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Ultralocal Quantum Field Theory in Terms of Currents

I. The Free Theory

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Abstract. The paper considers the possibility of constructing ultralocal theories, whose Hamiltonians contain no gradient terms and are therefore diagonal in position space, entirely in terms of currents with an equal time current algebra replacing the canonical commutation relations. It is shown that the free current theory can be defined in terms of a certain representation of the current algebra related to the group, $SL(2, R)$. This representation is then constructed by using certain results of Araki and in the process a new infinitely divisible state on the universal covering group of $SL(2, R)$ is displayed. An ultralocal free theory can also be constructed for the canonical fields, and its relation to the free current theory is shown to involve a certain renormalization procedure reminiscent of the thermodynamic limit.

1. Introduction

In the quantum theory for finite degrees of freedom, it has long been known [1] that all irreducible representations of the quantum mechanical commutation relations, $[p_k, q_j] = -i\delta_{kj}$ $(k, j = 1, 2, \ldots, N)$, are unitarily equivalent[1]. Thus, regardless of the particular Hamiltonian being considered, one may use the standard representation: $p_k \to \frac{1}{i}\frac{d}{dx_k}$, $q_k \to x_k$ on $L^2(R^N, dx)$. In field theory with infinite degrees of freedom, there are many inequivalent irreducible representations of the canonical commutation relations (CCR) [2],

$$[\pi(x), \phi(y)] = -i\delta(x - y) \qquad (x, y \in R^s), \tag{1.1}$$

and it is generally expected that for each Hamiltonian, written in some heuristic fashion in terms of these time-zero fields, one must choose the appropriate representation so that the Hamiltonian can be defined as a bona fide operator on the representation space.

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[1] This result is rigorously true only for the Weyl form of the commutation relations.

The Hamiltonian of a relativistic theory is usually taken to be of the heuristic form,

$$\tfrac{1}{2}\int (\pi^2(x)+\mu^2\phi^2(x)+(\nabla\phi\cdot\nabla\phi)(x)+W(\phi(x)))\,dx\,, \tag{1.2}$$

with W a polynomial, for example. Several authors [3–6] have studied quantum field theories whose Hamiltonians have the simpler form,

$$H_W^\phi=\tfrac{1}{2}\int (\pi^2(x)+\mu^2\phi^2(x)+W(\phi(x)))\,dx\,, \tag{1.3}$$

and are thus diagonal in position space. The absence of gradient terms leads one to expect that different regions of space will propagate in time independently of each other and such models are accordingly called ultralocal. Aside from the inherent mathematical interest in such models, it is hoped that they may be useful in an alternate approach to constructive field theory in which the gradient term, $\frac{1}{2}\int(\nabla\phi\cdot\nabla\phi)(x)\,dx$, rather than $\frac{1}{2}\int W(\phi(x))\,dx$ is taken as a perturbation.

The search for an appropriate representation of the CCR, expressed in terms of π and ϕ, and for the ground state vector, in the representation space, for an ultralocal Hamiltonian is the continuous analogue of the corresponding problem expressed in terms of an infinite number of q_k's and p_k's with a Hamiltonian of the form,

$$H_W^q=\frac{1}{2}\sum_{k=1}^{\infty}(p_k^2+\mu^2q_k^2+W(q_k))\,. \tag{1.4}$$

This discrete problem has been thoroughly investigated by Reed [7] in the context of infinite tensor product (ITP) representations of the CCR and its straightforward solution involves the replacement of H_W^q by a renormalized Hamiltonian,

$$\bar{H}_W^q=\frac{1}{2}\sum_{k=1}^{\infty}(p_k^2+\mu^2q_k^2+W(q_k)-\varepsilon_W) \tag{1.5}$$

where ε_W is the ground state energy of $\frac{1}{2}(p^2+\mu^2q^2+W(q))$.

The ultralocal free canonical field theory with $W\equiv 0$ (for positive mass μ) may also be constructed in a straightforward manner (see Appendix A). If we denote the free canonical fields by ϕ_μ and π_μ, then the heuristic Hamiltonian,

$$H_\mu^\phi=\tfrac{1}{2}\int(\pi^2(x)+\mu^2\phi^2(x))\,dx\,, \tag{1.6}$$

is replaced by the rigorously defined renormalized Hamiltonian,

$$\bar{H}_\mu^\phi=\tfrac{1}{2}\int(\pi_\mu^2(x)+\mu^2\phi_\mu^2(x)-\mu)\,dx\,. \tag{1.7}$$

When W does not vanish, however, it appears that no representation of the CCR is appropriate for the ultralocal Hamiltonian, H_w^ϕ [5, 6, 8]. In order to allow for theories with "interaction", we therefore construct our ultralocal models using currents in place of fields with an equal time current algebra replacing the CCR.

The idea of using observable currents as the dynamical variables of a physical theory in place of fields and of replacing the CCR by an equal time current algebra,

$$[J_j(x), J_k(y)] = i \sum_{l=1}^{n} c_{jkl} J_l(x)\, \delta(x-y), \tag{1.8}$$

has been suggested by several authors [9–12]. The basic method is to express the Hamiltonian in terms of the currents and then to look for the representation of the current algebra appropriate to the particular Hamiltonian under consideration. In this paper we construct the ultralocal free theory for the currents $S(x)$ and $T(x)$, defined heuristically as

$$S(x) = \tfrac{1}{4}\phi^2(x) \tag{1.9a}$$

and

$$T(x) = \tfrac{1}{4}(\phi(x)\,\pi(x) + \pi(x)\,\phi(x)) \tag{1.9b}$$

and then relate it to the ultralocal free canonical field theory. In a succeeding paper, we construct ultralocal "interacting" theories for these same currents. It is hoped that this work will lead to a better understanding of both ultralocality and current theories.

In Section 2, the (S, T) current algebra is studied and it is shown that the free current theory is closely connected to the representations of a larger current algebra related to the Lie group, $SL(2, R)$. Section 3 is basically a review of Araki's work [13] on the construction of representations of current algebras in terms of generating functionals and the relation of "ultralocal generating functionals" to cocycles on the corresponding Lie groups. Section 4 consists of a construction of the appropriate cocycle for $SL(2, R)$ and a calculation of the corresponding generating functional which defines the ultralocal free current theory. In Section 5, the ultralocal free current, $S_\mu(x)$, is related to $\phi_\mu(x)$, and (1.9a) is made rigorous by means of a peculiar kind of renormalization reminiscent of a thermodynamic limit in momentum space. There are two appendices: Appendix A concerns the ultralocal free canonical field theory while Appendix B contains relevant material on $SL(2, R)$ and its universal covering group. Throughout the paper, we will be working in s-dimensional position space and, unless otherwise stated, all our results are independent of s.

2. The Current Algebra

The choice of the two currents defined by (1.9) as basic variables has been proposed previously [10, 14] and it seems to be appropriate for a neutral scalar boson theory. We proceed to determine, by formal application of the CCR, the current commutation relations and the form of the Hamiltonian density as a function of these currents. The current theories themselves use the results of these calculations as a starting point and are defined independently of the existence of the corresponding canonical field theories.

The current algebra, calculated in this manner, is easily seen to be:

$$[S(\boldsymbol{x}), S(\boldsymbol{y})] = 0 = [T(\boldsymbol{x}), T(\boldsymbol{y})] \tag{2.1a}$$

$$[T(\boldsymbol{x}), S(\boldsymbol{y})] = -i\delta(\boldsymbol{x}-\boldsymbol{y})\, S(\boldsymbol{x}). \tag{2.1b}$$

If we define $S(f) = \int S(\boldsymbol{x}) f(\boldsymbol{x})\, d\boldsymbol{x}$ and $T(g) = \int T(\boldsymbol{x})\, g(\boldsymbol{x})\, d\boldsymbol{x}$, then $\mathcal{U}(f) = \exp(iS(f))$ and $\mathcal{V}(g) = \exp(iT(g))$ are to be unitary operators for real valued f and g in some test function space. The multiplication rules for the $\mathcal{U}$ and $\mathcal{V}$ follow formally from the smeared current commutation relations,

$$[S(f), S(g)] = 0 = [T(f), \quad T(g)] \tag{2.2a}$$

$$[T(g), S(f)] = -iS(fg), \tag{2.2b}$$

by the use of the equation,

$$e^X Y e^{-X} = \sum_{n=0}^{\infty} \frac{1}{n!} (\operatorname{ad} X)^n Y, \tag{2.3}$$

where $(\operatorname{ad} X)\, Y = [X, Y]$. The calculation yields that

$$\mathcal{U}(f)\, \mathcal{U}(g) = \mathcal{U}(f+g), \tag{2.4a}$$

$$\mathcal{V}(f)\, \mathcal{V}(g) = \mathcal{V}(f+g), \tag{2.4b}$$

$$\mathcal{V}(g)\, \mathcal{U}(f) = \mathcal{U}(e^g f)\, \mathcal{V}(g). \tag{2.4c}$$

These results may be expressed by saying that the (S, T) current algebra is related to the Lie algebra with two self adjoint generators, S and T, satisfying $[T, S] = -iS$ and $[S, S] = 0 = [T, T]$, while the exponentiated smeared current algebra is related to the group $F = \{(z, w) \mid z, w \in R\}$ with multiplication law

$$(z_1, w_1) \cdot (z_2, w_2) = (z_1 + e^{w_1} z_2, w_1 + w_2). \tag{2.5}$$

In order to express (1.3) in terms of our currents, we first assume that W is an even function and replace $\mu^2 \phi^2(\boldsymbol{x}) + W(\phi(\boldsymbol{x}))$ by $V(S(\boldsymbol{x}))$, where $V(\lambda) = 4\mu^2\lambda + W(\sqrt{4\lambda})$, and then replace $\pi^2(\boldsymbol{x})$ by the peculiar combination $(TS^{-1}T)(x)$. We thus consider ultralocal Hamiltonians of the form

$$H_V = \tfrac{1}{2}\int ((TS^{-1}T)(x) + V(S(x)))\,dx\,. \tag{2.6}$$

The choice of $(TS^{-1}T)(x)$ to replace $\pi^2(x)$ has also been suggested by Sharp [10]; it is heuristically justified by considering the commutation relations of the free Hamiltonian,

$$H_\mu = \tfrac{1}{2}\int ((TS^{-1}T)(x) + 4\mu^2 S(x))\,dx\,, \tag{2.7}$$

with $S(x)$ and $T(x)$ as we do immediately below. The results of Sections 4 and 5 yield a more acceptable justification.

Using (2.1), we formally calculate that

$$[H_\mu, S(x)] = -iT(x)\,, \tag{2.8a}$$

$$[H_\mu, T(x)] = -\tfrac{i}{2}((TS^{-1}T)(x) - 4\mu^2 S(x))\,. \tag{2.8b}$$

An analogous calculation using (1.1) yields that

$$[H_\mu^\phi, \tfrac{1}{4}\phi^2(x)] = -i(\tfrac{1}{4}(\phi(x)\,\pi(x) + \pi(x)\,\phi(x)))\,, \tag{2.9a}$$

$$[H_\mu^\phi, \tfrac{1}{4}(\phi(x)\,\pi(x) + \pi(x)\,\phi(x))] = -\tfrac{i}{2}(\pi^2(x) - \mu^2\phi^2(x))\,. \tag{2.9b}$$

The obvious similarity between (2.8) and (2.9) indicates that at least on this algebraic level $(TS^{-1}T)(x)$ is a reasonable substitute for $\pi^2(x)$. That this substitution is a priori not totally reasonable can be seen by substituting (1.9) into $(TS^{-1}T)(x)$ and then using (1.1). The result of that calculation is that $(TS^{-1}T)(x) = \pi^2(x) + \tfrac{3}{4}\,\delta^2(0)\,\phi^{-2}(x)$, which would seem to indicate that $(TS^{-1}T)(x) - \tfrac{3}{16}\,\delta^2(0)\,S^{-1}(x)$ should be chosen to replace $\pi^2(x)$ in the ultralocal Hamiltonian. It is one of the implicit results of this paper that in fact $(TS^{-1}T)(x)$ is the proper substitution for $\pi^2(x)$ and that (2.7) is the correct replacement for (1.6).

We next note the important fact that if we consider the *free* Hamiltonian density,

$$H_\mu(x) = \tfrac{1}{2}((TS^{-1}T)(x) + 4\mu^2 S(x))\,, \tag{2.10}$$

and include it together with $S(x)$ and $T(x)$, we obtain a three current, current algebra with the commutation relations:

$$[T(x), S(y)] = -i\delta(x-y)\,S(x)\,, \tag{2.11a}$$

$$[H_\mu(x), S(y)] = -i\delta(x-y)\,T(x)\,, \tag{2.11b}$$

$$[H_\mu(x), T(y)] = -i\delta(x-y)\,(H_\mu(x) - 4\mu^2 S(x))\,. \tag{2.11c}$$

The search for the ultralocal free current theory can thus be interpreted as a search for the proper representation of this three current, current algebra. When an interaction term is added to the Hamiltonian, the currents

$S(\boldsymbol{x})$, $T(\boldsymbol{x})$, and $H_V(\boldsymbol{x})$ no longer form a closed algebra; the completely algebraic approach to the problem is then no longer applicable.

In these algebraic terms, it is also possible to see why an ambiguity arises in the choice of the term to replace $\pi^2(\boldsymbol{x})$. If we consider the three dimensional Lie algebra defined by,

$$[T, S] = -iS, \quad [L_\mu, S] = -iT, \quad [L_\mu, T] = -i(L_\mu - 4\mu^2 S), \tag{2.12}$$

we see that a formal representation is obtained by choosing for arbitrary real r,

$$S = \tfrac{1}{4}q^2, \quad T = \tfrac{1}{4}(qp + pq), \quad L_\mu = \tfrac{1}{2}(p^2 + rq^{-2} + \mu^2 q^2), \tag{2.13}$$

with p and q the usual canonical variables of quantum mechanics[2]. The value $r = 0$ corresponds to the CCR choice of $\pi^2(\boldsymbol{x})$ in H_μ^ϕ while the value $r = 3/4$ corresponds to the choice of $(TS^{-1}T)(\boldsymbol{x})$ in H_μ for the current theories.

