Commun. math. Phys. 23, 132—136 (1971)

Spherical Functions of the Lorentz Group on the Two Dimensional Complex Sphere of Zero Radius

M. HUSZÁR*

Joint Institute for Nuclear Research Moscow, USSR

Received April 26, 1971

Abstract. Spherical functions of the Lorentz group with respect to the horyspheric subgroup are derived and their relation to Gelfand's homogeneous functions are discussed.

Introduction

There exists a number of homogeneous spaces whose group of motion may serve for the definition of the Lorentz group. Of these homogeneous spaces the most familiar is the three dimensional hyperboloid. It has turned out, however, that in certain respects it is expedient to treat the Lorentz group as a group of motion of the two-(complex) dimensional complex sphere $S^2 = S_1{}^2 + S_2{}^2 + S_3{}^2$. Namely, it has been pointed out by H. Joos and R. Schrader [1] and by M. Huszár and J. Smorodinsky [2] that if the Lorentz group is considered in this spirit, matrix elements of its unitary representation take a rather simple form.

A three dimensional complex vector $\boldsymbol{S}$ is the self-dual part of the Lorentz covariant antisymmetric tensor $S_{\mu\nu}$, i. e. $S_k = S_{0k} + \frac{i}{2} \varepsilon_{klm} S_{lm}$, $(k, l, m = 1, 2, 3)$. Since the real and imaginary part of $\boldsymbol{S}$ transform like the electric and magnetic field, respectively, the invariance of $\boldsymbol{S}^2 \sim (\boldsymbol{E} + i\boldsymbol{B})^2$ under proper Lorentz transformations is evident. And conversely, it can be proved [3] that the connected part of three dimensional complex rotation group is isomorphic to the proper Lorentz group.

1. Little Groups on the Complex Sphere of Zero and Non-zero Radius

Let us associate to a three dimensional complex vector $\boldsymbol{S} = (S_1, S_2, S_3)$ the matrix $\hat{S} = \begin{pmatrix} S_3 & S_1 - iS_2 \\ S_1 + iS_2 & -S_3 \end{pmatrix}$. Under $g \in SL(2, C)$ $\boldsymbol{S}$ transforms as

* On leave of absence from the Central Research Institute for Physics, Budapest, Hungary.

$T_g\hat{S} \equiv \hat{S}' = g\hat{S}g^{-1}$ and clearly $S^2 = S_1{}^2 + S_2{}^2 + S_3{}^2$ is invariant. And conversely, it can be shown that if one excludes the point $S = (0, 0, 0)$ any two complex vectors S, S' of the same length can be translated to each other by means of a suitable $SL(2, C)$ transformation. Consider now the point $S_0 = (-iS, S, 0)$ $(S \neq 0)$ on the complex sphere of zero radius Σ_0. The little group of this point i. e. the subgroup satisfying $T_\eta \hat{S}_0 = \hat{S}_0$ constitute elements of the type $\eta = \begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix}$. This is the horyspheric subgroup [5, 6] isomorphic to the two dimensional translation group $T(2)$. An arbitrary other point $\hat{S} = T_g\hat{S}_0$ $(g \in SL(2, C))$ on Σ_0 has the little group $\eta_g = g\eta g^{-1}$. The converse statement is also true i.e. any three dimensional complex vector having the horyspheric little group η_g is on the sphere of the zero radius. It can be shown in an analogous way that the little group of a vector on the complex sphere of non-zero radius is the group $H = SO(2) \times SO(1, 1)$ [2]. Spherical functions of the Lorentz group with respect to the subgroup H have been studied in [2]. Here we derive the spherical functions with respect to the horyspheric subgroup.

2. Spherical Functions on the Complex Sphere of Zero Radius

Consider the state $|\rangle$ satisfying

$$T_\eta |\rangle = |\rangle$$

where T_η is the unitary representation of the horyspheric subgroup. Then spherical functions of the Lorentz group with respect to the subgroup η are defined as

$$f_\Phi(\Sigma_0) = \langle \Phi | T_g | \rangle^* . \tag{1}$$

Here T_g is the unitary representation of the Lorentz group and $|\Phi\rangle$ is a basis vector specified below. The quantity Σ_0 indicates that $f_\Phi(\Sigma_0)$ is a function over the factor space g/η, i.e. it is defined over the complex sphere of zero radius Σ_0.

Explicit form of the spherical functions (1) can be found by solving the eigenvalue equation of the Casimir operators. To this end introduce the combinations

$$J = \frac{1}{2}(M + iN), \qquad K = \frac{1}{2}(M - iN)$$

where M and N are the infinitesimal generators of the spatial and hyperbolic rotations. At first the basis $|\Phi\rangle$ will be labelled by the eigenvalues of J_3 and K_3 i.e. by $m = (\mu + i\nu)/2$, $m^* = (\mu - i\nu)/2$ $(\mu = 0, \pm 1, \pm 2, \ldots,$ $-\infty < \nu < \infty$ continuous).

