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"Weighted Dispersion Relations" and Cut to Cut Extrapolations

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Abstract. Using error-affected data for the scattering amplitude on a part of the cuts, we construct a stable extrapolation procedure for it to the remaining parts of the cuts (higher energies or crossed reactions), using Carleman-weighted dispersion relations. To this end, it is assumed that the amplitude satisfies on the cut some smoothness condition of the Hölder type.

1. Introduction and Statement of the Problem

The results contained in this paper were obtained during our stay at Nordita in 1969 and have already been presented in a seminar at the 1969-Lund Conference. This delay in publishing them is due to the fact we tried to optimize our extrapolation procedures. So far we managed to do this optimization only for extrapolations to the interior points [1]; the problem seems to be more difficult for boundary points and so we decided to publish these available partial results.

One often meets need of extrapolating the available experimental data either along the same cut in order to get informations about the high or intermediate energy behaviour from the low energy region, or to the crossed cut to obtain e.g. the partial waves of the crossed reaction.

One way of performing this would, of course, be that of first bringing parts of the second sheet onto the first sheet by a suitable conformal mapping such that the points of the cut become interior points of the analyticity domain D under consideration, and then applying the produre described in [1]. Unfortunately, one is faced with the lack of knowledge about the position of the singularities on the second sheet so that the usual analyticity requirements might be disobeyed. Although one can first proceed to the location of these singularities as is done in [2] and the results of the above extrapolation might be quite good, the aim of this paper is to present an extrapolation method to the cuts based solely upon some smoothness conditions to be satisfied by the scattering amplitude on these cuts.

Therefore, assume the amplitude f is analytic in a domain D with boundary Γ, and that the data are known with an error ε on some limited part Γ_1 of Γ in the form of a complex function $h(z)$. Via a suitable conformal mapping [3] one can always take the domain D to be

a) the unit disk $|\zeta|<1$, if D is simply connected;

b) a circular ring $1<|\zeta|<R$, if D is doubly connected.

Thus, on the known cut Γ_1, the amplitude $f(\zeta)$ passes through the error-channel

$$|f(\zeta)-h(\zeta)|\leqq\varepsilon\,,\qquad \zeta\in\Gamma_1 \tag{1}$$

but in order to make the extrapolation stable one usually has to assume that the amplitude is bounded on the unknown cut too:

$$|f(\zeta)|\leqq M\,,\qquad \zeta\in\Gamma_2\equiv\Gamma\setminus\Gamma_1\,. \tag{2}$$

Indeed, from the Nevanlinna principle, it follows that any two functions $f_1(\zeta), f_2(\zeta)$ holomorphic in D and satisfying (1) and (2), differ, at any point ζ inside D, at most by

$$|f_1(\zeta)-f_2(\zeta)|\leqq(2\varepsilon)^{1-\omega(\zeta)}(2M)^{\omega(\zeta)} \tag{3}$$

where $\omega(\zeta)$ is the usual harmonic measure, i.e. that function, harmonic in D which is zero on Γ_1 and equal to one on Γ_2. If the error ε on Γ_1 goes to zero, the r.h.s. vanishes for each interior point (as $\omega(\zeta)\neq 1$) which proves the stability.

However, on Γ_2 according to (3) we can only state that the difference between f_1 and f_2 is less than $2M$, no matter how small ε is, since its exponent vanishes here identically. Moreover, by taking an apropiate example, one can show that among all the f's consistent with (1) and (2) and without any aditional assumptions, there are some whose difference *is* $2M$ on Γ_2; an example of this kind is furnished in the case of an identically vanishing data function $h(\zeta)$ on Γ_1, by

$$f_\pm(\zeta)=\pm\varepsilon^{1-(\omega+i\tilde\omega)}\,M^{\omega+i\tilde\omega}\,.$$

(D is supposed to be simply connected and $\tilde\omega$ is the harmonic conjugate of ω.) On Γ_2, the difference between the two "possible" amplitudes f_+ and f_- is $|f_+(\zeta)-f_-(\zeta)|=2M$, independentely of ε.

