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Invariant Tensors in $SU(3)$. II

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Abstract. Invariant $SU(3)$ octet tensors are constructed in terms of λ matrices and applied to the problem of forming tensors from a single octet used repeatedly. Next a similar problem but with two octets is considered which demonstrates that different outer products of invariant tensors may be related. Finally, a theorem is proved which shows that the number of invariant tensors is essentially finite, and that relations of ranks greater than six exist on outer products of these tensors.

I. Introduction

The octet representation of $SU(3)$ appears to be of considerable significance in the consideration of symmetries of strong interactions, in that many of the stable particles correspond to this representation. Consequently, in many theories involving Lagrangians, for which $SU(3)$ or $SU(3)\otimes SU(3)$ are exact or approximate symmetries, octet fields arise, and it is frequently necessary to combine these fields to form scalars, vectors or tensors. This often occurs when considering non-linear realisations [1] of the chiral group or when constructing in a non-linear manner particular forms of symmetry breaking Lagrangians [2–4]; in many of these problems, however, it is not always a trivial matter to deduce just from simple observation either whether the most general form of a certain tensor has been found or, alternatively, whether certain tensors are linearly independent.

In an earlier paper [5] (hereafter referred to as (I)) we have made use of independent sets of invariant $8\otimes 8\otimes\cdots\otimes 8$ tensors to construct sets of independent tensors out of given finite numbers of octet fields. In this paper we proceed further with our considerations of these numerically invariant $SU(3)$ tensors, our analysis splitting broadly into four sections, which can be summarised as follows. First, in Section II, we show how the invariant tensors are just products of traces of products of λ matrices. In the next section we apply this to the problem of constructing the most general tensor formed from a single octet field to all orders in this field. This had been attempted previously [6, 7] by other workers in this field who required the form of the most general second rank tensor formed from a single octet. Using intuitive reasoning, they obtained a form for this second rank tensor but, as they themselves point out, this falls somewhat short of a proof; we shall prove in this section, however, that their tensor was indeed the most general one. The case of two octets used repeatedly is considered in Section IV, using similar techniques. There exists an additional problem in this case and the solution is made considerably more complicated, so we merely outline the method and quote the result for scalars. This result is then used to illustrate an interesting phenomenon concerning independence of products of independent scalars. This leads us to consider independence of outer products of invariant tensors and in Section V we prove a theorem which shows (a) that all invariant tensors are outer products of a small finite set of tensors and, (b) that relations exist between outer products of these tensors, these relations being of rank $\geqq 7$.

We point out that, although we restrict our considerations here to $SU(2)$ and $SU(3)$, our techniques apply equally well to any $SU(n)$.

II. A Spanning Set for Invariant $SU(3)$ Tensors

As in (I) we shall consider coupling $SU(3)$ octet vectors by means of invariant tensors. A vector A_i transforms by the law

$$[Q_i, A_j] = i f_{ijk} A_k \tag{1}$$

where the Q_i are the group generators, and the invariant tensors $H_{jk\ldots q}$ satisfy

$$f_{ijt} H_{tk\ldots q} + f_{ikt} H_{jt\ldots q} + \cdots + f_{iqt} H_{jk\ldots t} = 0\,. \tag{2}$$

Suppose that A_i is an $SU(3)$ octet vector, then by the usual procedure we can construct a corresponding traceless 3×3 matrix

$$A_\alpha{}^\beta = A_i(\lambda_i)_\alpha{}^\beta\,. \tag{3}$$

We shall use the well known relation

$$\lambda_i \lambda_j = \tfrac{2}{3}\delta_{ij} + (d_{ijk} + i f_{ijk})\lambda_k \tag{4}$$

which, together with the fact that the λ_i are traceless, leads immediately to the following