Before concluding this section and beginning the rigorous mathematical discussions of the next, we determine the Lie algebra defined by the S, T and L_μ above. This is most easily accomplished by considering the skew adjoint basis defined by

$$A = \frac{i}{2\mu} L_\mu, \quad B = iT, \quad C = \frac{i}{2\mu} L_\mu - i2\mu S. \tag{2.14}$$

The commutation relations of this basis are

$$[A, B] = C, \quad [B, C] = -A, \quad [C, A] = B. \tag{2.15}$$

These are the commutation relations of the Lie algebra of $SO(2, 1)$, the proper three dimensional Lorentz group, or equivalently of the Lie algebra of $SL(2, R)$, the group of 2×2 real matrices of determinant one (see Appendix B). Considered as $SO(2, 1)$, A is the generator of spatial rotations while B and C are the generators of pure Lorentz boosts; $iS = (A - C)/2\mu$ generates the one parameter group of Lorentz transformations leaving a lightlike particle invariant (the little group of the zero-mass particle).

We may thus consider the skew adjoint current algebra with basis:

$$A(\boldsymbol{x}) = \frac{i}{2\mu} H_\mu(\boldsymbol{x}), \quad B(\boldsymbol{x}) = iT(\boldsymbol{x}), \quad C(\boldsymbol{x}) = \frac{i}{2\mu} H_\mu(\boldsymbol{x}) - i2\mu S(\boldsymbol{x}).$$

The search for the ultralocal free (S, T) current theory is thus a search for the appropriate representation of this $SL(2, R)$ current algebra.

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[2] It follows from Theorem B.4 and the discussion preceding Proposition 20 that on $L^2((0, \infty), dq)$ this is a valid representation when $r \geqq -\frac{1}{4}$.

3. Current Algebra Representations

In this section, G will always be a finite dimensional connected Lie group with Lie algebra $\boldsymbol{X}$. We denote elements of G by g, elements of $\boldsymbol{X}$ by X, and the identity of G by id. If one fixes a basis, (X_i) $(i=1, \ldots, n)$, of elements of $\boldsymbol{X}$, the structure constants of $\boldsymbol{X}$ are the real numbers, c_{ijk}, satisfying $[X_i, X_j] = -\sum_{k=1}^{n} c_{ijk} X_k$. We also consider the (self-adjoint) generators, $J_k = -iX_k$, satisfying $[J_i, J_j] = i\sum_{k=1}^{n} c_{ijk} J_k$.

Given a continuous unitary representation U, of G, on a Hilbert space $\mathfrak{K}$, a representation of the Lie algebra is obtained by defining for $X \in \boldsymbol{X}$, $U(X) = \left.\frac{d}{dt}\right|_{t=0} U(\exp(tX))$ or equivalently $U\left(\sum_{i=1}^{n} r_i J_i\right) = \frac{1}{i}\left.\frac{d}{dt}\right|_{t=0} U\left(\exp\left(it\sum_{i=1}^{n} r_i J_i\right)\right)$.

A (physical) current algebra is a finite set of objects, $(J_i(\boldsymbol{x}))$ $(i=1, \ldots, n)$, satisfying the commutation relations, (1.8), where c_{ijk} are the structure constants of some Lie algebra. To give a rigorous mathematical definition of this notion requires some preliminary definitions [13].

We define G^*, the current group of G, as the group of bounded, measurable[3], G-valued functions on R^s of compact support equipped with the pointwise group operations of G. A G-valued function, $g^*(\boldsymbol{x})$, is bounded if its range is contained in a compact subset of G; its support is defined as $\{\boldsymbol{x} \mid g^*(\boldsymbol{x}) \neq \mathrm{id}\}$. Analogously, we define $\boldsymbol{X}^*$, the current algebra of $\boldsymbol{X}$, as the (infinite dimensional) Lie algebra consisting of the set of bounded measurable $\boldsymbol{X}$-valued functions on R^s of compact support equipped with the pointwise Lie algebra operations of $\boldsymbol{X}$. If $X^* \in \boldsymbol{X}^*$, we denote by $\exp(X^*)$ that element of G^* defined by $(\exp(X^*))(\boldsymbol{x}) = \exp(X^*(\boldsymbol{x}))$.

A unitary representation, $\mathscr{U}$, of G^* on a Hilbert space $\mathfrak{H}$ is said to be continuous if, for all $X^* \in \boldsymbol{X}^*$, $\mathscr{U}(\exp(tX^*))$ is weakly (or equivalently, strongly) continuous in t. Given such a continuous unitary representation of G^*, we define $\mathscr{U}(X^*)$, for $X^* \in \boldsymbol{X}^*$, as $\left.\frac{d}{dt}\right|_{t=0} \mathscr{U}(\exp(tX^*))$. Given a basis for $\boldsymbol{X}$ as above, we define for f a bounded measurable function on R^s of compact support, $\mathscr{J}_k(f) = -i\mathscr{U}(X^*)$ where X^* is taken to be $X^*(\boldsymbol{x}) = f(\boldsymbol{x})\, X_k = if(\boldsymbol{x})\, J_k$. $\mathscr{J}_k(f)$ is then a formal representation of the smeared (physical) current algebra with $\mathscr{J}_k(f) = \int J_k(\boldsymbol{x}) f(\boldsymbol{x})\, d\boldsymbol{x}$. We will not discuss questions concerning the rigorous validity of the current commutation relations for such representations (see [14]).

[3] In this context, measurable refers to the Borel sets in R^s and to the σ-algebra of sets generated by the compacts in G.

If U is a representation of G on $\mathfrak{K}$, then $\psi \in \mathfrak{K}$ is said to be cyclic (with respect to U) if $\{U(g)\psi | g \in G\}$ is a total set in $\mathfrak{K}$[4]. A similar definition holds for a representation of G^*. A *state* on G is a complex valued continuous function, $E(g)$, with $E(\mathrm{id}) = 1$, such that for any positive integer N and any choice of N elements, $(g_1, \ldots, g_N)$, in G the $N \times N$ matrix, $E(g_i^{-1} g_j)$, is positive semidefinite [15]. The importance of states is due to the following well known fact.

Theorem 1. *$E(g)$ is a state on G if and only if there exists a continuous unitary representation, $U(g)$, of G on a Hilbert space $\mathfrak{K}$, with cyclic vector Ω, such that $E(g) = \langle \Omega, U(g)\Omega \rangle$ ($\langle .,. \rangle$ is the inner product in $\mathfrak{K}$). If $(U', \mathfrak{K}', \Omega')$ is another triple satisfying these conclusions, then there exists a unitary operator $W: \mathfrak{K}' \to \mathfrak{K}$ such that $W\Omega' = \Omega$ and $U(g) = W U'(g) W^{-1}$ for all $g \in G$.*

For the current group, the object analogous to a state is a generating functional.

Definition 2. A *generating functional* on G^* is a complex valued function on G^* satisfying:

(i) $E^*(\mathrm{id}^*) = 1$, where $(\mathrm{id}^*)(\boldsymbol{x}) = \mathrm{id}$ for all $\boldsymbol{x} \in R^s$.

(ii) $E^*(g_1^* \exp(tX^*) g_2^*)$ is continuous in t for all $X^* \in \boldsymbol{X}^*$ and all $g_1^*, g_2^* \in G^*$ and

(iii) the matrix $E^*(g_i^{*-1} g_j^*)$ is positive semidefinite for all N and all $g_1^*, \ldots, g_N^* \in G^*$.

In analogy with Theorem 1, and proven in a basically identical manner, we have

Theorem 3. *E^* is a generating functional on G^* if and only if there exists a continuous unitary representation, $\mathcal{U}$, of G^* on a Hilbert space $\mathfrak{H}$ with cyclic vector Ω such that $E^*(g^*) = \langle \Omega, \mathcal{U}(g^*)\Omega \rangle$.*

If $(\mathcal{U}', \mathfrak{H}', \Omega')$ is another triple satisfying these conclusions, then there exists a unitary operator $W: \mathfrak{H}' \to \mathfrak{H}$ such that $W\Omega' = \Omega$ and $\mathcal{U}(g^*) = W \mathcal{U}'(g^*) W^{-1}$ for all $g^* \in G^*$.

The moral of this theorem is that one can implicitly construct continuous unitary representations of the current group, G^*, and thus self adjoint (formal) representations, $\mathcal{J}_k(f)$, of the current algebra by determining generating functionals on G^*. This is quite useful in that the latter is often a simpler task than the former. In addition we note that many other properties of the representation can be easily stated in terms of its generating functional – for example, symmetry properties.

If $g^* \in G^*$ and $\boldsymbol{a} \in R^s$, we define $g_{\boldsymbol{a}}^* \in G^*$ as $g_{\boldsymbol{a}}^*(\boldsymbol{x}) = g^*(\boldsymbol{x} - \boldsymbol{a})$. A generating functional, E^*, is said to be translation invariant in case $E^*(g^*) = E^*(g_{\boldsymbol{a}}^*)$ for all $g^* \in G^*$ and all $\boldsymbol{a} \in R^s$. If $\boldsymbol{R} \in SO(s)$ and $\boldsymbol{a} \in R^s$,

[4] A total set in a Hilbert space is a set of vectors whose finite linear combinations, with arbitrary complex coefficients, are dense.

we define $g^*_{(\boldsymbol{R},\boldsymbol{a})}(\boldsymbol{x}) = g^*(\boldsymbol{R}^{-1}(\boldsymbol{x}-\boldsymbol{a}))$ and we say that E^* is Euclidean invariant in case $E^*(g^*) = E^*(g^*_{(\boldsymbol{R},\boldsymbol{a})})$ for all $(\boldsymbol{R},\boldsymbol{a}) \in E(s)$, the Euclidean group. The following theorem is then easily proven by standard methods.

Theorem 4. *E^* is Euclidean invariant if and only if there exists a unitary representation, $U(\boldsymbol{R},\boldsymbol{a})$, of $E(s)$ on $\mathfrak{H}$ such that $U(\boldsymbol{R},\boldsymbol{a})\,\Omega = \Omega$ and $U(\boldsymbol{R},\boldsymbol{a})\,\mathscr{U}(g^*)\,U((\boldsymbol{R},\boldsymbol{a})^{-1}) = \mathscr{U}(g^*_{(\boldsymbol{R},\boldsymbol{a})})$ for all $(\boldsymbol{R},\boldsymbol{a}) \in E(s)$ and all $g^* \in G^*$. (Here $(\mathscr{U}, \mathfrak{H}, \Omega)$ is the triple associated with E^* by Theorem 3.) In addition, $U(\boldsymbol{R},\boldsymbol{a})$ is a continuous representation of $E(s)$ if and only if $E^*(g^* h^*_{(\boldsymbol{R},\boldsymbol{a})})$ is continuous in $(\boldsymbol{R},\boldsymbol{a})$ for all $g^*, h^* \in G^*$.*

In the models we are concerned with, the cyclic vector plays the same role that the physical vacuum vector does in a Relativistic Quantum Field Theory. Physical Euclidean invariance thus suggests that we concentrate on Euclidean invariant (or at least translation invariant) generating functionals. In addition, the form of ultralocal Hamiltonians suggests that for such models, disjoint regions of space should be independent. The proper notion of disjointness for generating functionals is that of being factorizable [13, p. 365].

A generating functional is said to be *factorizable* if $E^*(g_1^* g_2^*) = E^*(g_1^*) \cdot E^*(g_2^*)$ whenever the supports of g_1^* and g_2^* are disjoint. Modulo certain technical niceties, one may say that factorizable generating functionals are always of the form $E^*(g^*) = \exp(\int d\nu(\boldsymbol{x})\, D(g^*(\boldsymbol{x}); \boldsymbol{x}))$ [13, Theorem 5.1]. If we require translation invariance as well, this form reduces to $\exp(\int d\boldsymbol{x} D(g^*(\boldsymbol{x})))$, where D is some function on G. The following theorem follows easily from the results of [13, pp. 380–381].

Theorem 5. *Let D be a continuous complex valued function on G. Then $E^*(g^*) = \exp(\int d\boldsymbol{x}\, D(g^*(\boldsymbol{x})))$ is a generating functional on G^* if and only if $\exp(aD(g))$ is a state on G for all positive a. In that case, E^* is Euclidean invariant and factorizable, and $E^*(g^* h^*_{(\boldsymbol{R},\boldsymbol{a})})$ is continuous as a function of $(\boldsymbol{R},\boldsymbol{a})$ in the Euclidean group for all $g^*, h^* \in G^*$.*

We mention that necessity is proven by considering elements of G^* which are identically equal to id outside a set of measure a and which are constant inside that set. In accord with Theorem 5, we make the following

Definition 6. A state E on G is said to be *infinitely divisible* if there exists a continuous complex valued function, $D(g)$, such that $E(g) = \exp(D(g))$, and $\exp(aD(g))$ is a state on G for all $a > 0$. Any such D will be said to *determine E*.

We now see that a search for the proper current algebra representations in which to define ultralocal Hamiltonians requires an investigation of the infinitely divisible states on the corresponding group, G. The classification of infinitely divisible states is accomplished most easily by use of the following theorem due to Araki [13, Theorem 4.4].

Theorem 7. *$D(g)$ determines an infinitely divisible state on G if and only if there exists a continuous unitary representation, $U(g)$, of G on a Hilbert space $\mathfrak{K}$; a strongly continuous $\mathfrak{K}$-valued function, $\delta(g)$; and a continuous real valued function, $P(g)$, such that the following three conditions hold:*

$$\text{(i)} \quad U(g')\,\delta(g) = \delta(g'g) - \delta(g') \qquad \text{for all } g, g' \in G. \tag{3.1}$$

$$\text{(ii)} \quad P(g'g) - P(g') - P(g) = \operatorname{Im}\langle \delta(g'^{-1}), \delta(g)\rangle_{\mathfrak{K}} \qquad \text{for all } g, g' \in G. \tag{3.2}$$

$$\text{(iii)} \quad D(g) = -\tfrac{1}{2}\langle \delta(g), \delta(g)\rangle_{\mathfrak{K}} + iP(g)\,. \tag{3.3}$$

We may assume that $\{\delta(g)\}$ is a total set in $\mathfrak{K}$, in which case, if $(\delta', U', \mathfrak{K}')$ also satisfy the hypotheses of the theorem, then there is a unitary operator $W: \mathfrak{K}' \to \mathfrak{K}$ such that $W\delta'(g) = \delta(g)$ and $U(g) = W U'(g) W^{-1}$.