Nevertheless, one can turn the problem of extrapolation to the cuts into a stable one if one assumes that the amplitude satisfies a Hölder condition on some p'th derivative on Γ_2:

$$|f^{(p)}(\zeta)-f^{(p)}(\zeta')|\leqq A|\zeta-\zeta'|^\alpha\,,\qquad \zeta,\zeta'\in\Gamma_2$$
$$p=0,1,2\,\ldots,\qquad 0<\alpha\leqq 1 \tag{4}$$

Any function satisfying (1) and (4) might be the true amplitude; call then the set of all these admissible amplitudes $\mathscr{F}(\varepsilon, h, \not{p}, \alpha, A)$ and let the set of complex values these amplitudes take on, at a certain point ζ of the boundary be called $T(\zeta)$. Ideally the complete extrapolation theory should provide a complete knowledge of the set of values $T(\zeta)$. This problem was solved for interior points under conditions (1) and (2) in Ref. [1], where it was also proved that $T(\zeta)$ is a disk and where explicit formulae for its centre – which obviously represents the optimal extrapolated value for the amplitude – and for its radius, were given.

Unfortunately, we were not yet able to find this kind of solutions also for the present case. Nevertheless it was possible to give a stable extrapolation procedure up to the boundary and to obtain two functions – an extrapolated amplitude $\check{h}(\zeta)$ and a majorant $\eta(\zeta)$ of the radius of the set of possible values $T(\zeta)$.

$$\sup_{f \in \mathscr{F}} |f(\zeta) - \check{h}(\zeta)| \leqq \eta(\zeta) \tag{5}$$

so that even on the unknown cut $\zeta \in \Gamma_2, \eta(\zeta) \to 0$ if $\varepsilon \to 0$.

The result is given in the form of a dispersion relation weighted with a "truncated Carleman function" defined to satisfy

$$|\mathscr{C}(\zeta)| = 1 \text{ for } \zeta \in \Gamma_1, \qquad |\mathscr{C}(\zeta)| = e^{-\lambda} \text{ for } \zeta \in \Gamma_2 \tag{6}$$

as in the Poisson weighted dispersion relations used in [3, 4] for extrapolations to interior points, *the difference* to the latter being that the Poisson kernel is replaced by a conventional Cauchy one. As it is known, the effect of the Carleman[1] weight function for interior points is the lowering of the contribution of the unknown cut Γ_2 by a factor $e^{-\lambda}/|\mathscr{C}(\zeta)|$; this factor tends to *one* as ζ tends to Γ_2, nevertheless, as it will be shown in the following sections, the stability of the extrapolation is still secured, owing to the damping produced by the *quick oscillatory* behaviour of the Carleman function on Γ_2.

In Sections 2, 3 we treat some simple cases of doubly, respectively simply connected domains. In the Section 4 we give an outline of proof for more complex situations.

[1] This kind of weight functions are known to mathematicians under the name of exterior functions or outer functions.

The name of "truncated Carleman functions" was introduced by us at the 1969-Lund Conference, in analogy with that of "Carleman functions" (more correct-"kernels") introduced by Lavrentiev [5], but this name appears to have produced much confusion: references to Carleman's book on quasianalytic functions are floating around although there is no trace in it of "Carleman functions" at all! May be it would be better to name them "exterior function weight factors".

2. Doubly Connected Domains

If, in the energy or momentum transfer plane the amplitude has two disjoint cuts, using the mapping (3.4) of Ref. [6] the holomorphy domain D becomes the ring $1<|\zeta|<R$. Further, if the data are known on the

