

On Event Horizons in Static Space-Times^{*}

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Abstract. A proof of the (vacuum) Israel theorem on event horizons in static space-times is given employing the Newman-Penrose formalism. The theorem is extended to include the case of a static, massive, complex, scalar field.

1. Introduction

In recent years properties of event horizons have been studied in some detail. Israel [1, 2] has shown that among the asymptotically flat, static, vacuum fields only the Schwarzschild solutions with $m > 0$, and among the corresponding electrovac space-times only the Reissner-Nordström solutions with $m^2 \geq \gamma e^2$ have regular event horizons. Chase [3] has found that in the presence of a static, asymptotically flat, massless scalar field the event horizon has to be singular.

It has been conjectured [1, 2, 4], and Carter [5] has essentially proved for the case of axial symmetry, that among the asymptotically flat stationary vacuum space-times only the Kerr solutions with $m \geq a$ have nonsingular event horizons. Unfortunately, Israel's proof of the static case, relying heavily on a three-dimensional formalism, does not easily generalize to the stationary case (where the Killing field is not orthogonal to a family of hypersurfaces). In Sections 3 and 4 of the present paper we give a fairly straightforward proof of the vacuum Israel theorem using the well-known Newman-Penrose formalism [6]. Minimal use is made of the hypersurfaces. It is hoped that this method will generalize to the stationary case. Moreover, the differential equations derived in Section 3 should prove useful in solving various other problems involving static fields.

In Section 5 we extend the horizon theorem to include the case of a general (possibly massive and complex) scalar field. Every such field which is gravitationally coupled, static, and asymptotically flat is found to become singular at a simply connected event horizon.

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2. Notation

The signature of space-time is taken to be $(1, -1, -1, -1)$. Greek indices run from 0 to 3, Latin indices from 2 to 3. The summation convention is used throughout. Ordinary differentiation is denoted by a comma, (two-dimensional) covariant differentiation by a semicolon, symmetrization by round brackets, and complex conjugation by a bar.

The Newman-Penrose formalism is too well known to require much elaboration. The definition of the intrinsic derivatives $(D, \Delta, \delta, \bar{\delta})$, the spin-coefficients $(\kappa, \sigma, \rho, \tau, \varepsilon, \alpha, \beta, \gamma, \pi, \lambda, \mu, \nu)$, the Weyl tensor components $(\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4)$ and the Ricci tensor components $(\Phi_{00}, \Phi_{01}, \Phi_{02}, \Phi_{11}, \Phi_{12}, \Phi_{22}, \Lambda)$ relative to an arbitrary null tetrad $(l^\alpha, n^\alpha, m^\alpha, \bar{m}^\alpha)$ may be found in Ref. [6].

3. Basic Formulas

A static space-time possesses a regular hypersurface-orthogonal Killing vector ξ which is time-like over some domain. In this region we may take the metric to be [1, 2]

$$ds^2 = V^2 dt^2 - \omega^2 dV^2 - P^2 d\theta_2^2 - Q^2 d\theta_3^2, \tag{3.1}$$

where $V \equiv (\xi_\alpha \xi^\alpha)^{1/2}$, and ω, P , and Q are functions of V, θ_2 and θ_3 .

At each point of this domain we take the null tetrad

$$\begin{aligned} l^\alpha &= 2^{-1/2} (V^{-1} \delta_0^\alpha - \omega^{-1} \delta_1^\alpha) \\ n^\alpha &= 2^{-1/2} (V^{-1} \delta_0^\alpha + \omega^{-1} \delta_1^\alpha) \\ m^\alpha &= 2^{-1/2} (P^{-1} \delta_2^\alpha + iQ^{-1} \delta_3^\alpha) \end{aligned} \tag{3.2}$$

and calculate the corresponding spin coefficients from their definition:

$$\kappa = \bar{\pi} = -\tau = -\bar{\nu} = (2\omega)^{-1} \delta\omega, \tag{3.3a}$$

$$\sigma = \lambda = (2^{3/2}\omega)^{-1} [\ln(PQ^{-1})]_{,1}, \tag{3.3b}$$

$$\rho = \mu = (2^{3/2}\omega)^{-1} [\ln(PQ)]_{,1}, \tag{3.3c}$$

$$\alpha = -\bar{\beta} = -(2^{3/2}PQ)^{-1} (Q_{,2} - iP_{,3}), \tag{3.3d}$$

$$\varepsilon = \gamma = -(2^{3/2}\omega V)^{-1}. \tag{3.3e}$$

Taking into account the static nature of the fields the intrinsic derivatives become

$$D = -\Delta = -(2^{1/2}\omega)^{-1} \partial/\partial V \tag{3.4}$$

$$\delta = 2^{-1/2} (P^{-1} \partial/\partial\theta_2 + iQ^{-1} \partial/\partial\theta_3). \tag{3.5}$$