$$\mathrm{Tr}(\lambda_i\lambda_j) = 2\delta_{ij}\,, \tag{5}$$

$$4 d_{ijk} = \mathrm{Tr}\{\lambda_i, \lambda_j\}\lambda_k\,, \tag{6}$$

$$4 i f_{ijk} = \mathrm{Tr}[\lambda_i, \lambda_j]\lambda_k\,. \tag{7}$$

Eq. (5) can be used to invert Eq. (3) yielding

$$A_i = \tfrac{1}{2}\,\mathrm{Tr}(A\lambda_i)\,. \tag{8}$$

Now consider trying to form scalars out of the octets $A_\alpha{}^\beta, \ldots, G_\gamma{}^\delta$. When combining these matrices in a group covariant manner to form scalars, the only allowed operations are contractions with the tensors δ^α_β, $\varepsilon^{\alpha\beta\gamma}$ and $\varepsilon_{\alpha\beta\gamma}$ and we must keep account of upper and lower indices. For the greek labels are quark (or antiquark) labels, and the problem amounts to reducing out *all* the singlet representations from an outer product of quarks (and antiquarks). It is well known that symmetrising and antisymmetrising does totally reduce such an outer product of basic representations. So, by contracting with δ and ε tensors, we are sure to find all the available scalars. But the ε tensors must always appear in pairs because we have equal numbers of upper and lower indices to saturate. Consequently we may make use of the identity

$$\varepsilon_{\alpha\beta\gamma}\varepsilon^{\lambda\mu\nu} = \begin{vmatrix} \delta^\lambda_\alpha & \delta^\mu_\alpha & \delta^\nu_\alpha \\ \delta^\lambda_\beta & \delta^\mu_\beta & \delta^\nu_\beta \\ \delta^\lambda_\gamma & \delta^\mu_\gamma & \delta^\nu_\gamma \end{vmatrix} \tag{9}$$

and deduce that the only way to make scalars out of the octets $A_\alpha{}^\beta \ldots G_\gamma{}^\delta$ is to saturate with sums of δ tensors. Hence, a spanning set for the set of scalars formed out of the octets $A_\alpha{}^\beta \ldots G_\gamma{}^\delta$ consists of the terms of the form

$$(\delta_s{}^t \ldots \delta_u{}^v)\, A_\alpha{}^\beta \ldots G_\gamma{}^\delta \tag{10}$$

where the indices on the δ's take all possible combinations of the indices $\alpha, \beta, \ldots, \gamma, \delta$, with proper regard still being paid, of course, to the lower or upper nature of these indices. Using Eq. (3) the terms (10) may be written as

$$(\delta_s{}^t \ldots \delta_u{}^v)\,(\lambda_i)_\alpha{}^\beta \ldots (\lambda_k)_\gamma{}^\delta A_i \ldots G_k \tag{11}$$

which can be simply re-written as

$$[\mathrm{Tr}(\lambda_i \ldots \lambda_m) \ldots \mathrm{Tr}(\lambda_n \ldots \lambda_p)]\, A_i \ldots G_k \tag{12}$$

where the square brackets contain all possible products of traces of all products of the λ matrices $\lambda_i \ldots \lambda_k$. We conclude from this that a spanning set for the invariant octet tensors consists of all such products of traces of products of λ matrices. Note that we do not also have to consider products of traces of λ's of the form

$$\mathrm{Tr}(\lambda_i \ldots \lambda_m \lambda_s)\,\mathrm{Tr}(\lambda_n \ldots \lambda_p \lambda_s)$$

where indices are contracted between trace terms. Such invariant tensors are implicitly included in our set. Repeated applications of Eqs. (4)–(7) lead quickly to the result that the trace of a product of λ matrices is either δ_{ij} or the sum of contracted products of d and f tensors. Therefore, we also conclude that a spanning set for the invariant tensors consists of inner and outer products of δ, d and f tensors.

This result can be applied immediately in $SU(2)$ with regard to forming tensors out of a single vector used repeatedly. For $SU(2)$ the d_{ijk} tensors are zero and the structure constants are ε_{ijk}. The most general $SU(2)$ invariant (triplet) tensor is therefore a product of ε and δ tensors. For an even rank tensor we may eliminate all the ε tensors in favour of δ's using the identity (9), and for an odd rank tensor we may eliminate all but one. Consequently the general even rank tensor is simply a product of δ tensors, and the most general odd rank tensor a product of a single ε_{ijk} with repeated δ tensors. Now, if we saturate these tensors with an $SU(2)$ vector, like, say, the pion field π_i, we arrive at all the familiar $SU(2)$ results. For example, the most general second rank $SU(2)$ tensor of even order in pion fields is [1]

$$f(\pi^2)\delta_{ij} + g(\pi^2)\pi_i\pi_j\,. \tag{13}$$

Unfortunately it is impossible to proceed with corresponding problems in $SU(3)$ in the same way. This is because the general contracted product of f and d tensors does not obviously reduce to simple outer products of low order tensors as was the case with $SU(2)$. Instead we are obliged to resort to induction proofs.

