operatorname{Im} z > 0;$$

we have for $\operatorname{Im} z > 0$

$$\begin{aligned} \|A_i R(z) A_i u\|_2 &\leqq \int_0^\infty \|A_i e^{-iTt} A_i u\|_2\,dt \\ &\leqq \int_0^{a_{ii}} \|A_i e^{-iT_i t} A_i u\|_2\,dt + \int_{a_{ii}}^\infty \|A_i e^{-iT_i t} A_i u\|_2\,dt\,. \end{aligned} \tag{2.6}$$

For $\operatorname{Im} z < 0$ we obtain a similar majorization. We obtain (2.2) by substituting (2.5) with $p$ for the first integrand of (2.6) and (2.5) with $q$ replacing $p$ in the second integrand of (2.6). We now derive (2.5). Denote the coordinates by $(x, x_R)$ where $x \in R^3$ is the relative coordinate of the pair $i$ and $x_R$ are the other $3n-6$ coordinates. We have for $t>0$

$$\begin{aligned} &(e^{-iT_i t} A_i u)(x, x_R) \\ &\quad = (4\pi i t/2m_i)^{-3/2} \int \exp[-2m_i|x-y|^2/4it]\, A_i(y)\, u(y, x_R)\,dy\,. \end{aligned} \tag{2.7}$$

The spatial integrations are understood to be over $R^3$ and are to be interpreted as limit in mean relations. With $v(x, x_R) = \exp(-2m_i|x|^2/4it) \cdot A_i(x)\, u(x, x_R)$ we have, using the Hausdorff-Young inequality [1],

$$\left[\int |(e^{-iT_i t} A_i u)(x, x_R)|^{r'} dx\right]^{1/r'} \leqq C_i(r', t) \left[\int |v(x, x_R)|^r dx\right]^{1/r} \tag{2.8}$$

with $r'^{-1} + r^{-1} = 1$, $C_i(r', t) = (4\pi t/2m_i)^{3(r^{-1}-2^{-1})}$. But

$$\left[\int |v(x, x_R)|^r dx\right]^{1/r} \leqq \|A_i\|_{L^{2p}(R^3)} \left[\int |u(x, x_R)|^2 dx\right]^{1/2} \tag{2.9}$$

with $r^{-1} = 2^{-1} + (2p)^{-1}$. Thus

$$\begin{aligned} \int |(A_i e^{-iT_i t} A_i u)(x, x_R)|^2\,dx\,dx_R \leqq \int &\left[\int |A_i(x)|^{2s} dx\right]^{1/s} \\ &\cdot \left[\int |(e^{-iT_i t} A_i u)(x, x_R)|^{2s'} dx\right]^{1/s'} dx_R \end{aligned} \tag{2.10}$$

with $s^{-1} + s'^{-1} = 1$, $2s' = r'$, which upon substituting (2.8) and (2.9) in (2.10) gives (2.5).

*Case 2.* $i \neq j$ but $i$ and $j$ not disjoint pairs. It is sufficient and for definiteness we consider $i=(23)$, $j=(12)$. In analogy with case 1, (2.4), we have

$$\|A_{23}e^{-iTt}A_{12}u'\| \leqq \|A_{23}e^{-iT_jt}A_{12}u\|, \qquad u = e^{-i(T-T_j)t}u' \tag{2.11}$$

where in this case $T_j$ is the relative kinetic energy of the pair $j$. The relative coordinates of 1 and 2 we denote by $y_1 \in R^3$ and $y_1$ plays the role of the variable $x$ of case 1. We denote the coordinates by $(y_1, x_R)$. We have with $\|A_{12}u(\cdot, x_R)\|_{L^r(R^3, dy_1)} < \infty$,

$$v(y_1, x_R) = e^{-|y_1|^2 2m_j/4it}(A_{12}u)(y_1, x_R) \tag{2.12}$$

$m_j^{-1} = m_1^{-1} + m_2^{-1}$, and from the Hausdorff-Young inequality

$$\|e^{-iT_jt}A_{12}u(\,, x_R)\|_{L^r(R^3, dy_1)} \leqq C_j(r', t)\|v(\,, x_R)\|_{L^{r}(R^3, dy_1)} \tag{2.13}$$

where $r'^{-1} + r^{-1} = 1$. Then

$$\|A_{23}e^{-iT_jt}A_{12}u\|_2^2 = \int |A_{23}|^2 \, |(e^{-iT_jt}A_{12}u)(y_1, x_R)|^2 \, dy_1 \, dx_R \,. \tag{2.14}$$