With the aid of Eqs. (3.3)–(3.5) the “Ricci identities” (Eqs. (4.2) of Ref. [6]) reduce to

$$\begin{aligned}
 \Psi_0 &= \bar{\Psi}_4 = -4\varepsilon\sigma + \Phi_{02} \\
 \Psi_1 &= -\bar{\Psi}_3 = 4\varepsilon\kappa - \Phi_{01} \\
 \Psi_2 &= 4\varepsilon\rho + \Phi_{00} - 2\Lambda \\
 \Phi_{22} &= \Phi_{00} \\
 \Phi_{12} &= -\Phi_{01}
 \end{aligned} \tag{3.6}$$

and

$$D\rho = \bar{\delta}\kappa + \rho^2 + \sigma^2 + 2\varepsilon\rho + 2\kappa\bar{\kappa} - 2\kappa\alpha + \Phi_{00}, \tag{3.7a}$$

$$D\sigma = \delta\kappa + 2\rho\sigma - 2\varepsilon\sigma + 2\kappa^2 + 2\kappa\bar{\alpha} + \Phi_{02}, \tag{3.7b}$$

$$D\alpha = \rho\alpha - \sigma\bar{\alpha} - \kappa\sigma + \rho\bar{\kappa} - 2\varepsilon\bar{\kappa} + \Phi_{10}, \tag{3.7c}$$

$$D\varepsilon = -2\varepsilon^2 + 2\varepsilon\rho + \frac{1}{2}(\Phi_{00} - 3\Lambda + \Phi_{11}), \tag{3.7d}$$

$$\delta\varepsilon = -2\varepsilon\kappa, \tag{3.7e}$$

$$\delta\rho - \bar{\delta}\sigma = -4\sigma\alpha - 4\varepsilon\kappa + 2\Phi_{01}, \tag{3.7f}$$

$$\delta\alpha + \bar{\delta}\bar{\alpha} = \rho^2 - \sigma^2 + 4\alpha\bar{\alpha} - 4\varepsilon\rho - \Phi_{00} + 3\Lambda + \Phi_{11}. \tag{3.7g}$$

The “Bianchi identities” [7] yield the further equations

$$\begin{aligned}
 2D\Phi_{01} - \delta\Phi_{00} + \bar{\delta}\Phi_{02} - 4\varepsilon(D\kappa + \delta\rho + \kappa\rho - \bar{\kappa}\sigma) - 6\kappa\Phi_{00} \\
 + (4\varepsilon - 6\rho)\Phi_{01} + 12\kappa\Lambda - 2\sigma\Phi_{10} + (2\bar{\kappa} - 4\alpha)\Phi_{02} = 0 \\
 D\Phi_{11} + 3D\Lambda - D\Phi_{00} - \delta\Phi_{10} - \bar{\delta}\Phi_{01} + (2\rho - 4\varepsilon)\Phi_{00} \\
 + (2\alpha - 4\bar{\kappa})\Phi_{01} + (2\bar{\alpha} - 4\kappa)\Phi_{10} - \sigma(\Phi_{02} + \Phi_{20}) - 4\rho\Phi_{11} = 0 \tag{3.8} \\
 2D\Phi_{01} + \bar{\delta}\Phi_{02} - 2\delta\Phi_{11} - 3\delta\Phi_{00} + 6\delta\Lambda - 14\kappa\Phi_{00} + 36\kappa\Lambda \\
 - 12\varepsilon(D\kappa + \delta\rho - \bar{\kappa}\sigma + \kappa\rho) - 8\kappa\Phi_{11} - 2\sigma\Phi_{10} \\
 + (4\varepsilon - 6\rho)\Phi_{01} + (2\bar{\kappa} - 4\alpha)\Phi_{02} = 0.
 \end{aligned}$$

Due to Eqs. (3.6) the invariant square of the Riemann tensor becomes

$$\begin{aligned}
 \frac{1}{16}R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} &= |\Phi_{02} - 4\varepsilon\sigma|^2 + 4|\Phi_{01} - 4\varepsilon\kappa|^2 + 3(\Phi_{00} - 2\Lambda + 4\varepsilon\rho)^2 \\
 &+ \Phi_{00}^2 + 4|\Phi_{01}|^2 + |\Phi_{02}|^2 + 2\Phi_{11}^2 + 6\Lambda^2.
 \end{aligned} \tag{3.9}$$

We shall also need the equations

$$D\omega = -2\omega\rho + \sqrt{2}\omega^2V(\Phi_{00} - 3\Lambda + \Phi_{11}), \tag{3.10}$$

$$D(PQ) = -2\rho PQ, \tag{3.11}$$

$$D(PQ\omega^{-1}) = -\sqrt{2}PQV(\Phi_{00} - 3\Lambda + \Phi_{11}), \tag{3.12}$$

which are easily derived from Eqs. (3.3b), (3.3c), (3.3e), (3.4) and (3.7d).