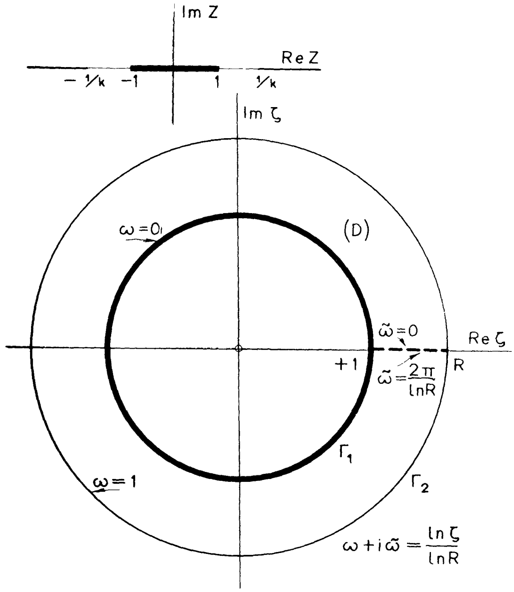

Fig. 1. The doubly connected z-cut plane and the corresponding ζ-ring. Here $\zeta(z)=i\exp\left\{-\frac{i\pi}{2K}\int\limits_0^z[(1-t^2)(1-k^2t^2)]^{-1/2}\,dt\right\}$ where $K=\int\limits_0^1[(1-t^2)(1-k^2t^2)]^{-1/2}\,dt$ and $R=\zeta(\frac{1}{k})$. If $\lambda=n\ln R$, then $\exp\{\lambda(\omega(\zeta)+i\tilde{\omega}(\zeta))\}$ is uniform and equals ζ^n

inner circle, we take a weight function of the following form

$$\mathscr{C}(\zeta)=\exp\left(-\lambda\frac{\ln\zeta}{\ln R}\right)\equiv\zeta^{-n} \tag{7}$$

where $n\equiv\lambda/\ln R$ is an integer. Then, denoting by

$$\check{h}(\zeta)=\frac{\zeta^n}{2\pi i}\int\limits_{|\zeta'|=1}\frac{h(\zeta')}{\zeta'^n(\zeta'-\zeta)}\,d\zeta' \tag{8}$$

the extrapolated amplitude and using the Cauchy integral formula, we get for $1<|\zeta|<R$ and every admissible amplitude $f\,(f\in\mathscr{F})$

$$|f(\zeta)-\check{h}(\zeta)|\,|\zeta|^{-n}\leqq\varepsilon K(\zeta)+\left|\frac{1}{2\pi i}\int\limits_{|\zeta'|=R}\frac{f(\zeta')}{\zeta'^n(\zeta'-\zeta)}\,d\zeta'\right| \tag{9}$$

where

$$K(\zeta) = \frac{1}{2\pi} \int_{|\zeta'|=1} \frac{|d\zeta'|}{|\zeta' - \zeta|}.$$

The key of the proof of the stability of the extrapolation (8) on Γ_2 rests in a more careful evaluation of the last term in Eq. (9). For $|\zeta| < R$, one can expand $\frac{1}{\zeta' - \zeta}$ in a series

$$\frac{1}{\zeta' - \zeta} = \frac{1}{\zeta'} \sum_0^\infty \left(\frac{\zeta}{\zeta'}\right)^k. \tag{10}$$

Using (10) and the fact that the sum and the integral can be interchanged, we notice the following sequence of equalities

$$\begin{aligned} &\frac{1}{2\pi i} \int_{|\zeta'|=R} f(\zeta') \frac{1}{\zeta'^n(\zeta' - \zeta)} d\zeta' \\ &\quad = \frac{1}{2\pi} \int_{|\zeta'|=R} \left(\sum_{l=-\infty}^{\infty} f_l \zeta'^l\right) \zeta'^{-n} \left(\sum_{k=0}^{\infty} \left(\frac{\zeta}{\zeta'}\right)^k\right) d\theta \\ &\quad = \sum_{k=0}^{\infty} f_{k+n} \zeta^k = \frac{1}{\zeta^n} \sum_{k=n}^{\infty} f_k \zeta^k \equiv \frac{1}{\zeta^n} R_n(\zeta) \end{aligned}$$

where $R_n(\zeta)$ is the positive frequency remainder ($k \geqq n$) of the Laurent series of the amplitude. From this evaluation and from Eq. (9) it results:

$$|f(\zeta) - h(\zeta)| \leqq \varepsilon K(\zeta) |\zeta|^n + |R_n(\zeta)|. \tag{11}$$

A known theorem from Fourier series theory [7] ensures that under condition (4) there exists a constant B such that

$$|R_n(\zeta)| \leqq B \frac{|\zeta|^n}{R^n} \frac{\ln n}{n^{\alpha+\not{h}}}. \tag{12}$$

So on Γ_2 we have

$$|f(\zeta) - \check{h}(\zeta)| \leqq \varepsilon K(\zeta) R^n + B \frac{\ln n}{n^{\alpha+\not{h}}}. \tag{13}$$