This theorem suggests a definition as follows:

Definition 8. A triple, $(\delta, U, \mathfrak{K})$, consisting of a continuous unitary representation, $U(g)$, of G on a Hilbert space, $\mathfrak{K}$, and a strongly continuous $\mathfrak{K}$-valued function, δ, satisfying (3.1) for all $g, g' \in G$ is called a *cocycle* for G.

In a certain sense [16, Theorem 2.1] the classification of infinitely divisible states reduces to the classification of cocycles. In particular, the possibility of there existing different $P(g)$'s for the same cocycle is largely answered by the following easily proven proposition.

Proposition 9. *If $(\delta, U, \mathfrak{K})$ is a cocycle for G and $P_k(g)$ $(k = 1, 2)$ are continuous real valued functions on G satisfying $P_k(g'g) - P_k(g') - P_k(g) = \operatorname{Im}\langle \delta(g'^{-1}), \delta(g)\rangle_{\mathfrak{K}}$ for all $g, g' \in G$, then $P_1(g) - P_2(g)$ is a homomorphism from G into R, the additive group of reals.*

The classification problem for infinitely divisible states on abelian groups has been completely solved [17]. In the case $G = R$, one obtains the classical Lévy-Khinchine formula of Probability Theory for infinitely divisible characteristic functions [18]. In addition, Araki [13, Cor. 7.6] has classified all the cocycles for any solvable Lie group; he also has reduced the general problem to the case of semisimple (noncompact) groups and irreducible representations, $(U, \mathfrak{K})$. There seems to be very little known for such groups [19], but in Section 4 we will construct a nontrivial cocycle for an irreducible representation of $SL(2, R)$, a semisimple group. We will then be interested in the existence of a $P(g)$ for that cocycle. The following theorem suggests that in that case we consider the universal covering group of $SL(2, R)$, which is simply connected as well as semisimple.

Theorem 10. *Suppose G is semisimple, connected, and simply connected, and $(\delta, U, \mathfrak{K})$ is a cocycle for G. Then there exists a unique continuous real valued function, $P(g)$, on G satisfying (3.2).*

Proof. Let $m(g_1, g_2) = \mathrm{Im}\langle \delta(g_1^{-1}), \delta(g_2)\rangle_{\mathfrak{K}}$. A simple calculation shows that m is an additive multiplier [16] on G; i.e., $m(g_1 g_2, g_3) + m(g_1, g_2) = m(g_1, g_2 g_3) + m(g_2, g_3)$. Then $M(g_1, g_2) \equiv \exp(im(g_1, g_2))$ is a (multiplicative) multiplier on G, and a well known theorem [20, p. 40] states that all multipliers on such a group are trivial; i.e., $M(g_1, g_2)$ is given by $\alpha(g_1 g_2) \cdot (\alpha(g_1))^{-1} \cdot (\alpha(g_2))^{-1}$ for some continuous function α with values on the unit circle in C. Since G is simply connected, we may define $P(g)$ by continuity to be the continuous function such that $\alpha(g) = \exp(iP(g))$. We then have the desired result.

The uniqueness of $P(g)$ follows from Proposition 9 since a semisimple group only has the trivial homomorphism into R. This is so because the commutator subgroup of G (the subgroup generated by elements of the form $ghg^{-1}h^{-1}$) is mapped into zero by any homomorphism into R, and if G is semisimple, then G is equal to its commutator subgroup. This completes the proof of the theorem.

In order to construct representations of the (S, T) current algebra discussed in Section 2, we must consider the infinitely divisible states on the group F defined by (2.5). Since F is a semidirect product, $R \circledS R$, and is therefore solvable, we can use Araki's results to obtain cocycles for F. In this way, we obtain the following (incomplete)[5] set of infinitely divisible states on F:

Proposition 11. *Let* $\mathfrak{K} = L^2(R, d\lambda)$ *and let* $U(z, w)$ *be the representation of* F *on* $\mathfrak{K}$ *defined by*

$$U(z, w): \psi(\lambda) \rightsquigarrow e^{iz\lambda} e^{w/2} \psi(e^w \lambda). \tag{3.4}$$

Let $Y = L^1(R, d\lambda)$ *and suppose* $c(\lambda)$ *is a measurable function on* R *satisfying*

(i) $\sqrt{\frac{|\lambda|}{1+|\lambda|}}\, c(\lambda) \in \mathfrak{K}$,

(ii) $(c(\lambda) - e^{w/2} c(e^w \lambda))$ *is a strongly continuous* $\mathfrak{K}$*-valued function of* $w \in R$ *and*

(iii) $\mathrm{Im}(\overline{c(\lambda)}\, c(e^w \lambda))$ *is a stongly continuous* Y*-valued function of* $w \in R$.

We define

$$\delta_c(z, w) = (e^{iz\lambda} e^{w/2} c(e^w \lambda) - c(\lambda)) \tag{3.5}$$

and

$$P_c(z, w) = e^{w/2} \int_{-\infty}^{+\infty} \mathrm{Im}(e^{iz\lambda}\, \overline{c(\lambda)}\, c(e^w \lambda))\, d\lambda. \tag{3.6}$$

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[5] The incompleteness of this classification is due to the fact that we do not include the cocycles corresponding to those representations of F in which $U(z, 0) = 1$ for all $z \in R$. Such a cocycle yields $D(z, w) = D_0(w)$, where D_0 satisfies the Lévy-Khinchine formula. The most general $D(z, w)$ is then of the form $D_0(w) + D_c(z, w)$, with D_c as given in Proposition 11.

Then $(\delta_c, U, \mathfrak{K}, P_c)$ *satisfy the conditions of Theorem 7 and we may obtain the corresponding* $D_c(z, w)$ *which determines an infinitely divisible state on* F.

We note that if we choose $c(\lambda) = d(\lambda)/\lambda^{\frac{1}{2}}$ with $d(\lambda) \in C^\infty([0, \infty))$, $d(\lambda) = 0$ for $\lambda < 0$, and $\int^{+\infty} |d(\lambda)|^2 \frac{d\lambda}{\lambda} < \infty$, then c satisfies requirements (i), (ii), and (iii) above. In this case, we have

$$D_c(z, w) = \int_0^\infty \frac{d\lambda}{\lambda} \left(d(\lambda)\, e^{iz\lambda} d(e^w \lambda) - \frac{|d(\lambda)|^2 + |d(e^w \lambda)|^2}{2} \right) \tag{3.7}$$

determining an infinitely divisible state on F. This choice of c yields a cocycle for the irreducible subrepresentation U_0, of U, defined on $\mathfrak{K}_0 \equiv L^2((0, \infty), d\lambda) \subset \mathfrak{K}$, in which the infinitesimal generator of $U_0(z, 0)$ is positive.

4. The Free Theory Generating Functional

According to our discussion of Section 2, in order to investigate the ultralocal free current theory, we must find the appropriate representation of the $SL(2, R)$ current algebra generated by $S(\boldsymbol{x})$, $T(\boldsymbol{x})$, and $H_\mu(\boldsymbol{x}) = 1/2((TS^{-1}T)(\boldsymbol{x}) + 4\mu^2 S(\boldsymbol{x}))$. Since the commutation relations of the current algebra do not uniquely determine the relevant group (except in a neighborhood of the identity), it is not yet clear whether we should be concerned with the current group of $SL(2, R)$, or perhaps of some covering group of $SL(2, R)$. In the end, we will naturally be led to consider the universal covering group of $SL(2, R)$.

In line with Theorem 3, we will actually search for the appropriate generating functional for the (S, T, H_μ) current algebra in which the cyclic vector, Ω, is the ground state of H_μ. We should thus investigate the infinitely divisible states on $SL(2, R)$ (or some covering group) in order to obtain the appropriate Euclidean invariant factorizable generating functional as indicated by Theorem 5. Unfortunately, since $SL(2, R)$ is semisimple, it is not included in the work of Araki [13] and there is no other systematic investigation of its infinitely divisible states [19]. Rather than working directly on $SL(2, R)$, we will first construct the correct generating functional for the (S, T) current subalgebra and then extend it to a generating functional for the total (S, T, H_μ) current algebra.

The current group for the (S, T) current algebra is F^*, where $F = \{(z, w) \,|\, z, w \in R\}$ with multiplication law given by (2.5). We express elements of F^* as $g^*(\boldsymbol{x}) = (f(\boldsymbol{x}), g(\boldsymbol{x}))$ and denote a continuous unitary representation of F^* by $\mathscr{U}((f(\boldsymbol{x}), g(\boldsymbol{x}))) = \exp(iS(f))\exp(iT(g))$, where

$S(f)$ and $T(g)$ are respectively the self adjoint generators of $\mathscr{U}((tf(\mathbf{x}), 0))$ and $\mathscr{U}((0, tg(\mathbf{x})))$. We may use Proposition 11 to list a large class of generating functionals on F^*, from which we shall choose the appropriate one corresponding to H_μ (and to the various interacting Hamiltonians, H_V). We choose $c(\lambda)=0$ for $\lambda<0$ to insure that $S(f)\geqq 0$ for $f\geqq 0$ as is expected for a current heuristically defined as $\frac{1}{4}\phi^2(\mathbf{x})$. We then have

Theorem 12. *Suppose* $c(\lambda)=d(\lambda)/\lambda^{\frac{1}{2}}$ *on* $(0,\infty)$ *with* $\int\limits^{\infty} |c(\lambda)|^2\, d\lambda<\infty$ *and* $d(\lambda)\in C^\infty([0,\infty))$; *then for f and g real valued, bounded, measurable functions of compact support,* $E^*(f,g)=\exp\left(\int d\mathbf{x}\, D_c(f(\mathbf{x}), g(\mathbf{x}))\right)$ *is a Euclidean invariant factorizable generating functional on* F^*, *where* D_c *is defined by (3.7).*

As a generating functional, E^* *is given by* $E^*(f,g)=(\Omega, \exp(iS(f))\exp(iT(g))\,\Omega)$. *We then have that* $S(f)\geqq 0$ *when* $f\geqq 0$, *and* $S(f)$ *has absolutely continuous spectrum (for* $f\neq 0$*) if and only if* $d(0)\neq 0$. *In addition, the unitary representation of the Euclidean group induced on the representation space is continuous.*

Proof. The theorem follows directly from Theorem 5 and Proposition 11 except for the properties of positivity and absolute continuity for $S(f)$. To prove these, we need only consider the functions $E_\Phi(t) = (\Phi, \exp(itS(f))\,\Phi)$, as Φ ranges over a total set in the representation space. $E_\Phi(t)$, for each Φ, is a continuous function of positive type and thus by Bochner's theorem is the Fourier transform of a measure: $E_\Phi(t)=\int\limits_{-\infty}^{+\infty} e^{itq}\, d\varrho_\Phi(q)$.

To prove positivity for $S(f)$, we need only show (by the spectral theorem) that the ϱ_Φ all have support on $[0,\infty)$ as Φ ranges over a total set; to prove absolute continuity, we need only show that the ϱ_Φ are all absolutely continuous (with respect to Lebesgue measure) as Φ ranges over a total set. We choose as our total set, $\{e^{iS(h)}e^{iT(g)}\Omega\}$. A simple calculation using the multiplication law of F^* shows that for $\Phi=e^{iS(h)}e^{iT(g)}\Omega$, $E_\Phi(t) = \exp\left(\int d\mathbf{x}\, D_c(e^{-g(\mathbf{x})}tf(\mathbf{x}), 0)\right)$. It is therefore clear that we need only show that $E_f(t)\equiv\exp\left(\int d\mathbf{x}\, D_c(tf(\mathbf{x}),0)\right)$ is the Fourier transform of a measure concentrated on $[0,\infty)$ whenever $f\geqq 0$, and of an absolutely continuous measure (for $f\neq 0$) if and only if $d(0)\neq 0$. We have

$$E_f(t)=\exp\left(\int\limits_{R^s} d\mathbf{x} \int\limits_0^\infty (e^{itf(\mathbf{x})\lambda}-1)\, d\sigma(\lambda)\right) \tag{4.1}$$

where $\frac{d\sigma}{d\lambda}=|c(\lambda)|^2$. (4.1) may be rewritten as

$$E_f(t)=\exp\left(\int\limits_{-\infty}^{+\infty} (e^{itu}-1)\, d\sigma_f(u)\right) \tag{4.2}$$

where σ_f is defined by

$$\sigma_f(u) = \int_{\mathrm{supp}(f)} dx\, \sigma\left(\frac{u}{f(x)}\right). \tag{4.3}$$

We first note that if $f \geqq 0$, then since $\mathrm{supp}(\sigma) \subset [0, \infty)$, it follows from (4.3) that $\mathrm{supp}(\sigma_f)$ is also contained in $[0, \infty)$. It is a well known fact in probability theory [21] that a characteristic function of the form (4.2) with σ_f concentrated on $[0, \infty)$ is the Fourier transform of a measure concentrated on $[0, \infty)$. If σ_f is of finite total mass (i.e. $d(0) = 0$), this can be seen directly, since then $E_f(t) = \int_{-\infty}^{+\infty} e^{itq} d\varrho_f(q)$, with

$$\varrho_f = e^{-\int_{-\infty}^{+\infty} d\sigma_f} \sum_{n=0}^{\infty} \frac{1}{n!} \sigma_f^{*n} \tag{4.4}$$

(where * denotes convolution); and the convolution of two measures with support in $[0, \infty)$ again has support in $[0, \infty)$. If σ_f has infinite total mass (i.e. $d(0) \neq 0$), one can approximate σ_f by measures of finite total mass. ϱ_f will then be a weak limit of measures concentrated on $[0, \infty)$ and will therefore also be concentrated on $[0, \infty)$.

We next note that (4.3) implies that σ_f has finite total mass if and only if σ has finite total mass, and thus if and only if $d(0) = 0$. From (4.4) we can easily see that when σ_f has finite total mass, ϱ_f has a point mass at the origin and therefore $d(0) = 0$ implies that $S(f)$ does not have absolutely continuous spectrum. The converse follows from the theorem in probability theory that for a characteristic function of the form (4.2) (with σ_f absolutely continuous), ϱ_f is absolutely continuous if σ_f has infinite total mass [22]. This completes the proof of the theorem.