III. Tensors Formed from a Single Octet Used Repeatedly

Suppose we are forming tensors from the octet A_i. We shall use the notation

$$X = A_i A_i\,, \tag{14}$$

$$Y = d_{ijk} A_i A_j A_k\,. \tag{15}$$

Then A satisfies its own characteristic equation giving [6]

$$F(A) \equiv A^3 - XA - \tfrac{2}{3}Y = 0\,. \tag{16}$$

The general scalar formed from A_i is found by contracting all the indices on the general invariant tensor against the vectors A_i. The previous section tells us, therefore, that it consists of sums of products of the terms $\mathrm{Tr}(A^s)$. But

$$\mathrm{Tr}(A^s) = X\,\mathrm{Tr}(A^{s-2}) + \tfrac{2}{3}Y\,\mathrm{Tr}(A^{s-3}) \tag{17}$$

using (16). Also

$$\mathrm{Tr}(A^0)=3$$
$$\mathrm{Tr}(A^1)=0$$
$$\mathrm{Tr}(A^2)=2X\,.$$

Therefore a simple induction argument tells us that the general scalar is simply a function of X and Y. Similarly we see that the general vector takes the form

$$f(X,Y)A_i+g(X,Y)d_{ijk}A_jA_k \tag{18}$$

also a spanning set for the second rank tensors formed from A_i consists of

$$\mathrm{Tr}(\lambda_i A^s \lambda_j B^t) \quad \text{and} \quad \mathrm{Tr}(\lambda_i A^p)\,\mathrm{Tr}(\lambda_j A^q) \tag{19}$$

to within functions of X and Y. The latter terms are just outer products of the vectors given in (18) and the former reduce down to

$$\mathrm{Tr}(\lambda_i A^s \lambda_j A^t): s,t=0,1,2$$

using Eq. (16). Evaluating all these terms leads to the ten second rank tensors found by the authors of Ref. [6] and [7], and is thus a justification for their choice.

IV. Scalars Formed from Two Octets

Applying the approach of the previous section to the question of forming tensors out of two octets A_i and B_j is far more complicated. We illustrate the problem for the case of scalars. The method of Section III quickly leads to the result that the general scalar from A_i and B_j is a sum of products of terms like

$$\mathrm{Tr}(A^s B^t \ldots A^p B^q): s,t,\ldots,p,q=1 \quad \text{or} \quad 2 \tag{20}$$

to within A_iA_i, B_iB_i, $d_{ijk}A_iA_jA_k$ and $d_{ijk}B_iB_jB_k$.

The added difficulties are now apparent; these terms may contain arbitrarily large numbers of couples A^lB^m and although for any given l and m we can use the characteristic equation for A^lB^m this is still not sufficient to reduce us to a finite set. However, we may make use of the identity

$$[AB,BA]=2(d_{ijk}A_iB_jB_k)A-2(d_{ijk}A_iA_jB_k)B+(B_iB_i)A^2-(A_iA_i)B^2 \tag{21}$$

to show that the general term (20) is equivalent to a term of the form

$$\mathrm{Tr}\{(A^2B^2)\ldots(A^2B^2)(AB)\ldots(AB)((AB^2))^n\} \quad \text{where} \quad n=0 \text{ or } 1\,. \tag{22}$$

Again this is most easily seen by setting up an induction argument. At this stage the characteristic equations for A^2B^2 and AB can be applied to reduce us to a finite set, and this can be reduced still further by means of Eq. (21). We deduce finally that the general scalar formed from A_i and B_j is built out of the following terms:

$$\begin{gathered} A_iA_i,\ A_iB_i,\ B_iB_i,\ d_{ijk}\,A_iA_jA_k,\ d_{ijk}\,A_iA_jB_k\,, \\ d_{ijk}\,A_iB_jB_k,\ d_{ijk}\,B_iB_jB_k,\ C_iC_i,\ d_{ijk}\,C_iC_jC_k \end{gathered} \tag{23}$$

where for brevity $C_i \equiv d_{ijk}A_jB_k$. The need for all the nine terms in (23) can be checked using the results of (I). In that paper we have shown that the number of scalars formed from A_i and B_j, each being used just three times, is the frequency of the trivial representation in the decomposition of