Let us look at formula (13). If ε is small, the higher is n and the smaller is the r.h.s. of Eq. (13). When n reaches the critical value defined by

$$\frac{d}{dn}\left(\varepsilon K(\zeta) R^n + B \frac{\ln n}{n^{\alpha+\not{h}}}\right) = 0, \tag{14}$$

i.e. [6]

$$n_c \approx \ln \frac{B(\not{h} + \alpha)}{\varepsilon K(R) \ln R} \Big/ \ln [R(1.44)^{\alpha+\not{h}}]$$

the r.h.s. of Eq. (13) reaches its lowest value

$$|f(\zeta)-\check{h}(\zeta)| \leqq B\frac{\ln n_c}{n_c^{\alpha+\not{p}}}\left[1+\frac{\alpha+\not{p}}{n_c \ln R}\right] \tag{15}$$

which proves the stability on the "*unknown*" cut Γ_2.

3. Simply Connected Domains

If the scattering amplitude is holomorphic in a complex plane (energy or momentum transfer) with nondisjoint cuts and if the data are given on a connected part of the cuts, then using the mappings (2.2)–(2.6) of Ref. [3], the holomorphy domain becomes the unit disk

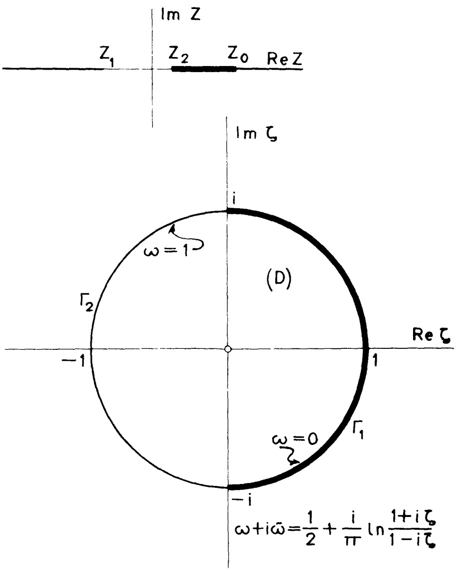

Fig. 2. Simply connected z-cut plane and the corresponding canonical ζ-domain. The Amplitude is supposed to be known on Γ_1 between z_2 and z_0. Here $\zeta(z)=(1-\sqrt{1-u^2})/u$ with $u=[(z_1+z_2-2z_0)\,z+z_0(z_1+z_2)-2z_1z_2]/[(z-z_0)\,(z_2-z_1)]$ and $\omega+i\tilde{\omega}=-i/\pi\ln\{[(u+1)^{1/2}+(u-1)^{1/2}]/[(u+1)^{1/2}-(u-1)^{1/2}]\}$

$|\zeta|<1$, in such a way that the data come on the right-hand semicircle. In this case

$$W(\zeta)\equiv\omega(\zeta)+i\tilde{\omega}(\zeta)=\frac{1}{2}-\frac{2}{\pi}\operatorname{arctg}\zeta=\frac{1}{2}+i\ln\frac{1+i\zeta}{1-i\zeta}. \tag{16}$$

As for the doubly connected case, we start with the following weighted Cauchy integral formula

$$f(\zeta)=\frac{1}{2\pi i}\int_{|\zeta'|=1}\frac{f(\zeta')\mathscr{C}_\lambda(\zeta')}{(\zeta'-\zeta)\mathscr{C}_\lambda(\zeta)}\,d\zeta' \qquad (17)$$

$$\mathscr{C}_\lambda(\zeta)=\exp\{-\lambda W(\zeta)\}$$

Then denoting by

$$\check{h}(\zeta)=\frac{1}{2\pi i}\int_{\Gamma_1}\frac{h(\zeta')\mathscr{C}_\lambda(\zeta')}{(\zeta'-\zeta)\mathscr{C}_\lambda(\zeta)}\,d\zeta' \qquad (18)$$

the interpolated amplitude, we have for every $|\zeta|\leqq 1$

$$|f(\zeta)-\check{h}(\zeta)|\leqq\frac{1}{|\mathscr{C}_\lambda(\zeta)|}\left\{\varepsilon K(\zeta)+\left|\frac{1}{2\pi i}\int_{\Gamma_2}\frac{f(\zeta')\mathscr{C}_\lambda(\zeta')}{\zeta'-\zeta}\,d\zeta'\right|\right\} \qquad (19)$$