It would be quite unphysical to have all the currents, $S(f)$, with point spectrum at the origin; we therefore expect that in any physically relevant representation of the (S, T) current algebra given by Theorem 12, we should have $d(0) \neq 0$. Beyond this requirement, however, we have as yet no way of choosing the proper $c(\lambda)$ to yield the appropriate generating functional for the ultralocal Hamiltonian H_V. We therefore make the following

Ansatz 13. The appropriate generating functional on F^*, yielding a representation of the (S, T) current algebra in which the cyclic vector Ω is the ground state of $H_V = \frac{1}{2} \int ((TS^{-1}T)(x) + V(S(x)))\, dx$, is that given by $E^*(f, g) = \exp(\int dx\, D_c(f(x), g(x)))$, as in Theorem 12, with $c(\lambda)$ the solution of the *differential equation*,

$$\tfrac{1}{2}(TS^{-1}T + V(S))\, c(\lambda) = 0\,, \tag{4.5}$$

where $S = \lambda$, $T = \dfrac{1}{2i}\left(\lambda \dfrac{d}{d\lambda} + \dfrac{d}{d\lambda} \lambda\right)$, and $\lambda \in (0, \infty)$.

We note that $c(\lambda)$ is chosen as that solution of the differential equation which is square integrable at ∞. We also mention that the above definitions of S and T are just those obtained from the representation, U_0, of F on $\mathfrak{K}_0$ defined at the end of Section 3. In this paper, we will utilize this ansatz only for the case of the ultralocal free Hamiltonian, H_μ, with $V(\lambda)=4\mu^2\lambda$. The result is:

Proposition 14. *The solution, which is square integrable at* ∞, *to the differential equation,* $\frac{1}{2}(TS^{-1}T+4\mu^2 S)\,c_\mu(\lambda)=0$, *is* $c_\mu(\lambda)=\frac{K}{\sqrt{2}}\frac{e^{-2\mu\lambda}}{\lambda^{\frac{1}{2}}}$. *The corresponding* $D_{c_\mu}(z,w)$, *as defined by (3.7), is then*

$$D_{c_\mu}(z,w)=|K|^2 D_\mu(z,w)=-\frac{|K|^2}{2}\ln\left(\cosh\frac{w}{2}-ie^{-w/2}\frac{z}{4\mu}\right) \tag{4.6}$$

and the corresponding generating functional on F^* *is*

$$E^*_{\mu,K}(f,g)=\exp\left(|K|^2\int d\mathbf{x}\,D_\mu(f(\mathbf{x}),g(\mathbf{x}))\right). \tag{4.7}$$

Proof. Using the definitions of S and T given in Ansatz 13 and rearranging terms, we find that $TS^{-1}T=-\sqrt{\lambda}\frac{d^2}{d\lambda^2}\sqrt{\lambda}$. If we let $c_\mu(\lambda)=d_\mu(\lambda)/\lambda^{\frac{1}{2}}$, the differential equation for c_μ reduces to

$$\left(-\frac{d^2}{d\lambda^2}+4\mu^2\right)d_\mu(\lambda)=0\,,$$

and to make c_μ square integrable at ∞, we must choose the solution, $d_\mu(\lambda)=K'e^{-2\mu\lambda}$, where K' is an arbitrary complex number; for later convenience we choose $K'=\frac{K}{\sqrt{2}}$. It only remains to calculate $D_\mu(z,w)$ as defined by (3.7), viz.

$$D_\mu(z,w)=\frac{1}{2}\int_0^\infty\frac{d\lambda}{\lambda}\left(e^{-2\mu\lambda-2\mu e^w\lambda+iz\lambda}-\frac{e^{-4\mu\lambda}+e^{-4\mu e^w\lambda}}{2}\right).$$

Letting $a_1=2\mu$, $a_2=2\mu e^w$, $z_1=-2\mu e^w+iz$, and $z_2=-2\mu+iz$, we find that

$$D_\mu(z,w)=\frac{1}{4}\left(\int_0^\infty e^{-a_1\lambda}(e^{z_1\lambda}-e^{-a_1\lambda})\frac{d\lambda}{\lambda}+\int_0^\infty e^{-a_2\lambda}(e^{z_2\lambda}-e^{-a_2\lambda})\frac{d\lambda}{\lambda}\right). \tag{4.8}$$

We may use the fact [23, formula 3.551–1] that

$$\int_0^\infty\lambda^{\mu-1}e^{-\beta\lambda}(\sinh\gamma\lambda)\,d\lambda=\frac{1}{2}\Gamma(\mu)\left((\beta-\gamma)^{-\mu}-(\beta+\gamma)^{-\mu}\right)$$

for $\operatorname{Re}\mu>-1$ and $\operatorname{Re}\beta>|\operatorname{Re}\gamma|$ to obtain, after taking suitable limits, that

$$\int_0^\infty e^{-a\lambda}(e^{y\lambda}-e^{-a\lambda})\frac{d\lambda}{\lambda}=\ln\left(\frac{2a}{a-y}\right)$$

for $\operatorname{Re}\left(\frac{3a-y}{2}\right) > \left|\operatorname{Re}\frac{y+a}{2}\right|$. By substituting into (4.8), we find that

$$D_\mu(z,w) = -\frac{1}{4}\ln\left(\frac{(a_1-z_1)(a_2-z_2)}{4a_1a_2}\right) = -\frac{1}{4}\ln\left(\cosh\frac{w}{2} - ie^{-w/2}\frac{z}{4\mu}\right)^2,$$

which immediately yields (4.6) and thus completes the proof of the proposition.

Formula (4.7) completely determines the ultralocal free current theory generating functional on F^* up to a choice of $|K|^2$. This choice will be made in Section 5, but meanwhile we wish to extend $E^*_{\mu,K}$ to the entire (S, T, H_μ) algebra by extending the cocycle,

$$(\delta_\mu(z,w))(\lambda) = \frac{e^{iz\lambda}d_\mu(e^w\lambda) - d_\mu(\lambda)}{\lambda^{\frac{1}{2}}}, \tag{4.9}$$

from F to all of $SL(2,R)$. It is first necessary to extend the representation U_0, as defined at the end of Section 3, from a representation of F to a representation of $SL(2,R)$; this may be done by using the results of Appendix B.

We take the standard basis (A, B, and $D = \frac{A-C}{2}$ as defined in Appendix B) for the Lie algebra of $G_0 = SL(2,R)$ and then invert (2.14) to obtain the self adjoint basis:

$$S = -\frac{i}{\mu}D, \qquad T = -iB, \qquad L_\mu = -i2\mu A. \tag{4.10}$$

F is thus isomorphic to the subgroup of G_0 generated by S and T (or equivalently by B and D). Ansatz 13 suggests that on $\mathfrak{K}_0 = L^2((0,\infty), d\lambda)$, a representation of G_0 is obtained by representing S as λ, T as $\frac{1}{2i}\left(\lambda\frac{d}{d\lambda} + \frac{d}{d\lambda}\lambda\right)$, and L_μ as $\frac{1}{2}(TS^{-1}T + 4\mu^2 S) = \frac{1}{2}\left(-\sqrt{\lambda}\frac{d^2}{d\lambda^2}\sqrt{\lambda} + 4\mu^2\lambda\right)$. Proposition B.3 and Theorem B.4, for the case $h=1$, show that this does indeed define a valid representation:

Theorem 15. $\frac{1}{2}\left(-\sqrt{\lambda}\frac{d^2}{d\lambda^2}\sqrt{\lambda} + 4\mu^2\lambda\right)$ *is essentially self adjoint on* $C_0^\infty(0,\infty) \subset \mathfrak{K}_0$. *If we denote its self adjoint closure by* h_μ, *then there exists a continuous unitary irreducible representation,* $U_\mu(g)$, *of* G_0 *on* $\mathfrak{K}_0$, *such that*

$$U_\mu(S) = \lambda, \qquad U_\mu(T) = \frac{1}{2i}\left(\lambda\frac{d}{d\lambda} + \frac{d}{d\lambda}\lambda\right), \qquad U_\mu(L_\mu) = h_\mu. \tag{4.11}$$

$U_\mu(g)$ is unitarily equivalent to the positive discrete representation $U^+(g,1)$ (as defined in Appendix B). The normalized eigenbasis of h_μ is given by:

$$h_\mu \Psi_m(\lambda) = 2\mu(m+1)\,\Psi_m(\lambda) \qquad (m=0,1,2,\ldots), \tag{4.12}$$

with

$$\Psi_m(\lambda) = \frac{4\mu(-1)^m}{\sqrt{m+1}} \lambda^{\frac{1}{2}} e^{-2\mu\lambda} L_m^{(1)}(4\mu\lambda), \tag{4.13}$$

where $L_m^{(1)}$ is an associated Laguerre polynomial [23, p. 1037].

We wish to extend the cocycle, $\delta_\mu(z,w) \equiv \delta_\mu(e^{izS} e^{iwT})$, defined by (4.9) from F to all of G_0. According to (B.2) any $g \in G_0$ may be expressed as $e^{zD} e^{wB} e^{tA}$ and we thus need only define $\delta_\mu(g)$ for such a g and then show that it is in fact a cocycle (with respect to the representation U_μ). Now, heuristically, $\delta_\mu(z,w)$ is of the form $(e^{izS} s^{iwT} - 1)\, c_\mu(\lambda)$, and thus to extend δ_μ, we should consider $(e^{izS} e^{iwT} e^{ith_\mu} - 1)\, c_\mu(\lambda)$. Since our ansatz suggests that we consider c_μ as a kind of pseudo ground state of h_μ with zero eigenvalue, it is reasonable to replace $e^{ith_\mu} c_\mu(\lambda)$ in the above expression by $c_\mu(\lambda)$. As the next theorem shows, this idea works.

Theorem 16. *For $g \in G_0$ of the form $e^{zD} e^{wB} e^{tA}$, we define*

$$\delta_\mu(g) = \delta_\mu(\mu z, w) = \left(\frac{e^{i\mu z \lambda} d_\mu(e^w \lambda) - d_\mu(\lambda)}{\lambda^{\frac{1}{2}}} \right) \in \mathfrak{R}_0 . \tag{4.14}$$

Then $(\delta_\mu, U_\mu, \mathfrak{R}_0)$ is a cocycle for G_0.

Proof. We first note that $\delta_\mu(g)$ is well defined since, as explained in Appendix B, the only ambiguity in the parametrization of g by z, w, and t is in the value of t which does not enter into the determination of $\delta_\mu(g)$. It is also immediately clear that if g'' has the special form $g'' = e^{zD} e^{wB}$, then since δ_μ is already a cocycle when restricted to F, we have $U_\mu(g'')\,\delta_\mu(g) = \delta_\mu(g''g) - \delta_\mu(g'')$ as required. Since an arbitrary $g' \in G$ can be expressed as $g' = g'' e^{tA}$ for some g'' of the above form and some $t \in R$, we need only show that $U_\mu(e^{tA})\,\delta_\mu(g) = \delta_\mu(e^{tA} g) - \delta_\mu(e^{tA})$. But since $\delta_\mu(e^{tA}) = \delta_\mu(0,0) = 0$, this reduces to the requirement that

$$U_\mu(e^{tA})\,\delta_\mu(\mu z, w) = \delta_\mu(\mu z', w'), \tag{4.15}$$

where z, w, t, z', w', t' satisfy $e^{tA} e^{zD} e^{wB} = e^{z'D} e^{w'B} e^{t'A}$.

To prove (4.15), it is clearly sufficient to show that for $m = 0, 1, 2, \ldots$

$$(\Psi_m, U_\mu(e^{tA})\,\delta_\mu(\mu z, w)) = (\Psi_m, \delta_\mu(\mu z', w')). \tag{4.16}$$

Using (4.12), we find that (4.16) is identical to the requirement that

$$e^{i(m+1)t}(\Psi_m, \delta_\mu(\mu z, w)) = (\Psi_m, \delta_\mu(\mu z', w')). \tag{4.17}$$

By using (4.9), (4.13), and the fact that for $\mathrm{Re}\, b > 0$,

$$\int_0^\infty e^{-b\lambda} L_m^{(1)}(\lambda)\, d\lambda = \left(1 - \left(\frac{b-1}{b}\right)^{m+1}\right) \qquad (m = 0, 1, 2, \ldots),$$

it can be directly calculated that

$$(\Psi_m, \delta_\mu(\mu z, w)) = (-1)^{m+1} \frac{K}{\sqrt{2(m+1)}} \left(\frac{\overline{y(z,w)} - 1}{\overline{y(z,w)}}\right)^{m+1},$$

where $y(z,w) = \frac{1+e^w}{2} + i\frac{z}{4}$. A comparison of this last equation with (4.17) indicates that it suffices to show that

$$e^{-i(m+1)t} \left(\frac{y(z,w)-1}{y(z,w)}\right)^{m+1} = \left(\frac{y(z',w')-1}{y(z',w')}\right)^{m+1}.$$

Since this final equation follows immediately from Proposition B.1, we have completed the proof of the theorem.

Now that we have a cocycle for G_0, a semisimple group, Theorem 10 would guarantee a corresponding $P(g)$ and therefore an infinitely divisible state on G_0, if only G_0 were simply connected. G_0 itself, however, is not simply connected and it is therefore necessary to consider G_C, the universal covering group of $SL(2, R)$, which is simply connected. Since G_C is a covering group of G_0, it follows that the representation $(U_\mu, \mathfrak{K}_0)$ of Theorem 15 also yields a representation of G_C, and that $(\delta_\mu, U_\mu, \mathfrak{K}_0)$ is also a cocycle for G_C. Since G_C satisfies all the requirements of Theorem 10, we are guaranteed the existence and uniqueness of a $P(g)$ corresponding to δ_μ and thus a non-trivial infinitely divisible state on G_C. This infinitely divisible state can be determined by considering the positive discrete series of representations of G_C, as defined in Appendix B. These representations, denoted by $U^+(g,h)$ for $h>0$, are defined on certain Hilbert spaces, $\mathfrak{H}_{(h)}$, of functions analytic on the unit disk in C, with inner product $\langle .,. \rangle_h$. We then have

Proposition 17. $U^+(A,h) \equiv \left.\frac{d}{dt}\right|_{t=0} U^+(e^{tA}, h)$ *is a skew adjoint operator on* $\mathfrak{H}_{(h)}$. *Its normalized eigenbasis is given by*

$$U^+(A,h)\, \Phi_{n,h}(y) = i(n+h)\, \Phi_{n,h}(y) \qquad (n = 0, 1, 2, \ldots) \tag{4.18}$$

where

$$\Phi_{n,h}(y) = \left(\frac{\Gamma(2h+n)}{\Gamma(2h)\,\Gamma(n+1)}\right)^{\frac{1}{2}} y^n. \tag{4.19}$$

In addition, we have for $g = e^{zD} e^{wB} e^{tA}$, *that*

$$\langle \Phi_{0,h}, U^+(g,h)\, \Phi_{0,h} \rangle_h = e^{-2h\left(\ln\left(\cosh\frac{w}{2} - ie^{-w/2}\frac{z}{4}\right) - i\frac{t}{2}\right)}. \tag{4.20}$$

Proof. $U^+(A, h)$ is of course skew adjoint. Formula (B.12a) clearly shows that $\{\Phi_{n,h}\}$ are the eigenvectors of $U^+(A, h)$ and (B.11) shows that they are properly normalized. Since $\Phi_{0,h}(y) \equiv 1$, we see from (B.12) that

$$(U^+(g, h)\, \Phi_{0,h})\,(y) = e^{iht} e^{2i\omega(z,w)h}(1 - |\gamma(z, w)|^2)^h\,(1 + \overline{\gamma(z, w)}\, y)^{-2h}.$$

Since the first term in the Taylor series for $(1 + \bar{\gamma} y)^{-2h}$ is 1, it immediately follows from (B.11) that

$$\langle \Phi_{0,h}, U^+(g, h)\, \Phi_{0,h}\rangle_h = e^{2ih\left(\omega(z,w) + \frac{t}{2}\right)}(1 - |\gamma(z, w)|^2)^h.$$

By substituting (B.6) into this equation, we obtain (4.20), which completes the proof of the proposition.