We use the Holder inequality in $y_1$ with $s^{-1} + s'^{-1} = 1$ to obtain

$$\begin{aligned} \|A_{23}e^{-iT_jt}A_{12}u\|_2^2 &\leqq \int [\int |A_{23}|^{2s} dy_1]^{1/s} [\int |(e^{-iT_jt}A_{12}u) \\ &\qquad \cdot (y_1, x_R)|^{2s'} dy_1]^{1/s'} dx_R \\ &\leqq (m_{12}/m_1)^{3/s} \|A_{23}\|_{L^{2s}(R^3)}^2 C_j(r', t)^2 \\ &\qquad \cdot \int \|v(\,, x_R)\|_{L^r(R^3, dy_1)}^2 dx_R \end{aligned} \tag{2.15}$$

with $r' = 2s'$. In arriving at (2.15) we have used the fact that $A_{23}(x_2' - x_3') = A_{23}(m_1 m_{12}^{-1} y_1 + \sum_i b_i x_R^i)$, $x_2'$, $x_3'$ denote the coordinates of 2 and 3, $m_{12} = m_1 + m_2$, and where $\sum_i b_i x_R^i$ is a linear combination of the other $x_R$ coordinates (see Appendix 1 of Simon [2]). Thus we have

$$|\int |A_{23}|^{2s} dy_1|^{1/s} = (m_{12}/m_1)^{3/s} \|A_{23}\|_{L^{2s}(R^3)}^2 \,.$$

In the case where $A_{13}$ occurs in place of $A_{23}$ we have $A_{13}(x_1' - x_3') = A_{13}\left(m_2 m_{12}^{-1} y_1 + \sum_i c_i x_R^i\right)$. Using the Holder inequality again in (2.15) we obtain

$$\begin{aligned} \|v(\,, x_R)\|_{L^2(R^3, dy_1)}^2 &= [\int |A_{12}u(y_1, x_R)|^r dy_1]^{2/r} \\ &\leqq [\int |A_{12}|^{rk} dy_1]^{2/rk} [\int |u(y_1, x_R)|^{rk'} dy_1]^{2/rk'} \end{aligned} \tag{2.16}$$

where $k^{-1} + k'^{-1} = 1$. Let $rk' = 2$ then

$$\|A_{23}e^{-iT_jt}A_{12}u\|_2^2 \leqq (m_{12}/m_1)^{6/q} \|A_{23}\|_{L^q(R^3)}^2 \|A_{12}\|_{L^q(R^3)}^2 C^2(r', t) \|u\|_2^2 \tag{2.17}$$

with $q = 2r(2-r)^{-1}$ or as $\|u'\|_2 = \|u\|_2$ we have

$$\begin{aligned} &\|A_{23}e^{-iTt} A_{12}u'\|_2 \\ &\leqq (m_{12}/m_1)^{3/q} \|A_{23}\|_{L^q(R^3)} \|A_{12}\|_{L^q(R^3)} (4\pi t/2m_j)^{-3/q} \|u'\|_2 \end{aligned} \tag{2.18}$$

which is analogous to (2.5). Proceeding in the same manner as from (2.5) to (2.6) we obtain (2.2) for this case.

*Case 3.* $i \neq j$ with the pair $i$ disjoint from the pair $j$. We have for $\operatorname{Im} z > 0$ with the sup taken over $v \in D(A_j)$, $\|v\| = 1$,

$$\begin{aligned}
\|A_i R(z) A_j u\| &\leqq \sup \left| \int_0^\infty (v, A_i e^{iTt} A_j u)\, dt \right| \\
&= \sup \left| \int_0^\infty (e^{iT_j t} v, A_i e^{-i(T-T_j)t} A_j u)\, dt \right| \\
&= \sup \left| \int_0^\infty (A_j^* e^{iT_j t} v, A_i e^{-i(T-T_j)t} u)\, dt \right| \\
&\leqq \left( \sup \left( \int_0^\infty \|A_j^* e^{iT_j t} v\|^2\, dt \right)^{1/2} \right) \left( \int_0^\infty \|A_i e^{-iT_i t} u\|^2\, dt \right)^{1/2}
\end{aligned} \tag{2.19}$$

where $T_i$, $T_j$ are the relative kinetic energies of the pairs $i, j$ respectively. From Lemma 3.6, Eq. (3.9) and Theorem 5.1, Eq. (5.3) of Kato [1] we have (recalling that $A_k : D(A_k) \subset H \to H$)