where

$$K(\zeta)=\frac{1}{2\pi}\int_{\Gamma_1}\frac{|d\zeta'|}{|\zeta'-\zeta|}\,.$$

Once again we prove the stability by examining more closely the last term of inequality (19). Let us suppose that $\zeta\to\zeta_0\in\Gamma_2$. We divide Γ_2 into two parts: the first, Δ, including the point ζ_0 and strictly included in Γ_2 and the second, $\Gamma_2\setminus\Delta$, the remainder of Γ_2. As for every $\zeta'\in\Gamma_2$, $\omega(\zeta')=1$, and for every $\zeta\notin\Gamma_2$, $\omega(\zeta)<1$, we can write

$$\frac{1}{\zeta'-\zeta}=\frac{W(\zeta')-W(\zeta)}{\zeta'-\zeta}\int_0^\infty dx\exp\{-x(W(\zeta')-W(\zeta))\}. \qquad (20)$$

We now introduce (20) into the last term of inequality (19) and using the fact that we are allowed to interchange the integrals, one obtains

$$\frac{1}{2\pi i}\int_{\Gamma_2}\frac{f(\zeta')\mathscr{C}_\lambda(\zeta')}{(\zeta'-\zeta)\mathscr{C}_\lambda(\zeta)}\,d\zeta'=\frac{1}{2\pi i}\int_{\Gamma_2\setminus\Delta}\frac{f(\zeta')\mathscr{C}_\lambda(\zeta')}{(\zeta'-\zeta)\mathscr{C}_\lambda(\zeta)}\,d\zeta'+R_\lambda(\zeta) \qquad (21)$$

where

$$R_\lambda(\zeta)=\int_\lambda^\infty dy\,e^{yW(\zeta)}\left(\frac{1}{2\pi i}\int_\Delta\frac{f(\zeta')(W(\zeta')-W(\zeta))}{\zeta'-\zeta}\,e^{-yW(\zeta')}d\zeta'\right). \qquad (22)$$

Both terms in r.h.s. of Eq. (21) go to zero as $\lambda\to\infty$ even for $\zeta\to\zeta_0$. To see this let us make the following change of variables $u=\tilde{\omega}(\zeta')$ $\left(\text{as } \frac{\partial\tilde{\omega}}{\partial\zeta'}=\frac{\partial\omega}{\partial n}\neq 0 \text{ the change of variable is nonsingular}\right)$. After this, the first term in the r.h.s. of Eq. (21) can be regarded as the Fourier transform $G_{\zeta_0}(\lambda)$ of the following function

$$g_{\zeta_0}(\zeta)=\begin{cases}\dfrac{f(\zeta'(u))\,(u-\tilde{\omega}(\zeta_0))}{e^{iu_0}(\zeta'(u)-\zeta_0)\dfrac{\partial}{\partial\zeta'}\tilde{\omega}(\zeta'(u))}; & \zeta'\in\Gamma_2-\Delta\\ 0 & \zeta'\in\Delta\end{cases}$$

which is obviously absolutely integrable and piecewise satisfies a Hölder condition of order $\alpha+\not{p}$, wherefrom it follows that $G_{\zeta_0}(\lambda)\underset{\lambda\to\infty}{\simeq}\dfrac{1}{\lambda^{\alpha+\not{p}}}$.
The second term can be written as

$$R_\lambda(\zeta_0)=\int_\lambda^\infty dy e^{-iy\tilde{\omega}(\zeta_0)}\left(\int_{-\infty}^{\infty} g'_{\zeta_0}(u)\,e^{iuy}du\right) \tag{23}$$

where

$$g'_{\zeta_0}(u)=\begin{cases}0 & ; \quad \zeta'(u)\in\Gamma_2\setminus\Delta\\ \dfrac{f(\zeta'(u))\,(u-\tilde{\omega}(\zeta_0))}{(\zeta'(u)-\zeta_0)\dfrac{\partial}{\partial\zeta'}\tilde{\omega}(\zeta'(u))}; & \zeta'(u)\in\Delta\end{cases}$$