Theorem 18. *The function on* G_C,

$$E(e^{zD} e^{wB} e^{tA}) = e^{-\frac{|K|^2}{2}\left(\ln\left(\cosh\frac{w}{2} - ie^{-w/2}\frac{z}{4}\right) - i\frac{t}{2}\right)}, \tag{4.21}$$

is the unique infinitely divisible state corresponding to the cocycle, $(\delta_\mu, U_\mu, \mathfrak{K}_0)$.

Proof. If we write $E(g) = \exp(D(g))$ in the usual way, then since $e^{aD(g)} = \langle \Phi_{0,h}, U^+(g, h)\, \Phi_{0,h}\rangle_h$ for $h = \dfrac{a|K|^2}{4}$, we have by Theorem 1 that $e^{aD(g)}$ is a state on G_C. This being true for all $a > 0$, it follows that $E(g)$ is an infinitely divisible state. $D(g)$ corresponds to $\delta_\mu(g)$ since it is easily seen that $\operatorname{Re} D(g) = -\frac{1}{2}(\delta_\mu(g), \delta_\mu(g))_{\mathfrak{K}_0}$; since $P(g)$ is uniquely determined by Theorem 10, we must have $P(g) = \operatorname{Im} D(g)$. By Theorem 7, $(\delta_\mu, U_\mu, \mathfrak{K}_0)$ is also uniquely determined (up to unitary equivalence).

Corollary 19. *Let* D_1 *be defined as*

$$D_1(z, w, t) = -\frac{1}{2}\left(\ln\left(\cosh\frac{w}{2} - \mathrm{i}e^{-w/2}\frac{z}{4}\right) - i\frac{t}{2}\right) \tag{4.22}$$

and let f, g, *and* k *be real valued, bounded, measurable functions of compact support; then*

$$E^*_{\mu,K}(f, g, k) \equiv \exp\left(|K|^2 \int d\boldsymbol{x}\, D_1\left(\frac{f(\boldsymbol{x})}{\mu}, g(\boldsymbol{x}), 2\mu k(\boldsymbol{x})\right)\right) \tag{4.23}$$

is a generating functional on G^*_C *which yields (by Theorem 3) a cyclic representation in which*

$$E^*_{\mu,K}(f, g, k) = (\Omega_{\mu,K}, e^{iS(f)} e^{iT(g)} e^{iH_\mu(k)}\, \Omega_{\mu,K}) \tag{4.24}$$

with $H_\mu(k) = \int H_\mu(\boldsymbol{x})\, k(\boldsymbol{x})\, d\boldsymbol{x}$.

This generating functional defines (neglecting temporarily the choice of $|K|^2$) the ultralocal free current theory for mass μ. In the next section we choose $|K|^2$ and then determine the relation of this current theory to the ultralocal free canonical field theory.

5. The Relation of the Free Currents to the Free Canonical Fields

We see from the form of $E^*_{\mu,K}(f,g,k)$ given in Corollary 19 that the representation thus defined for G^*_C has the property that $\exp(iH_\mu(k))\,\Omega_{\mu,K} = \exp\left(i|K|^2\frac{\mu}{2}\int k(x)\,dx\right)\Omega_{\mu,K}$; in other words, $H_\mu(x)\,\Omega_{\mu,K}=|K|^2\frac{\mu}{2}\,\Omega_{\mu,K}$. The choice of $|K|^2$ therefore corresponds to a choice of vacuum energy renormalization, since the renormalized Hamiltonian density clearly must be $\bar{H}_\mu(x)=\left(H_\mu(x)-|K|^2\frac{\mu}{2}\right)$. Since $H_\mu(x)$ is heuristically equivalent to the unrenormalized field theory Hamiltonian density, $\frac{1}{2}(\pi^2(x)+\mu^2\phi^2(x))$, the proper choice of $|K|^2$ is determined by a kind of correspondence principle with respect to the ultralocal free canonical field theory. The relation between (1.7) and (1.6) in the field theory case clearly indicates that we must take $\bar{H}_\mu(x)=\left(H_\mu(x)-\frac{\mu}{2}\right)$ and we thus choose $|K|^2=1$. We therefore consider the representation of G^*_C defined by (4.24) with $|K|^2=1$ as the ultralocal free current theory with the corresponding currents denoted by $S_\mu(x)$ and $T_\mu(x)$, the generating functional by E^*_μ, and the cyclic vector by Ω_μ. The work of this section is intended to show that this choice of $|K|^2$ together with the choices already dictated by Ansatz 13, yield a current theory which is closely related to the ultralocal free field theory (defined in Appendix A).

The first step in this process is to determine the relation between the corresponding single degree of freedom problems. We note that the choice of $|K|^2=1$ is related, by Proposition 17, to the positive discrete representation, $U^+(g,\frac{1}{4})$, of G_C and to the ground state, $\Phi_{0,\frac{1}{4}}$, of $U^+(L_\mu,\frac{1}{4})$ in $\mathfrak{H}_{(\frac{1}{4})}$. In particular, we have, for $|K|^2=1$, that

$$(\Omega_\mu, e^{iS_\mu(f)}e^{iT_\mu(g)}\Omega_\mu)=\exp\left(\int D_\mu(f(x),g(x))\,dx\right) \tag{5.1}$$

with

$$e^{D_\mu(z,w)}=\langle \Phi_{0,\frac{1}{4}}, U^+(e^{izS}e^{iwT},\tfrac{1}{4})\,\Phi_{0,\frac{1}{4}}\rangle_{\frac{1}{4}}\,. \tag{5.2}$$

The form of the generating functional, $E^{\phi*}_\mu$, as given by (A.4) and (A.5) shows that in an analogous way, the ultralocal free field theory is related to the usual representation of p and q and to Φ_μ, the ground state of $\frac{1}{2}(p^2+\mu^2q^2)$. In order to relate the quantum mechanical harmonic oscillator to the $U^+(g,\frac{1}{4})$ representation of G_C, we need a realization of this representation different from both (B.12) and (B.17).

We have from Theorem B.4 that $U_\mu(S,h)=\lambda$ and

$$U_\mu(T,h)=\frac{1}{2i}\left(\lambda\frac{d}{d\lambda}+\frac{d}{d\lambda}\lambda\right).$$

In order to compare these representations to the harmonic oscillator problem, we need a representation in which $S \to q^2/4$ and $T \to \frac{pq+qp}{4}$ on $L^2(R, dq)$. Toward this end, we consider the mapping, $W_\varepsilon : \mathfrak{K}_0 \to L^2(R, dq)$ where, for $\varepsilon = 0$ or 1,

$$(W_\varepsilon \Phi)(q) = (\operatorname{sgn} q)^\varepsilon \frac{\sqrt{|q|}}{2} \Phi\left(\frac{q^2}{4}\right).$$

W_ε is a unitary transformation from $\mathfrak{K}_0$ onto $L^2_{\text{even}} = \{\psi(q) | \psi(q) = +\psi(-q)\} \subset L^2(R, dq)$, for $\varepsilon = 0$, and onto L^2_{odd} for $\varepsilon = 1$. Moreover we have:

$$W_\varepsilon \lambda W_\varepsilon^{-1} = \frac{q^2}{4}, \quad W_\varepsilon \left(\frac{1}{2i}\left(\lambda \frac{d}{d\lambda} + \frac{d}{d\lambda}\lambda\right)\right) W_\varepsilon^{-1} = \frac{pq+qp}{4}.$$

If we neglect the "boundary conditions" used in defining $U_\mu(L_\mu, h)$, then heuristically we have

$$W_\varepsilon U_\mu(L_\mu, h) W_\varepsilon^{-1} = \frac{1}{2}\left(p^2 + \frac{(2h-\frac{1}{2})(2h-\frac{3}{2})}{q^2} + \mu^2 q^2\right);$$

in actuality, we have a certain self adjoint extension of this operator (defined initially on functions vanishing near the origin). We note that for $h = \frac{1}{4}$ or $h = \frac{3}{4}$, the operator reduces to $\frac{1}{2}(p^2 + \mu^2 q^2)$; in these particular cases the boundary conditions are automatically satisfied by choosing $\varepsilon = 0$ for $h = \frac{1}{4}$ and $\varepsilon = 1$ for $h = \frac{3}{4}$. We thus have

Proposition 20. *On $L^2_{\text{even}}(R, dq)$, the mappings,*

$$S \to \frac{q^2}{4}, \quad T \to \frac{qp+pq}{4}, \quad L_\mu \to \frac{1}{2}(p^2 + \mu^2 q^2), \tag{5.3}$$

define a representation of G_C, the universal covering group of $SL(2, R)$, which is unitarily equivalent to $U^+(g, \frac{1}{4})$. On $L^2_{\text{odd}}(R, dq)$ the mappings, (5.3), *define a representation of G_C unitarily equivalent to $U^+(g, \frac{3}{4})$.*

We thus see that the infinitely divisible state on G_C which defines the ultralocal free current theory (with $|K|^2 = 1$), which was shown to be

$$E_\mu(g) = \langle \Phi_{0,\frac{1}{4}}, U^+(g, \tfrac{1}{4}) \Phi_{0,\frac{1}{4}} \rangle_{\frac{1}{4}}, \tag{5.4}$$

may also be given by replacing $U^+(g, \frac{1}{4})$ in (5.4) with the representation defined by (5.3) and $\Phi_{0,\frac{1}{4}}$ with Φ_μ, the ground state of $\frac{1}{2}(p^2 + \mu^2 q^2)$. If $|K|^2$ had been taken to be 3 instead of 1, we would use the first excited state of $\frac{1}{2}(p^2+\mu^2 q^2)$ in place of the ground state. We have thus demonstrated

Corollary 21. *In the ultralocal free current theory representation (defined by Corollary 19, with $|K|^2=1$) we have*

$$(\Omega_\mu, e^{iS_\mu(f_1)} e^{iT_\mu(g_1)} e^{iH_\mu(k_1)} \dots e^{iS_\mu(f_N)} e^{iT_\mu(g_N)} e^{iH_\mu(k_N)} \Omega_\mu) \\ = \exp\left(\int dx\, D_\mu(f_1(x), g_1(x), k_1(x), \dots, k_N(x))\right), \tag{5.5}$$

where

$$e^{D_\mu(z_1, w_1, t_1, \dots, t_N)} = \langle \Phi_\mu, e^{iz_1 S} e^{iw_1 T} e^{it_1 h_\mu^q} \dots e^{it_N h_\mu^q} \Phi_\mu \rangle_{L^2(R, dq)} \tag{5.6}$$

with $S=q^2/4$, $T=\dfrac{pq+qp}{4}$, $h_\mu^q=\frac{1}{2}(p^2+\mu^2 q^2)$, and Φ_μ the ground state of h_μ^q.

This corollary is our first indication of the relation between the ultralocal free current theory and the ultralocal free field theory, since the analogous statement for the field theory case with $\phi_\mu(x)$, $\pi_\mu(x)$, q, and p replacing $S_\mu(x)$, $T_\mu(x)$, S, and T is also true. It is therefore clear that the choice of $|K|^2=1$ is not a trivial one; it actually happens that two different choices of $|K|^2$ yield representations of the (S, T) current algebra which are not even locally equivalent to each other. An interesting question is whether the other values of $|K|^2$ (in particular 2, 3, and 4) have any particular physical relevance.

To continue investigating the connection between the ultralocal free current theory and the ultralocal free field theory, we will concentrate on the relationship between the current, $S_\mu(x)$, of the free current theory representation and the field, $\phi_\mu(x)$, on the ultralocal Fock space of the free field theory. In particular, we are interested in seeing whether $S_\mu(x)$ can be interpreted as $\dfrac{\phi_\mu^2(x)}{4}$ in any sense. The analogous question in the one degree of freedom case is answered affirmatively by Proposition 20 which shows that

$$E_\mu(z) \equiv \langle \Phi_{0,\frac{1}{4}}, U^+(e^{izS}, \tfrac{1}{4})\, \Phi_{0,\frac{1}{4}} \rangle_{\frac{1}{4}} = e^{-\frac{1}{2}\ln\left(1 - i\frac{z}{4\mu}\right)} \\ = \left\langle \Phi_\mu, e^{iz\frac{q^2}{4}} \Phi_\mu \right\rangle_{L^2(R, dq)}. \tag{5.7}$$

The remainder of this section is devoted to analyzing the relation between

$$E_\mu^*(f) = e^{-\frac{1}{2}\int dx \ln\left(1 - i\frac{f(x)}{4\mu}\right)} = (\Omega_\mu, e^{iS_\mu(f)} \Omega_\mu) \tag{5.8}$$

and

$$\left(\Omega_\mu^\phi, e^{i\frac{\phi_\mu^2(f)}{4}} \Omega_\mu^\phi\right) \tag{5.9}$$

where Ω_μ^ϕ is the ground state of the ultralocal free field theory Hamiltonian and

$$\phi_\mu^2(f) = \int \phi_\mu^2(x) f(x)\, dx = \int \phi_\mu(x)\, \phi_\mu(y)\, f(x)\, \delta(x-y)\, dx\, dy\,. \quad (5.10)$$

Since the multiplication operator M_f, with kernel $f(x)\,\delta(x-y)$, on $L^2(R^s, dx)$ is not even compact, let alone trace class, it is clear from Theorem A.1 that (5.10) is a purely heuristic equation and that $\phi_\mu^2(f)$ does not really exist as an operator on the ultralocal Fock space. If we nevertheless apply Theorem A.1 formally in order to compute (5.9), we find that since $\ln(1 - iM_f/\mu)$ is the multiplication operator with kernel $\ln(1 - if(x)/\mu)\,\delta(x-y)$, we then have that formally,

$$(\Omega_\mu^\phi, e^{i\phi_\mu^2(f)}\Omega_\mu^\phi) = e^{-\frac{1}{2}\delta(0)\int dx \ln\left(1 - i\frac{f(x)}{\mu}\right)}. \quad (5.11)$$

Comparing (5.11) with (5.8), we find that the heuristic relation between $S_\mu(f)$ in the ultralocal current theory and $\phi_\mu^2(f)/4$ in the ultralocal Fock space is a kind of renormalization which somehow "divides out" the $\delta(0)$ infinity involved in the definition of $\phi_\mu^2(x)$. The next theorem makes this renormalization somewhat more precise and shows that $\delta(0)$ is equivalent to the infinite volume of momentum space and is removed by a kind of thermodynamic limit.