$$(8\otimes 8\otimes 8)_s \otimes (8\otimes 8\otimes 8)_s \tag{24}$$

and this is easily found to be six. Using the set (23) we can form only

$$\begin{aligned} & A_iA_iA_jB_jB_kB_k \\ & A_iB_iA_jB_jA_kB_k \\ & d_{ijk}A_iA_jA_k\,d_{pqr}B_pB_qB_r \\ & d_{ijk}A_iA_jB_k\,d_{pqr}B_pB_qA_r \\ & C_iC_iA_jB_j \\ & d_{ijk}C_iC_jC_k \end{aligned} \tag{25}$$

and this therefore shows that $d_{ijk}C_iC_jC_k$, for instance, is not dependent on the others. We can continue checking in this way for scalars formed from p vectors A_i and q vectors B_j, and, for values of p and q less than six, the numbers of scalars found by the method in (I) agree exactly with the counts of products of the terms (23). However, when p and q are both six we find a discrepancy. There are forty three different products of the terms (23), but the method of (I) says that only forty two of these are independent. This implies that, although the nine terms in (23) are all independent, in the sense that no single one of them can be written as a sum of products of the others, there exist relations between them at higher orders.

V. Relations between Outer Products of Tensors

The relationships between products of scalars indicate that there exist relationships between outer products of invariant $SU(3)$ tensors. The aim of this final section is to show how such outer products arise, and as a result of this investigation we shall deduce that the number of invariant tensors is essentially finite, for any invariant tensor can be written as a sum of outer products of a finite set of tensors.

In Section II we remarked that the invariant tensors are simply products of traces of products of λ matrices. Consequently, any relations which exist on the invariant tensors arise either through trivial relations among the traces of products of matrices, or through non-trivial relations on the λ matrices themselves. Effectively, the only relations at our disposal are Eq. (4) and the requirements that the λ_i are hermitian and traceless. Applying Eq. (4) to the trace of a product of λ matrices, however, serves only to convert this factor into its equivalent of contracted f_{ijk} and d_{ijk} tensors. Clearly, to obtain relations between outer products of the trace factors, we must use relations on λ matrices involving no contractions. Such a relation is obtained from Eq. (16) by removing the vectors A_i:

$$\begin{aligned}&\lambda_i\lambda_j\lambda_k+\lambda_i\lambda_k\lambda_j+\lambda_j\lambda_k\lambda_i+\lambda_j\lambda_i\lambda_k+\lambda_k\lambda_i\lambda_j+\lambda_k\lambda_j\lambda_i\\&\quad=2(\delta_{ij}\lambda_k+\delta_{jk}\lambda_i+\delta_{ki}\lambda_j)+4d_{ijk}\,.\end{aligned}\tag{26}$$

It seems likely that all other outer product relations on the λ_i can be deduced from Eq. (26), since this is the statement that any (not necessarily traceless) 3×3 hermitian matrix satisfies its own characteristic equation. If, however, by some chance, there should exist other independent relations, the theorem which follows will be unaffected, since it does not depend on the particular form of these relations. Hence, without loss of generality in the following, we shall be able to assume that Eq. (26) is the only uncontracted relation on the λ_i.

Theorem. (a) All invariant tensors are outer products of members of a small finite set of tensors. This set consists of certain contractions of d_{ijk} and f_{ijk} tensors up to sixth rank.

(b) There exist relations on uncontracted products of members of this set at seventh and higher ranks.

Proof. (1) Suppose we have constructed sets of independent invariant tensors up to the k^{th} rank. Some of these will be written as outer products of tensors and some cannot; call all the ones that cannot "primitive invariant tensors" (primitives).

(2) Assume that there exist no identities of up to k^{th} rank on outer products of the primitives we have found of rank less than k.

(3) Consider the $(k+1)^{\text{th}}$ rank tensors. A spanning set for these is all traces of $(k+1)$ λ's (let us call these $\mathrm{Tr}(\lambda^{(k+1)})$ terms), together with all outer products of the primitives we have found up to k^{th} rank. We try to pick the independent ones out of these by writing down sets of relations using Eq. (26).

(4) There is no point in applying Eq. (26) to the outer products of primitives. This would merely give us relations which we have used at rank less than $(k+1)$ to pick the primitives we have. Consequently we apply Eq. (26) in all possible ways on the $\mathrm{Tr}(\lambda^{(k+1)})$ terms, and obtain constraint equations on our spanning set.