$$\begin{gathered}
\int_{-\infty}^{\infty} \|A_k e^{-itT_k} w\|^2\, dt \leqq 2 \|w\|^2 \sup \|A_k R(z, T_k) A_k^* u'\| / \|u'\| \\
\operatorname{Im} z \neq 0, \qquad u' \in D(A_k^*), \qquad u' \neq 0
\end{gathered} \tag{2.20}$$

for $k = i$ or $j$, $w \in H$ so that

$$\begin{aligned}
\|A_i R(z) A_j u\| \leqq 2 &[\sup \|A_i R(z, T_i) A_i^* v'\| / \|v'\|]^{1/2} \\
&\cdot [\sup \|A_j R(z, T_j) A_j^* u'\| / \|u'\|]^{1/2} \|u\|
\end{aligned} \tag{2.21}$$

where the sup in (2.21) is taken over $\operatorname{Im} z \neq 0$, $v' \in D(A_i^*)$, $u' \in D(A_j^*)$, $v' \neq 0$, $u' \neq 0$. The resolvents $R(z, T_m)$ in (2.21) are the resolvents of the operators $T_m$. By noting that each term in the brackets of (2.21) just reduces to the type given by (2.6) we arrive at (2.3).

*Proof of Theorem 1.1.* Since $T$ is self-adjoint it is obvious that all the hypotheses of Theorems 1.5, 3.9 and 4.1 of Kato [1] are satisfied except for the $T$-smoothness [1, Definition 1.2] of $A$ and $B$ and that

$$\|A R(z) B^* u\| \leqq N \|u\|, \qquad \operatorname{Im} z \neq 0, \qquad u \in D(B^*) \subset H'. \tag{2.22}$$

Thus our proof consists in showing that these two hypotheses are satisfied. From Remark 1.10 of Kato [1] we see that $T$-smoothness of $A_i$ and $B_i$ imply that $A$ and $B$ are $T$-smooth. From the hypothesis (1.3) we see (referring to Lemma 1.1) that $\|[A_i R(z) A_i]\|$ is bounded uniformly for $\operatorname{Im} z \neq 0$. Since $A_i = A_i^* \geqq 0$, $B_i = A_i U_i = U_i A_i$ this implies the $T$-smoothness of $A_i$ and $B_i$ using Theorem 5.1, Eq. (5.3) of Kato [1].

Eq. (2.22) is valid since

$$\begin{aligned}
\|AR(z)B^*u\| &\leqq \sup_{\|v\|=1} |(v, AR(z)B^*u)| \\
&\leqq \sup_{\|v\|=1} \left[\sum_{ij} |(v_i, A_i R(z) B_j^* u_j)|\right] \\
&\leqq \sup_{\|v\|=1} \left[\left(\sum_{ij} \|v_i\|^2 \|u_j\|^2\right)^{1/2} \left(\sum_{ij} \|[A_i R(z) B_j^*]\|^2\right)^{1/2}\right] \\
&\leqq \max_{i,j} \|[A_i R(z) B_j^*]\| \, (n(n-1)/2) \, \|u\|
\end{aligned} \tag{2.23}$$

and by the hypotheses (1.3) the right-hand side of (2.23) is less than $N\|u\|$ for $\operatorname{Im} z \neq 0$.

## References


1. Kato, T.: Wave Operators and Similarity for Some Non-selfadjoint Operators. Math. Ann. **162**, 258—279 (1966).
2. Simon, B.: On the Infinitude or Finiteness of the Number of Bound States of an N-Body Quantum System, I. Helv. Phys. Acta **43**, (6) 607—630 (1970).
3. Hepp, K.: On the Quantum Mechanical $N$-Body Problem. Helv. Phys. Acta **42**, (3) 425—458 (1969).
4. Lavine, R.: Commutators and Scattering Theory. I. Repulsive Interactions. Commun. math. Phys. **21**, 301—323 (1971).
5. Balslev, E., Combes, J. M.: Spectral Properties of Many-body Schroedinger Operators with Dilatation-analytic Interactions. Commun. math. Phys. **22**, 280—294 (1971).
6. Albeverio, S.: (to appear in Ann. Phy.).
7. Simon, B.: Quantum Mechanics for Hamiltonians defined as Quadratic Forms. Princeton, New Jersey: Princeton University Press. 1971.



R. J. Iorio, Jr.
Departamento de Matemática
Ponificia Universidade Católica
Rio de Janeiro, Brasil