Theorem 22. *In the ultralocal free representation of mass μ of the* CCR, *let $\phi_\kappa(x)$ denote the momentum cut off field defined by*

$$\phi_\kappa(g) = \int \phi_\kappa(x)\, g(x)\, dx = \phi_\mu(g_\kappa) \quad (5.12)$$

where $g_\kappa(x)$ is defined by

$$\hat{g_\kappa}(k) = \begin{cases} \hat{g}(k)\,, & |k| \leqq \kappa \\ 0\,, & |k| > \kappa \end{cases},$$

where ^ denotes the Fourier transform. Then for $f \in \mathscr{S}(R^s)$[6],

$$\phi_\kappa^2(f) \equiv \int \phi_\kappa(x)\, \phi_\kappa(y)\, f(x)\, \delta(x-y)\, dx\, dy \quad (5.13)$$

exists as a self adjoint operator on the representation space for all $\kappa < \infty$ and

$$\lim_{\kappa\to\infty} \frac{1}{V(\kappa)} \ln(\Omega_\mu^\phi, e^{i\phi_\kappa^2(f)}\Omega_\mu^\phi) = -\frac{1}{2}\int dx \ln\left(1 - i\frac{f(x)}{\mu}\right), \quad (5.14)$$

where $V(\kappa)$ denotes the volume of the ball of radius κ in R^s and $\mathscr{S}(R^s)$ is Schwartz space in s variables.

Proof. We denote by P_κ, the projection operator on $L^2(R^s, dx)$ which maps g into g_κ. It follows from (5.12) and (5.13) that

$$\phi_\kappa^2(f) = \int \phi_\mu(x)\, \phi_\mu(y)\, M_f'(x, y)\, dx\, dy$$

[6] f is chosen in Schwartz space purely for convenience; the theorem is clearly true for more general test functions.

where $M_f'(x, y)$ is the kernel of the operator $P_\kappa M_f P_\kappa$, with M_f the operator of multiplication by $f(x)$ on $L^2(R^s, dx)$. For $f \in \mathcal{S}(R^s)$, $P_\kappa M_f P_\kappa$ is easily seen to be trace class with a real symmetric kernel, and thus by Theorem A.1, $\phi_\kappa^2(f)$ exists as a self adjoint operator and

$$(\Omega_\mu^\phi, e^{i\phi_\kappa^2(f)}\Omega_\mu^\phi) = e^{-\frac{1}{2}\operatorname{tr}\left(\ln\left(1 - i\frac{P_\kappa M_f P_\kappa}{\mu}\right)\right)}. \tag{5.15}$$

To complete the proof of the theorem, we need only show that for any $g \in \mathcal{S}(R^s)$,

$$\lim_{\kappa \to \infty} \frac{1}{V(\kappa)} \operatorname{tr}(\ln(1 - iP_\kappa M_g P_\kappa)) = \int dx \ln(1 - ig(x)).$$

This fact will follow from the next three lemmas.

Lemma 23. *Suppose A is a self adjoint trace class operator with $\|A\| \leqq B$. Suppose F_1 and F_2 are two functions on R such that $|F_1(r) - F_2(r)| \leqq \varepsilon |r|$ for all $|r| \leqq B$. Then $|\operatorname{tr}(F_1(A) - F_2(A))| \leqq \varepsilon \operatorname{tr}|A|$.*

Proof. Let $\{\lambda_k\}$ be the eigenvalues of A; then $|\lambda_k| \leqq B$ for all k and

$$\begin{aligned}|\operatorname{tr}(F_1(A) - F_2(A))| &= \left|\sum_{k=1}^{\infty} ((F_1(\lambda_k) - F_2(\lambda_k))\right| \leqq \sum_{k=1}^{\infty} |F_1(\lambda_k) - F_2(\lambda_k)| \\ &\leqq \sum_{k=1}^{\infty} \varepsilon |\lambda_k| = \varepsilon \operatorname{tr}|A|.\end{aligned}$$

Lemma 24. *Suppose $f \in \mathcal{S}(R^s)$ is real valued and P_κ is as above. Then, if G is any polynomial of the form $G(r) = \sum_{n=1}^{N} c_n r^n$, we have*

$$\lim_{\kappa \to \infty} \frac{1}{V(\kappa)} \operatorname{tr}(G(P_\kappa M_f P_\kappa)) = \int_{R^s} dx\, G(f(x)).$$

Proof. It clearly suffices to consider the case $G(r) = r^n (n \geqq 1)$. A trace class integral operator with kernel $K(k, k')$ has trace $\int_{R^s} K(k, k)\, dk$, and thus by taking Fourier transforms and changing variables, we obtain

$$\begin{aligned}\operatorname{tr}((P_\kappa M_f P_\kappa)^n) = \int dP\, dp_1 \ldots dp_{n-1}\, &\chi_\kappa(P) \chi_\kappa(P + p_1) \ldots \chi_\kappa(P + p_{n-1}) \\ &\cdot \hat{f}(-p_{n-1}) \hat{f}(p_{n-1} - p_{n-2}) \ldots \hat{f}(p_2 - p_1) \hat{f}(p_1),\end{aligned}$$

where χ_κ is the characteristic function of O_κ, the ball of radius κ in R^s.

Let H_κ be defined as

$$H_\kappa(p_1, \ldots, p_{n-1}) = \frac{1}{V(\kappa)} \int dP\, \chi_\kappa(P)\, \chi_\kappa(P + p_1) \ldots \chi_\kappa(P + p_{n-1}).$$

Then if m denotes Lebesgue measure on R^s and $O^{(p)}$ for a set $O \subset R^s$ denotes the set translated by the vector $p \in R^s$, we have

$H_\kappa = \frac{1}{V(\kappa)} m\left(\left(\bigcap_{k=1}^{n-1} O_\kappa^{(-p_k)}\right) \cap O_\kappa\right)$. Since $m(O_\kappa) = V(\kappa)$ we clearly have that $0 \leqq H_\kappa \leqq 1$ and it is easily seen geometrically that for fixed $p_1, p_2, \ldots, p_{n-1}$; $\lim_{\kappa\to\infty} H_\kappa = 1$. We thus have by the Dominated Convergence Theorem that

$$\begin{aligned}\lim_{\kappa\to\infty} \frac{1}{V(\kappa)} \operatorname{tr}((P_\kappa M_f P_\kappa)^n) &= \int \hat{f}(-p_{n-1}) \hat{f}(p_{n-1} - p_{n-2}) \ldots \hat{f}(p_2 - p_1) \hat{f}(p_1)\, dp_1 \ldots dp_{n-1} \\ &= \int (f(x))^n\, dx\,.\end{aligned}$$

Lemma 25. *Let f and P_κ be as in Lemma 24, then*

$$\limsup_{\kappa\to\infty} \frac{1}{V(\kappa)} \operatorname{tr}(|P_\kappa M_f P_\kappa|) \leqq \int_{R^s} |f(x)|\, dx\,.$$

Proof. We decompose f into its positive and negative parts, $f(x) = f_+(x) - f_-(x)$ with f_+ and f_- both positive functions. We may then write $M_f = M_+ - M_-$ in the obvious way and using the fact that for any two trace class operators A and B, $\operatorname{tr}(|A+B|) \leqq \operatorname{tr}(|A|) + \operatorname{tr}(|B|)$ [24], we find that

$$\begin{aligned}&\limsup_{\kappa\to\infty} \frac{1}{V(\kappa)} \operatorname{tr}(|P_\kappa M_f P_\kappa|) \\ &\leqq \limsup_{\kappa\to\infty} \frac{1}{V(\kappa)} \operatorname{tr}(P_\kappa M_+ P_\kappa) + \limsup_{\kappa\to\infty} \frac{1}{V(\kappa)} \operatorname{tr}(P_\kappa M_- P_\kappa)\,,\end{aligned}$$

since $|P_\kappa M_\pm P_\kappa| = P_\kappa M_\pm P_\kappa$. Since the proof of Lemma 24 shows that its conclusions are still true even if f is only continuous (and rapidly decreasing) we have $\limsup_{\kappa\to\infty} \frac{1}{V(\kappa)} \operatorname{tr}(P_\kappa M_\pm P_\kappa) = \int_{R^s} f_\pm(x)\, dx$. This completes the proof of the lemma.

Completion of Proof of Theorem 22. We first note that $\frac{\ln(1-ir)}{r}$ is continuous on R and thus by the Weierstrass approximation theorem, we can find for any B and $\varepsilon > 0$ a polynomial $G'_{B,\varepsilon}(r)$ such that $\left|\frac{\ln(1-ir)}{r} - G'_{B,\varepsilon}(r)\right| < \varepsilon$ for all $|r| < B$. If we let $G_{B,\varepsilon}(r) = r G'_{B,\varepsilon}(r)$ then we have $|\ln(1-ir) - G_{B,\varepsilon}(r)| < \varepsilon |r|$ for all $|r| < B$. Thus by Lemma 23, if we choose $B > \|M_g\| \geqq \sup_\kappa \|P_\kappa M_g P_\kappa\|$, we have

$$\begin{aligned}&\limsup_{\kappa\to\infty} \left|\frac{1}{V(\kappa)} \operatorname{tr}(\ln(1 - iP_\kappa M_g P_\kappa)) - \frac{1}{V(\kappa)} \operatorname{tr}(G_{B,\varepsilon}(P_\kappa M_g P_\kappa))\right| \\ &\qquad \leqq \limsup_{\kappa\to\infty} \frac{\varepsilon}{V(\kappa)} \operatorname{tr}|P_\kappa M_g P_\kappa|\,.\end{aligned} \tag{5.16}$$

Now it follows from Lemma 25 that the right hand side of (5.16) is proportional to ε, and from Lemma 24 that $\lim_{\kappa\to\infty}\frac{1}{V(\kappa)}\operatorname{tr}(G_{B,\varepsilon}(P_\kappa M_g P_\kappa))$ exists and equals $\int dx\, G_{B,\varepsilon}(g(x))$. Thus we have that $\lim_{\kappa\to\infty}\frac{1}{V(\kappa)}\operatorname{tr}(\ln(1-iP_\kappa M_g P_\kappa))$ exists and equals

$$\lim_{\varepsilon\to 0}\int dx\, G_{B,\varepsilon}(g(x)) = \int dx \ln(1-ig(x)).$$

This completes the proof of the theorem.

The relevance of Theorem 22 can perhaps be better understood by considering the generating functional for $\phi_R^2(f)$ in the *relativistic* Fock space representation of the CCR. A calculation analogous to that of Theorem A.1 yields that in this case

$$(\Omega_R, e^{i\phi_R^2(f)}\Omega_R) = e^{-\frac{1}{2}\operatorname{tr}(\ln(1-iA_\mu M_f A_\mu))},$$

where M_f is the multiplication by f operator and $A_\mu = (\mu^2-\Delta)^{-\frac{1}{4}}$. Here Ω_R denotes the relativistic no-particle vacuum. An expansion of $\ln(1-iA_\mu M_f A_\mu)$ in a Taylor series shows, that in $(s+1)$-dimensional space-time, the operators $(A_\mu M_f A_\mu)^n$ are *not* trace class (although they are compact) when $n \leqq s$ while they are trace class for $n > s$. Thus, in two dimensional space time, Wick ordering subtracts the first term in the Taylor series and $:\phi_R^2(f):$ is in fact a well defined operator. In higher dimensions, Wick ordering is not sufficient and $:\phi_R^2(f):$ is not a well defined operator when smeared only in space; $(s-1)$ more renormalizations of some kind are needed to define $\phi_R^2(x)$ at fixed time.

In the ultralocal case, as we have seen, there are no momentum damping factors and thus all terms in the Taylor series are divergent with the same $\delta(0) = \int dk$ momentum-volume divergence in each term. Somehow the transition to the ultralocal current representation of $S_\mu(x)$ simultaneously renormalizes all the terms by dividing away the infinity (rather than by subtracting s terms as would be appropriate in the relativistic case). It can certainly be said that the situation is not unambiguous.

A final interesting question is whether this notion of "thermodynamic" limit, as given in Theorem 22, has anything to do with thermodynamics. One might expect that a statistical mechanics analysis of these representations would be more hospitable to the fact that varying $|K|^2$ yields locally inequivalent representations of the (S, T) current algebra. This would especially be true if the position locality in our present interpretation became a momentum space locality in the statistical mechanics interpretation. It would then be similar to the situation in representations of the CCR appropriate for statistical mechanics [25], in which different densities of particles in position space often correspond to infinite differences in numbers of particles in regions of momentum space.