(5) If there are not many independent constraints, we can eliminate some of the $\mathrm{Tr}(\lambda^{(k+1)})$ terms in favour of the others and the outer products of lower primitives. As a result we (a) pick new independent primitives of rank $(k+1)$, (b) obtain no $(k+1)^{\text{th}}$ rank identities on the outer products of lower primitives. In this case we may proceed to the next rank; our assumptions (1) and (2) hold for k replaced by $(k+1)$.

(6) But Eq. (26) may impose so many constraints that all the $\mathrm{Tr}(\lambda^{(k+1)})$ terms are constrained away and we obtain no $(k+1)^{\text{th}}$ rank primitives. In this case we may also obtain relations on $(k+1)^{\text{th}}$ rank outer products of primitives of rank $\leqq (k-1)$. Then this will be the lowest rank constraint on outer products of primitives, because of assumption (2).

(7) If part (6) holds, then all $\mathrm{Tr}(\lambda^n)$ terms with $n \geqq k+1$ may also be written as sums of outer products of lower rank primitives. For

$$\begin{aligned}\mathrm{Tr}(\lambda^{(n+1)}) &= \mathrm{Tr}(\lambda_i \dots \lambda_p \lambda_q \lambda_r) \\ &= \tfrac{2}{3}\,\delta_{qr}\,\mathrm{Tr}(\lambda_i \dots \lambda_p) + (d_{qrt} + i f_{qrt})\,\mathrm{Tr}(\lambda_i \dots \lambda_p \lambda_t)\,.\end{aligned} \tag{27}$$

This completes the first part of the proof, which may be summarised as follows. If, at a certain rank, there exist invariant tensors which are not outer products of lower rank tensors, then outer products at this rank are all independent. But, if there exists none, there exists none at any higher rank.

The second part of the proof is then simply a combination of the counting method described in (I) and the results above. We present the final answer in the following Table.

Rank	No. of independent tensors	No. of outer products	No. of new primitives
2	1	0	1
3	2	0	2
4	8	3	5
5	32	20	12
6	145	130	15
7	702	812	0
>7			0

This completes the proof of the theorem. It is clear that any invariant octet tensor can be written as a sum of outer products of 35 tensors.

It is also clear that at ranks $\geqq 7$ there exist relations on outer products of these. We are, of course, familiar with this situation in $SU(2)$, where there exist relations such as Eq. (9) or

$$\delta_{ip}\varepsilon_{jqr} + \delta_{iq}\varepsilon_{jrp} + \delta_{ir}\varepsilon_{jpq} = \delta_{ij}\varepsilon_{pqr}. \tag{28}$$

We conclude by discussing one further point which now arises. We have found that the general scalar formed from one octet must be a function of X and Y, the two basic scalars defined in Eqs. (14) and (15). But the work of the last two sections raises the following question. Can we be really sure that there does not exist some high order identity relating powers of X and Y? If the answer is no, we may have to rethink a lot of the earlier work on $SU(3)$ and $SU(3)\otimes SU(3)$. We can easily see, however, that the answer to this question is yes. We simply go through the proof of our theorem when it is applied to this particular case. First note that when we totally contract any equation on outer products of λ matrices with vectors A_i we arrive at Eq. (16). This is because (for general vectors) $f(\xi)$ is the minimal polynomial of A and hence divides any polynomial $g(\xi)$ satisfying

$$g(A) = 0. \tag{29}$$

Then to the terms $\mathrm{Tr}(\lambda^{(k+1)})$ there corresponds only one term, namely $\mathrm{Tr}(A^{(k+1)})$. So at each order we have only one constraint and this just eliminates the one possible primitive at this order but places no constraints on the products of X and Y.

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References

1. Weinberg, S.: Phys. Rev. **166**, 1568 (1968).
2. Gasiorowicz, S., Geffen, D. A.: Rev. Mod. Phys. **41**, 531 (1969).
3. Gell-Mann, M., Oakes, R., Renner, B.: Phys. Rev. **175**, 2195 (1968).
4. Barnes, K. J., Dittner, P., Dondi, P. H.: Nucl. Phys. B**31**, 195 (1971).
5. Dittner, P.: Commun. math. Phys. **22**, 238 (1971).
6. Macfarlane, A. J., Sudbery, A., Weisz, P. H.: Commun. math. Phys. **11**, 77 (1968).
7. Macfarlane, A. J., Sudbery, A., Weisz, P. H.: Proc. Roy. Soc. London A**314**, 217 (1970).

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