which is also a piecewise $\alpha+\not{p}$ Hölder function and (23) can be regarded as a remainder of a Fourier integral and thus goes to zero as $\dfrac{\ln\lambda}{\lambda^{\alpha+\not{p}}}$ as $\lambda\to\infty$. So we have proved that

$$\left|\frac{1}{2\pi i}\int_{\Gamma_2}\frac{f(\zeta')\,\mathscr{C}_\lambda(\zeta')}{(\zeta'-\zeta_0)\,\mathscr{C}_\lambda(\zeta_0)}\right|\leqq B\frac{\ln\lambda}{\lambda^{\alpha+\not{p}}}$$

and the remainder of the proof of the stability follows step by step the one for the doubly connected case.

4. More Complex Cases

In the previous sections we have treated some simple cases in which the part of the boundary where data were given was connected and also the precision of the measurement (i.e. ε) was constant. In more realistic cases ε is not constant and the parts of the boundary where data are given are disconnected. (i.e. Γ_1 is made up of disjoint pieces of boundary). In the following we give an outline of the proof of stability for these more general cases.

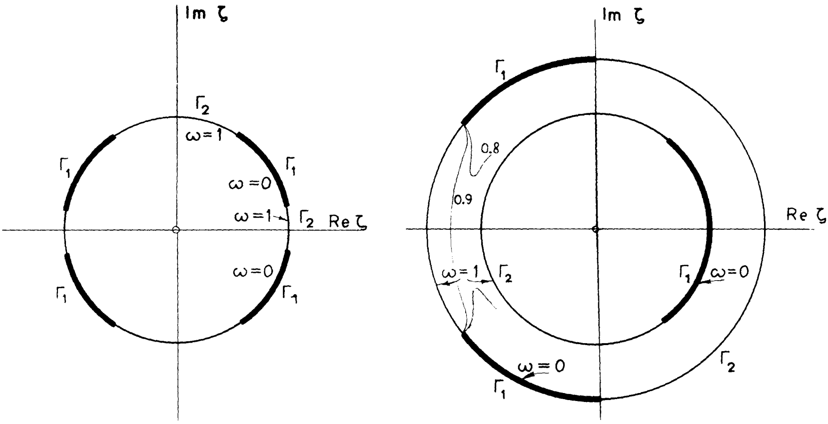

Fig. 3. More general data locations

Let $\varepsilon = \sup_{\zeta \in \Gamma_1} \varepsilon(\zeta)$, and $\mathfrak{w}(\zeta)$ the solution of the Dirichlet problem

$$\mathfrak{w}(\zeta) = \frac{1}{2\pi} \int_{\Gamma_1} \frac{\partial G}{\partial n} \ln \frac{\varepsilon(\zeta')}{\varepsilon} |d\zeta'|$$

and $\tilde{\mathfrak{w}}(\zeta)$ its harmonic conjugate. Further, let $\omega(\zeta)$ again be the harmonic measure, i.e.

$$\omega(\zeta) = \frac{1}{2\pi} \int_{\Gamma_2} \frac{\partial G}{\partial n} |d\zeta'| .$$

Then it is natural to choose a weight function of the form

$$\mathscr{C}_\lambda(\zeta) = e^{-\lambda(\omega(\zeta) + i\tilde{\omega}(\zeta))} e^{-(\mathfrak{w}(\zeta) + i\tilde{\mathfrak{w}}(\zeta))}$$

where λ is a positive parameter. For the doubly connected case we must keep only those values of λ for which $\mathscr{C}_\lambda(\zeta)$ is singlevalued (for example, in the case treated in Section 2, $\lambda = n \ln R$ where n is integer).

Now, in general, the proof of the stability rests on the evaluation of an integral of the following type:

$$\frac{1}{2\pi i} \int_{\Gamma_2 = \bigcup_i \Gamma_{2i}} \frac{f(\zeta')\, \mathscr{C}_\lambda(\zeta')}{(\zeta' - \zeta)\, \mathscr{C}_\lambda(\zeta)} d\zeta' .$$

The procedure developed in the previous section can be equally well applied, so a inequality like (15) is valid in this general case.

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