Appendix A. The Ultralocal Free Canonical Field Theory

There are several equivalent ways to define $\phi_\mu(\boldsymbol{x})$ and $\pi_\mu(\boldsymbol{x})$, the ultralocal free canonical fields of mass μ. If we let $\phi_\mu^R(\boldsymbol{x})$ and $\pi_\mu^R(\boldsymbol{x})$ denote the usual relativistic (Boson) Fock space representation of the CCR for mass μ and $P_{(\mu)}$ denote the positive operator $(\mu^2 - \nabla^2)$ on $L^2(R^s, d\boldsymbol{x})$, then we have

$$\phi_\mu(f) = \phi_\mu^R(P_{(\mu)}^{\frac{1}{4}} f), \qquad \pi_\mu(g) = \pi_\mu^R(P_{(\mu)}^{-\frac{1}{4}} g), \tag{A.1}$$

where $\phi_\mu(f) = \int \phi_\mu(\boldsymbol{x}) f(\boldsymbol{x})\, d\boldsymbol{x}$ and the other smeared fields are defined similarly.

This ultralocal representation of the CCR may also be defined in terms of its generating functional [26]. We let Ω_μ^ϕ be the ground state, in the representation space, of the ultralocal free Hamiltonian,

$$\begin{aligned} \bar{H}_\mu^\phi &= \mu \int a^\dagger(\boldsymbol{k})\, a(\boldsymbol{k})\, d\boldsymbol{k} \\ &= \tfrac{1}{2} \int :\pi_\mu^2(\boldsymbol{x}) + \mu^2 \phi_\mu^2(\boldsymbol{x}):\, d\boldsymbol{x} = \tfrac{1}{2} \int (\pi_\mu^2(\boldsymbol{x}) + \mu^2 \phi_\mu^2(\boldsymbol{x}) - \mu)\, d\boldsymbol{x}\,, \end{aligned} \tag{A.2}$$

where $a^\dagger(\boldsymbol{k})$ and $a(\boldsymbol{k})$ are the ultralocal creation and annihilation operators defined in a standard way. The generating functional is then defined as

$$E_\mu^{\phi *}(f, g) = (\Omega_\mu^\phi, e^{i\phi_\mu(f)} e^{i\pi_\mu(g)} \Omega_\mu^\phi)\,, \tag{A.3}$$

and is of the form

$$E_\mu^{\phi *}(f, g) = \exp\left(\int d\boldsymbol{x}\, D_\mu^\phi(f(\boldsymbol{x}), g(\boldsymbol{x}))\right), \tag{A.4}$$

with

$$D_\mu^\phi(z, w) = -i\frac{zw}{2} - \frac{z^2}{4\mu} - \mu\frac{w^2}{4}\,.$$

We note that D_μ^ϕ is also defined by the formula,

$$\exp(D_\mu^\phi(z, w)) = \langle \Phi_\mu, e^{izq} e^{iwp} \Phi_\mu \rangle\,, \tag{A.5}$$

where Φ_μ is the ground state of $\frac{1}{2}(p^2 + \mu^2 q^2)$ on $L^2(R, dq)$.

This representation of the CCR can also be realized on an infinite tensor product Hilbert space in a simple way by choosing an orthonomal basis of real functions, $\{f_k\}_{k=1}^\infty$, on $L^2(R^s, d\boldsymbol{x})$, and defining $q_k = \phi_\mu(f_k)$ and $p_k = \pi_\mu(f_k)$. The representation space is then the incomplete ITP space containing the vector,

$$\boldsymbol{\Phi}_\mu^q = \prod_{k=1}^{\infty} \otimes \Phi_\mu(q_k)\,, \tag{A.6}$$

and $\bar{H}_\mu^\phi$ is realized on this space as

$$\bar{H}_\mu^q = \sum_{k=1}^{\infty} \frac{1}{2}(p_k^2 + \mu^2 q_k^2 - \mu). \tag{A.7}$$

When $F(x, y)$ is of the form, $\sum_{k=1}^{N} r_k f_k(x)\, g_k(y)$, for f_k and g_k in $L^2(R^s, dx)$, we define

$$\int \phi_\mu(x)\, \phi_\mu(y)\, F(x, y)\, dx\, dy = \sum_{k=1}^{N} r_k \phi_\mu(f_k)\, \phi_\mu(g_k), \tag{A.8}$$

and for more general $F(x, y)$ by taking limits of such finite sums. We then have

Theorem A.1. *If F is a trace class self adjoint integral operator with a real symmetric kernel, $F(x, y)$, then $Q \equiv \int \phi_\mu(x)\, \phi_\mu(y)\, F(x, y)\, dx\, dy$ is a well defined self adjoint operator and*

$$(\Omega_\mu^\phi, e^{iQ}\Omega_\mu^\phi) = e^{-\frac{1}{2}\operatorname{tr}\left(\ln\left(1 - i\frac{F}{\mu}\right)\right)}. \tag{A.9}$$

Proof. We first choose an orthonormal basis of eigenvectors of F, $\{g_k\}_{k=1}^{\infty}$. The g_k may be chosen real because the kernel, $F(x, y)$, is real. We let $q_k = \phi_\mu(g_k)$ and $p_k = \pi_\mu(g_k)$ and realize the ultralocal Fock space as an ITP space, as described above. $F(x, y)$ is given by $F(x, y) = \sum_{k=1}^{\infty} \lambda_k g_k(x)\, g_k(y)$ with $\{\lambda_k\}$ the eigenvalues of F, and thus $Q = \lim_{N\to\infty} \sum_{k=1}^{N} \lambda_k q_k^2$, if this limit exists.

We note that $\sum_{k=1}^{\infty} |\lambda_k| < \infty$ since F is trace class and thus we also have that $\sum_{k=1}^{\infty} |\langle \Phi_\mu, \lambda_k q_k^2 \Phi_\mu \rangle| < \infty$ and $\sum_{k=1}^{\infty} |\langle \lambda_k q_k^2 \Phi_\mu, \lambda_k q_k^2 \Phi_\mu \rangle| < \infty$. Using certain results from the theory of ITP spaces [7, Cor. 2.5 and Thm. 2.6], we may define Q_0 as $\lim_{N\to\infty} \sum_{k=1}^{N} \lambda_k q_k^2$ on vectors of the form

$$\left(\prod_{k=1}^{M} \otimes \psi_k(q_k)\right) \otimes \left(\prod_{k=M+1}^{\infty} \otimes \Phi_\mu(q_k)\right),$$

with $\psi_k \in \operatorname{Dom}(q_k^2)$; it then follows that Q_0 is essentially self adjoint on the domain consisting of finite linear combinations of such vectors. We define Q as the closure of Q_0 and it also follows that $e^{iQ} = \text{strong} \lim_{N\to\infty} \left(\prod_{k=1}^{N} \exp(i\lambda_k q_k^2)\right)$. Thus

$$(\Phi_\mu^q, e^{iQ}\Phi_\mu^q) = \lim_{N\to\infty} \prod_{k=1}^{N} \langle \Phi_\mu, e^{i\lambda_k q_k^2} \Phi_\mu \rangle. \tag{A.10}$$

It can now be shown by direct calculation that

$$\langle \Phi_\mu(q), e^{i\lambda q^2}\Phi_\mu(q)\rangle = (1-i\lambda/\mu)^{-\frac{1}{2}} = e^{-\frac{1}{2}\ln(1-i\lambda/\mu)}.$$

Combining this with (A.10) and using the fact that $\sum_{k=1}^{\infty} |\lambda_k| < \infty$ implies that $\sum_{k=1}^{\infty} |\ln(1-i\lambda_k/\mu)| < \infty$, we find that

$$(\boldsymbol{\Phi}_\mu^q, e^{iQ}\boldsymbol{\Phi}_\mu^q) = \exp\left(-\frac{1}{2}\sum_{k=1}^{\infty} \ln(1-i\lambda_k/\mu)\right).$$

This completes the proof of the theorem, since

$$\operatorname{tr}(\ln(1-iF/\mu)) = \sum_{k=1}^{\infty} \ln(1-i\lambda_k/\mu).$$

Appendix B. $SL(2, R)$ and Its Universal Covering Group

We first review some facts concerning the (abstract) group $G_0 = SL(2, R)$, which we shall be using [27]. Considered as $SL(2, R)$, the Lie algebra consists of all real 2×2 traceless matrices. We choose as a basis:

$$A = \frac{1}{2}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \quad B = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \quad C = \frac{1}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}. \tag{B.1}$$

If we define $D = \dfrac{A-C}{2}$, then any $g\in G_0$ can be expressed as

$$g = e^{zD}e^{wB}e^{tA} \tag{B.2}$$

for some $z, w, t \in R$. This expression is unique except that $\exp(tA) = \exp((t+4\pi n)A)$ for all integers n. An alternative parametrization is

$$G_0 = \{(\gamma, \omega \mid \gamma \in C, |\gamma| < 1, -\pi < \omega \leqq \pi\}. \tag{B.3}$$

In this form the group multiplication is given by [27, p. 30] as $(\gamma,\omega)\cdot(\gamma',\omega') = (\gamma'',\omega'')$, where

$$\gamma'' = (\gamma + \gamma' e^{-2i\omega})(1+\bar{\gamma}\gamma' e^{-2i\omega})^{-1} \tag{B.4a}$$

and

$$\omega'' = \omega + \omega' + \frac{1}{2i}\log((1+\bar{\gamma}\gamma' e^{-2i\omega})(1+\gamma\bar{\gamma}' e^{2i\omega})^{-1}) \tag{B.4b}$$

with $\log x$ defined by its principal value and ω'' taken mod 2π.

The relation between the two parametrizations, (B.2) and (B.3), is given by

$$e^{zD}e^{wB}e^{tA} = \left(\gamma(z,w), \omega(z,w) + \frac{t}{2}\right) \tag{B.5}$$

where

$$\gamma(z,w)=\frac{-\sinh\frac{w}{2}-ie^{-\frac{w}{2}}\frac{z}{4}}{\cosh\frac{w}{2}+ie^{-\frac{w}{2}}\frac{z}{4}}, \tag{B.6a}$$

$$\omega(z,w)=\arg\left(\cosh\frac{w}{2}+ie^{-\frac{w}{2}}\frac{z}{4}\right), \tag{B.6b}$$

and $\omega(z,w)+\frac{t}{2}$ in (B.5) is understood to be taken mod 2π. These formulae may be used to obtain the following useful result.

Proposition B.1. *In G_0, we have*

$$(0,t)(\gamma,\omega)=(\gamma e^{-2it}, t+\omega) \tag{B.7a}$$

and

$$(\gamma',\omega')(0,t')=(\gamma',\omega'+t'), \tag{B.7b}$$

with $t+\omega$ and $\omega'+t'$ taken mod 2π. *Thus for any $t, z, w \in R$*

$$e^{tA}e^{zD}e^{wB}=e^{z'D}e^{w'B}e^{t'A} \tag{B.8}$$

for some $z', w', t' \in R$, with

$$e^{-it}\left(\frac{1-y(z,w)}{y(z,w)}\right)=\frac{1-y(z',w')}{y(z',w')} \tag{B.9}$$

where $y(z,w)=\frac{1+e^w}{2}+i\frac{z}{4}$.

Proof. (B.7) follows immediately from the multiplication law for G_0 given by (B.4). The parametrization of G_0 given by (B.2) clearly implies the existence of z', w', t' satisfying (B.8), and then by (B.7) and (B.5) we see that $e^{-it}\gamma(z,w)=\gamma(z',w')$. Since $\gamma(z,w)=\frac{1-y(z,w)}{y(z,w)}$, the proof is complete.

We next list the continuous unitary irreducible representations of G_0; they were first classified by Bargmann [28]:

(a) The *principal series*, $g\to U_h(g,s)$, $h=0$, $h=\frac{1}{2}$, $s\in iR$ (excluding $h=\frac{1}{2}$, $s=0$).

(b) The *complementary series*, $g\to U_h(g,\sigma)$, $h=0$, $0<\sigma<\frac{1}{2}$.

(c) The *discrete series*, $g\to U^+(g,h)$, $h=\frac{1}{2}, 1, \frac{3}{2}, \ldots$

$g\to U^-(g,h)$, $h=-\frac{1}{2}, -1, -\frac{3}{2}, \ldots$

We will be exclusively concerned with the discrete representations. We note that when h is an integer, $U^\pm(g,h)$ also defines a representation of $SO(2,1)$.

We now review some properties of G_C, the universal covering group of G_0 [27]. The Lie algebra of G_C is identical to that of G_0 and we label the corresponding elements again by A, B, C, and D. In G_C, any g may be *uniquely* expressed in the form (B.2). G_C may be parametrized in another form as

$$\{(\gamma, \omega) \mid \gamma \in C, |\gamma| < 1, -\infty < \omega < +\infty\} \tag{B.10}$$

with the same multiplication law (given by (B.4)) as for G_0 except that ω'' is *not* taken mod 2π. The canonical homomorphism of G_C onto G_0 is given by $(\gamma, \omega) \to (\gamma, \omega(\text{mod}\, 2\pi))$. The relation between these two parametrizations is still given by (B.5) and (B.6) except that in (B.5), $\omega(z, w) + t/2$ is also *not* taken mod 2π. We note that $F_C = \{e^{zD} e^{wB}\}$ is a subgroup of G_C (and similarly of G_0) isomorphic to the group F defined by (2.5).

The continuous unitary irreducible representations of G_C have been determined by Pukanszky [29]. The list is identical to the one given above for G_0 except that in the principal series, we allow $-\frac{1}{2} < h \leqq \frac{1}{2}$; in the complementary series, we allow $-\frac{1}{2} < h < \frac{1}{2}$; and in the discrete series we allow $h > 0$ for $U^+(g, h)$ and $h < 0$ for $U^-(g, h)$. We will again be concerned with the discrete series of representations.