The scalar R of intrinsic curvature of the two-space $t = \text{const}$, $V = \text{const}$ is given by [8]

$$-\frac{1}{4}R = -\Psi_2 + \Phi_{11} + \Lambda + \varrho^2 - \sigma^2, \tag{3.13}$$

with ϱ and σ being (apart from a constant factor) the components of extrinsic curvature of the two-surface with respect to the hyper-surface $t = \text{constant}$ (called Σ). From Eqs. (3.3e), (3.6) and (3.13) we obtain a formula we shall need later, namely

$$\varrho V^{-1} = -2^{-1/2}\omega\left[\frac{1}{4}R + \varrho^2 - \sigma^2 + \Phi_{11} + 3\Lambda - \Phi_{00}\right]. \tag{3.14}$$

4. The Israel Theorem

The Israel theorem may be stated as follows [9]: The Schwarzschild solutions with $m > 0$ are the only static, asymptotically flat, vacuum space-times with a family of simply connected equipotential surfaces which converge to a nonsingular event horizon with finite two-dimensional intrinsic geometry. The remainder of the proof of this theorem follows closely Israel's original proof. However, the necessary identities and the interior boundary conditions are obtained from the results of Section 3.

Since the space is asymptotically flat the metric has (in suitable coordinates) the asymptotic form

$$\begin{aligned} Pr^{-1} \rightarrow 1, \quad Q(r \sin \theta)^{-1} \rightarrow 1, \quad V \rightarrow 1 - mr^{-1} \quad (m = \text{const}), \\ \omega r^{-2} \rightarrow m^{-1}, \quad \varrho r \rightarrow 2^{-1/2}, \quad r \rightarrow \infty. \end{aligned} \tag{4.1}$$

With the aid of Eqs. (3.1), (3.3c), (3.4) and (3.10) the three-dimensional ($t = \text{const}$) Laplacian of V is found to vanish. If V has a positive lower bound on Σ we conclude from the harmonicity of V and its constant asymptotic value one that space-time is flat and the theorem is trivially true.

Therefore, we will assume that the greatest lower bound of V on Σ is zero. For a nonsingular event horizon $V = 0$ the invariant square of the Riemann tensor must remain bounded as $V \rightarrow 0+$. From Eqs. (3.9) and (3.14) we find the following boundary conditions at $V = 0+$:

$$\kappa = \sigma = \varrho = 0, \tag{4.2}$$

$$\varrho V^{-1} = -2^{-5/2}\omega R. \tag{4.3}$$

From Eqs. (4.2) and (3.3a) we deduce that on $V = 0+$, ω is a constant (ω_0).

Let us now integrate Eq. (3.12) over Σ . Using the asymptotic conditions Eq. (4.1) we arrive at the equality

$$\frac{S_0}{\omega_0} = 4\pi m, \tag{4.4}$$

where

$$S_0 = \int_{V=0+} dS = \int_{V=0+} PQ(0+, \theta_2, \theta_3) d\theta_2 d\theta_3$$

is the area of the two-space $V = 0+$, $t = \text{const}$ (Eq. (4.4) implies that m is non-negative). From the equation

$$4\bar{\delta}\kappa - 8\alpha\kappa = -\nabla^2 \ln \omega \tag{4.5}$$

as well as Eqs. (3.7a) and (3.10)–(3.12) we arrive at the identities

$$\frac{\partial}{\partial V} (PQ\varrho V^{-1} \omega^{-1/2}) = 2^{-3/2} PQV^{-1} [2\nabla^2 \omega^{1/2} - 4\omega^{1/2}(\kappa\bar{\kappa} + \sigma^2)], \tag{4.6}$$

$$\frac{\partial}{\partial V} [(\varrho V + 2^{1/2} \omega^{-1}) PQ\omega^{-1}] \tag{4.7}$$

$$= -2^{1/2} PQV^{-1} [-\frac{1}{4}\nabla^2 \ln \omega + 2\sigma^2 + 2\kappa\bar{\kappa} - \frac{1}{4}R],$$

where ∇^2 is the two-dimensional Laplacian. Upon integrating Eqs. (4.6) and (4.7) over Σ we obtain, with the aid of the boundary conditions Eqs. (4.1), (4.2) and (4.3) and the Gauss-Bonnet theorem, the inequalities $S_0 \geq \pi\omega_0^2$, $4m \leq \omega_0$, with equality if and only if $\kappa = \sigma = 0$. That equality must actually hold is seen by comparing these inequalities with Eq. (4.4). Defining r by $V = 2^{1/2}r\varrho$ it is not difficult to solve Eqs. (3.3), (3.7) and (3.8) (with $\kappa = \sigma = 0$) and find the exterior Schwarzschild metric in its usual form.