In the usual formulation, the positive discrete representation $U^+(g, h)$ is realized on $\mathfrak{H}_{(h)}$, a Hilbert space of functions analytic on the unit disk in C. If $f(y)$ and $g(y)$ are two analytic functions in $\mathfrak{H}_{(h)}$ with Taylor series, $\sum_{n=0}^{\infty} a_n y^n$ and $\sum_{n=0}^{\infty} b_n y^n$ respectively, then the inner product in $\mathfrak{H}_{(h)}$ is defined to be

$$\langle f, g \rangle_h = \sum_{n=0}^{\infty} \frac{\Gamma(2h)\, \Gamma(n+1)}{\Gamma(2h+n)} \bar{a}_n b_n . \tag{B.11}$$

$U^+(g, h)$ can be defined as follows on $\mathfrak{H}_{(h)}$:

$$U^+(e^{tA}, h) : f(y) \rightsquigarrow e^{ith} f(e^{it} y) \tag{B.12a}$$

$$U^+(e^{zD} e^{wB}, h) : f(y) \rightsquigarrow e^{2i\omega(z,w)h} (1 - |\gamma(z, w)|^2)^h \left(1 + \overline{\gamma(z, w)}\, y\right)^{-2h}$$

$$\cdot f\left(\frac{\left(\cosh \frac{w}{2} + i e^{-\frac{w}{2}} \frac{z}{4}\right) y + \left(-\sinh \frac{w}{2} - i e^{-\frac{w}{2}} \frac{z}{4}\right)}{\left(-\sinh \frac{w}{2} + i e^{-\frac{w}{2}} \frac{z}{4}\right) y + \left(\cosh \frac{w}{2} - i e^{-\frac{w}{2}} \frac{z}{4}\right)} \right). \tag{B.12b}$$

In this realization, we see that $U^+(A, h)$ takes on a particularly simple form while the subgroup F_C is represented in a considerably more complicated manner. The remainder of this appendix is devoted to obtaining different realizations of the positive discrete representations in which F_C is represented in a simple manner. These realizations will utilize $\mathfrak{K}_0 = L^2((0,\infty), d\lambda)$ as the representation space. For each fixed positive μ, D will be realized as $i\mu\lambda$ and B as $\frac{1}{2}\left(\lambda \frac{d}{d\lambda} + \frac{d}{d\lambda}\lambda\right)$ independent of h while A will be of the form $\frac{i}{4\mu}\cdot\left(-\sqrt{\lambda}\frac{d^2}{d\lambda^2}\sqrt{\lambda} + \frac{h(h-1)}{\lambda} + 4\mu^2\lambda\right)$. In order to define these realizations rigorously, we need some preliminary results.

Proposition B.2. *Let* $D_0 = C_0^\infty((0,\infty)) \subset \mathfrak{K}_0$. *Then the operator* Q_C, *defined on* D_0 *as* $\left(-\sqrt{\lambda}\frac{d^2}{d\lambda^2}\sqrt{\lambda} + \frac{C}{\lambda} + 4\mu^2\lambda\right)$, *is positive (as a quadratic form on* $D_0 \times D_0$*) for* $C \geqq -\frac{1}{4}$. Q_C *is essentially self adjoint when* $C \geqq 0$ *and for* $-\frac{1}{4} \leqq C < 0$, *it has (positive) self adjoint extensions.*

Proof. To prove positivity [30], we need only show that

$$\left(g, \left(-\sqrt{\lambda}\frac{d^2}{d\lambda^2}\sqrt{\lambda} - \frac{1}{4\lambda}\right)g\right) \geqq 0$$

for all $g \in D_0$. But for $g \in D_0$ we have that

$$\begin{aligned}\left(-\sqrt{\lambda}\frac{d^2}{d\lambda^2}\sqrt{\lambda} - \frac{1}{4\lambda}\right)g &= \sqrt{\lambda}\left(-\frac{d^2}{d\lambda^2} - \frac{1}{4\lambda^2}\right)\sqrt{\lambda}g \\ &= \sqrt{\lambda}\left(-\frac{d}{d\lambda} + \frac{1}{2\lambda}\right)\left(+\frac{d}{d\lambda} + \frac{1}{2\lambda}\right)\sqrt{\lambda}g;\end{aligned}$$

so we have, by integration by parts, that

$$\begin{aligned}&\left(g, \left(-\sqrt{\lambda}\frac{d^2}{d\lambda^2}\sqrt{\lambda} - \frac{1}{4\lambda}\right)g\right) \\ &\quad = \left(\left(+\frac{d}{d\lambda} + \frac{1}{2\lambda}\right)\sqrt{\lambda}g, \left(+\frac{d}{d\lambda} + \frac{1}{2\lambda}\right)\sqrt{\lambda}g\right) \geqq 0.\end{aligned}$$

Since positive quadratic forms have positive self adjoint extensions (for example, the Friedrichs extension), it only remains to show that Q_C is essentially self adjoint when $C \geqq 0$.

To show that Q_C is essentially self adjoint, it suffices to show that $Q_C^* \Phi = \pm i\Phi$ has no $L^2(0,\infty)$ solutions. By local regularity for differential equations, such a Φ is C^∞ on $(0,\infty)$ and satisfies (on $(0,\infty)$)

$$\left(-\sqrt{\lambda}\frac{d^2}{d\lambda^2}\sqrt{\lambda} + \frac{C}{\lambda} + 4\mu^2\lambda \mp i\right)\Phi(\lambda) = 0.$$

Let $f(y)=\sqrt{y}\,\Phi\left(\frac{y}{4\mu}\right)$; then f satisfies the differential equation

$$\left(\frac{d^2}{dy^2}+\left[-\frac{1}{4}-\frac{C}{y^2}\pm\frac{i}{4\mu y}\right]\right)f(y)=0\,. \tag{B.13}$$

The two independent solutions to (B.13) are $f_k(y)=\sqrt{y}\,\Phi_k\left(\frac{y}{4\mu}\right)$, where

$$f_1(y)=\boldsymbol{M}_{\pm\frac{i}{4\mu},\sqrt{C+\frac{1}{4}}}(y)\,,\qquad f_2(y)=\boldsymbol{W}_{\pm\frac{i}{4\mu},\sqrt{C+\frac{1}{4}}}(y)\,, \tag{B.14}$$

where $\boldsymbol{M}$ and $\boldsymbol{W}$ are Whittaker's functions [31]. Asymptotically, as $y\to\infty$, Φ_1 is not in L^2 while Φ_2 is in L^2; when $C\geqq 0$, Φ_1 is in L^2 near the origin, but Φ_2 is not. For $-\frac{1}{4}\leqq C<0$, $\Phi_2(y)$ is in $L^2((0,\infty))$ and Q_C is not essentially self adjoint. This completes the proof of the proposition.

We define R_h for $h\geqq 1$ as *the* self adjoint extension of $\frac{1}{2}Q_{h(h-1)}$. We will now define a *particular* self adjoint extension of $\frac{1}{2}Q_{h(h-1)}$ for $0<h<1$, as $h(h-1)$ varies between 0 and $-\frac{1}{4}$, by adding a vector to the Domain D_0. Let $g(\lambda)$ be in $C_0^\infty([0,\infty))$, such that $g(\lambda)=1$ on $[0,1)$, $g(\lambda)=0$ on $[2,\infty)$ and $0\leqq g(\lambda)\leqq 1$. Let $g_h(\lambda)=\lambda^{h-\frac{1}{2}}g(\lambda)$ for $h>0$. Then $g_h(\lambda)\in\mathfrak{K}_0$ for all $h>0$, and we define D_h as $\{\psi\,|\,\psi=cg_h(\lambda)+\Phi$ for some $\Phi\in D_0$ and some complex $c\}$. To extend $Q_{h(h-1)}$ from D_0 to D_h, we need only define it on g_h; we therefore *define*

$$(R'_h g_h)(\lambda)=\begin{cases}2\mu^2\lambda g_h(\lambda)\,, & \lambda\in[0,1]\\[4pt] \dfrac{1}{2}\left(-\sqrt{\lambda}\dfrac{d^2}{d\lambda^2}\sqrt{\lambda}+\dfrac{h(h-1)}{\lambda}+4\mu^2\lambda\right)g_h(\lambda)\,, & \lambda\notin[0,1]\end{cases}$$

and $R'_h\Phi=\frac{1}{2}Q_{h(h-1)}\Phi$, for all $\Phi\in D_0$; R'_h is thus defined on D_h. It is easily seen that R'_h is symmetric on $D_h\times D_h$ and in fact, letting $R'_h=\frac{1}{2}Q_{h(h-1)}$ and $D_h=D_0$ for $h\geqq 1$, we have,

Proposition B.3. *R'_h is essentially self adjoint on D_h for all $h>0$. The normalized eigenbasis of R_h, its self adjoint extension, is given by*

$$R_h\Psi_{m,h}(\lambda)=2\mu(m+h)\,\Psi_{m,h}(\lambda)\qquad(m=0,1,2,\ldots) \tag{B.15}$$

with

$$\Psi_{m,h}(\lambda)=(-1)^m\left(\frac{m!}{\Gamma(m+2h)}\right)^{\frac{1}{2}}\frac{(4\mu\lambda)^h}{\lambda^{\frac{1}{2}}}\,e^{-2\mu\lambda}L_m^{(2h-1)}(4\mu\lambda)\,, \tag{B.16}$$

where $L_m^{(\alpha)}(y)$ is an associated Laguerre polynomial [23, *p.* 1037].

Proof. We must show that $R'^*_h\Phi=\pm i\Phi$ has no solutions $\Phi\in\text{Domain}(R'^*_h)$. We are again led to the differential equation (B.13) and the solutions, f_i, of (B.14), with $\sqrt{C+\frac{1}{4}}=|h-\frac{1}{2}|$. Again Φ_1 has exponential growth at ∞ for *all* $h>0$ and is therefore not in $\mathfrak{K}_0$, let alone

$\mathrm{Dom}(R_h'^*)$. It thus suffices to show that for $0<h<1$, $\Phi_2 \notin \mathrm{Dom}(R_h'^*)$. This follows from the fact that near $y=0$, for $0<h<1\,(h \neq \frac{1}{2})$,

$$\frac{1}{\sqrt{y}} W_{\pm\frac{i}{4\mu},|h-\frac{1}{2}|}(y) \sim C_1 y^{h-\frac{1}{2}} + C_2 y^{-(h-\frac{1}{2})}$$

while

$$\frac{1}{\sqrt{y}} W_{\pm\frac{i}{4\mu},0}(y) \sim \ln y$$

It is easily seen that (near $y=0$), $y^{-(h-\frac{1}{2})} \notin \mathrm{Dom}(R_h'^*)$ (for $h \neq \frac{1}{2}$) and $\ln y \notin \mathrm{Dom}(R_{\frac{1}{2}}'^*)$ and the proof of essential self adjointness is completed. The determination of the eigenbasis is done in a straightforward manner using the small z behavior of the Whittaker function, $\boldsymbol{W}$. Alternately, one can directly verify that the $\Psi_{m,h}$ are indeed eigenvectors of R_h[7] and that they form a complete basis.

We are now in a position to prove the main result of this appendix.

Theorem B.4. *For each $h>0$ and $\mu>0$, there exists a continuous irreducible unitary representation, $U_\mu(g,h)$, of G_C, the universal covering group of $SL(2,R)$, on $L^2((0,\infty),d\lambda)$ such that*

$$U_\mu(A,h) = \frac{i}{2\mu} R_h\,, \qquad U_\mu(B,h) = \frac{1}{2}\left(\lambda \frac{d}{d\lambda} + \frac{d}{d\lambda}\lambda\right), \tag{B.17}$$
$$U_\mu(D,h) = i\mu\lambda\,.$$

$U_\mu(g,h)$ is unitarily equivalent to the positive discrete representation, $U^+(g,h)$. The normalized basis of R_h is as given in Proposition B.3.

Proof. Sally has constructed the "normalized discrete" representations, $R^+(g,h)$, of G_C on $\mathfrak{H}_{(\frac{1}{2})}$ which are unitarily equivalent to $U^+(g,h)$ on $\mathfrak{H}_{(h)}$ [27, Ch. 4]. These representations have the property that $R^+\left(e^{z\left(\frac{A+C}{2}\right)} e^{-wB}, h\right)$ is independent of h. We will transform these representations of Sally onto $\mathfrak{K}_0$ to yield $U_\mu(g,h)$.

Define $F_{\frac{1}{2}}: \mathfrak{H}_{(\frac{1}{2})} \to \mathfrak{K}_0$ by $\Phi_{n,\frac{1}{2}}(y) \to \Psi'_{n,\frac{1}{2}}(\lambda)$, where $\Phi_{n,\frac{1}{2}}(y) = y^n$ and $\Psi'_{n,\frac{1}{2}}(\lambda) = \sqrt{2}(-1)^n e^{-\lambda} L_n^{(0)}(2\lambda)$. Then, since $F_{\frac{1}{2}}$ maps one orthonormal basis onto another, it determines a unitary operator between $\mathfrak{H}_{(\frac{1}{2})}$ and $\mathfrak{K}_0$. We then define $\tilde{R}^+(g,h) = F_{\frac{1}{2}} R^+(g,h) F_{\frac{1}{2}}^{-1}$ as a representation realized on $\mathfrak{K}_0$. We are not yet finished since we require that $U_\mu\left(e^{z\left(\frac{A-C}{2}\right)} e^{wB}, h\right)$ be independent of h. To obtain this, we use the fact that in G_C,

$$e^{\pi A} e^{z\left(\frac{A-C}{2}\right)} e^{wB} e^{-\pi A} = e^{z\left(\frac{A+C}{2}\right)} e^{-wB}\,.$$

[7] It is easily shown that there is no difficulty with "boundary conditions" at $\lambda=\infty$.

We thus define

$$V(g,h)=\tilde{R}^+(e^{\pi A},h)\,\tilde{R}^+(g,h)\,\tilde{R}^+(e^{-\pi A},h)\,.$$

$V(g,h)$ is then unitarily equivalent to $U^+(g,h)$ and has the property that $V(B,h)$ and $V(D,h)$ are independent of h as desired. The only remaining difference between V and U_μ is a scaling which is accomplished by defining

$$U_\mu(g,h)=V(e^{(\ln(2\mu))B},h)\;V(g,h)\;V(e^{-(\ln(2\mu))B},h)\,.$$

$U_\mu(g,h)$ is clearly unitarily equivalent to $U^+(g,h)$.

Using the results of Sally, it is easily seen that

$$\tilde{R}^+\left(e^{z\left(\frac{A+C}{2}\right)},h\right):\Phi(\lambda)\rightsquigarrow e^{\iota\left(\frac{z}{2}\right)\lambda}\Phi(\lambda)$$

$$\tilde{R}^+(e^{-wB},h):\Phi(\lambda)\rightsquigarrow e^{+\frac{w}{2}}\Phi(e^w\lambda)$$

and that $\tilde{R}^+(e^{tA},h)$ has the (unnormalized) eigenstates $\Psi'_{n,h}(\lambda)=e^{-\lambda}(\lambda)^{h-\frac{1}{2}}L_n^{(2h-1)}(2\lambda)$ of eigenvalue $e^{it(n+h)}$. These facts immediately yield, with the help of Proposition B.3, the results of the theorem for the representation of the infinitesimal generators A, B, and D.

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