5. The Scalar Field

In this section we show that there are *no* static, asymptotically flat, “scalar” space-times with a family of simply connected equipotential surfaces which converge to a non-singular event horizon with finite two-dimensional intrinsic geometry. This theorem has previously been proved for the massless scalar field only [3].

The asymptotic form of the metric is given by Eq. (4.1), that of the scalar field ϕ by

$$\phi = \frac{k}{r} + 0(r^{-2}), \quad k = \text{const.}$$

Again, if V has a positive lower bound on Σ we can easily show that $\phi \equiv 0$ and the space is flat. We assume, therefore, that the equipotential surface $V = 0+$ forms an inner boundary of Σ .

The Ricci tensor for a (complex) scalar field ϕ satisfying the Klein-Gordon equation

$$\phi'^{\alpha}{}_{|\alpha} + \mu^2 \phi = 0 \tag{5.1}$$

is given by

$$R_{\alpha\beta} = -\phi_{,(\alpha}\bar{\phi}_{,\beta)} + \frac{1}{2}\mu^2\phi\bar{\phi}g_{\alpha\beta}.$$

Its dyad components with respect to the tetrad of Eq. (3.2) are easily calculated to be

$$\begin{aligned}\Phi_{00} &= \Phi_{22} = \frac{1}{4}\omega^{-2}|\phi_{,1}|^2 \\ \Phi_{11} &= -\frac{1}{8}\phi_{,\alpha}\bar{\phi}^{\alpha} - \Phi_{00} \\ \Phi_{01} &= -\Phi_{12} = -\frac{1}{8}\omega^{-1}\phi_{,1}(P^{-1}\bar{\phi}_{,2} + iQ^{-1}\bar{\phi}_{,3}) \\ &\quad -\frac{1}{8}\omega^{-1}\bar{\phi}_{,1}(P^{-1}\phi_{,2} + iQ^{-1}\phi_{,3}) \\ \Phi_{02} &= \frac{1}{4}(P^{-1}\Phi_{,2} + iQ^{-1}\phi_{,3})(P^{-1}\bar{\phi}_{,2} + iQ^{-1}\phi_{,3}) \\ 3\Lambda &= -\frac{1}{8}\phi_{,\alpha}\bar{\phi}^{\alpha} + \frac{1}{4}\mu^2\phi\bar{\phi}.\end{aligned}\tag{5.2}$$

From Eqs. (3.9) and (5.2) we infer that for a non-singular event horizon $\kappa, \sigma, \varrho \rightarrow 0$ and $\omega^{-1}\phi_{,1}$ and $\phi_{,a}$ (and hence ϕ itself) remain bounded as $V \rightarrow 0+$. (It should be noted that these conclusions are still valid and that the theorem is still true if we add a contribution from an electric field to the Ricci tensor.)

Writing Eq. (5.1) in the form

$$\frac{\partial}{\partial V}(VPQ\omega^{-1}\phi_{,1}) = PQ(V\omega\phi^{;a})_{,a} + \mu^2\phi V\omega PQ$$

we deduce that

$$\begin{aligned}\frac{\partial}{\partial V}(VPQ\omega^{-1}\phi_{,1}\bar{\phi}) &= PQ(V\omega\bar{\phi}\phi^{;a})_{,a} \\ &\quad + V\omega PQ(-\bar{\phi}_{,a}\phi^{;a} + \mu^2\phi\bar{\phi} + \omega^{-2}\phi_{,1}\bar{\phi}_{,1}).\end{aligned}\tag{5.3}$$

Integration of Eq. (5.3) over Σ yields

$$\int_{V=1} V\bar{\phi}\omega^{-1}\phi_{,1}dS - \int_{V=0+} V\bar{\phi}\omega^{-1}\phi_{,1}dS \geq 0,\tag{5.4}$$

with equality if and only if

$$\mu^2\phi\bar{\phi} = \bar{\phi}_{,a}\phi^{;a} = |\phi_{,1}|^2 = 0.$$

Since both integrals in Eq. (5.4) vanish, as seen from the exterior and interior boundary conditions, we conclude that $\phi \equiv 0$. (If $\mu = 0$ we have to use the fact that ϕ vanishes asymptotically.)

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