Introduce the following coordinate system on Σ_0:

$$S_1 = -i\cos\Theta\cos\Phi - \sin\Phi, \qquad S_2 = -i\cos\Theta\sin\Phi + \cos\Phi$$
$$S_3 = i\sin\Theta\,. \tag{2}$$

Here

$$\Theta = \Theta_1 + i\Theta_2, \qquad \Phi = \Phi_1 + i\Phi_2\,,$$
$$0 \leqq \Theta_1 < \pi,\ 0 \leqq \Phi_1 < 2\pi,\ -\infty < \Theta_2, \Phi_2 < \infty\,.$$

The spherical functions in unitary spinor basis satisfy the eigenvalue equations of the Casimir operators $\boldsymbol{J}^2, \boldsymbol{K}^2$ and the generators J_3, K_3. From (2) we obtain

$$\left[\mathrm{tg}^2\Theta\frac{\partial^2}{\partial\Theta^2} - \frac{1}{\cos^2\Theta}\frac{\partial^2}{\partial\Phi^2} + 2i\frac{\sin\Theta}{\cos^2\Theta}\frac{\partial^2}{\partial\Theta\,\partial\Phi}\right.$$
$$\left. + \mathrm{tg}\Theta(2+\mathrm{tg}^2\Theta)\frac{\partial}{\partial\Theta} + \frac{i}{\cos^3\Theta}\frac{\partial}{\partial\Phi}\right] f^{jj^*}_{mm^*} = j(j+1) f^{jj^*}_{mm^*} \tag{3}$$

$$\left[\mathrm{tg}^2\Theta^*\frac{\partial^2}{\partial\Theta^{*2}} - \frac{1}{\cos^2\Theta^*}\frac{\partial}{\partial\Phi^{*2}} - 2i\frac{\sin\Theta^*}{\cos^2\Theta^*}\frac{\partial^2}{\partial\Theta^*\,\partial\Phi^*}\right.$$
$$\left. + \mathrm{tg}\Theta^*(2+\mathrm{tg}^2\Theta^*)\frac{\partial}{\partial\Theta^*} - \frac{i}{\cos^3\Theta^*}\frac{\partial}{\partial\Phi^*}\right] f^{jj^*}_{mm^*} = j^*(j^*+1) f^{jj^*}_{mm^*} \tag{4}$$

$$\frac{1}{i}\frac{\partial}{\partial\Phi} f^{jj^*}_{mm^*} = m f^{jj^*}_{mm^*}, \qquad \frac{1}{i}\frac{\partial}{\partial\Phi^*} f^{jj^*}_{mm^*} = m^* f^{jj^*}_{mm^*}\,. \tag{5}$$

Here j is related to the familiar quantum numbers j_0, σ [4], as $j = \frac{1}{2}(j_0 - 1 + i\sigma)$ ($j_0 = 0, 1, 2, \ldots, -\infty < \sigma < \infty$ continuous).

The solution of Eqs. (3), (4), (5) can be written in the form

$$f^{jj^*}_{mm^*} = \frac{1}{2\sqrt{2}}\frac{1}{(2\pi)^2}\left(\cos\frac{\Theta}{2}\right)^{j-m}\left(\sin\frac{\Theta}{2}\right)^{j+m}\left(\cos\frac{\Theta^*}{2}\right)^{-j^*-1+m^*}$$
$$\cdot\left(\sin\frac{\Theta^*}{2}\right)^{-j^*-1-m^*} \cdot e^{i(m\Phi + m^*\Phi^*)}\,. \tag{6}$$

These functions are normalized as follows

$$\langle j'j'^*; m'm'^* \mid jj^*; mm^*\rangle$$
$$= \left(\frac{i}{2}\right)^2 \int \cos\Theta\cos\Theta^*\, d\Theta\, d\Theta^*\, d\Phi\, d\Phi^* (f^{j'j'^*}_{m'm'^*})^* f^{jj^*}_{mm^*}$$
$$= \delta_{j_0' j_0}\delta(\sigma'-\sigma)\,\delta_{\mu'\mu}\delta(\nu'-\nu)\,.$$

It is worthy of note that $f^{jj^*}_{mm^*}$ is a single valued function. If we cut the $\sin\frac{\Theta}{2}$ plane it is easily seen that as a consequence of integral valuedness of $j_0 \pm \mu$ the discontinuity over the cuts is equal to zero. Or conversely, the requirement of single valuedness leads to the quantization of j_0.

In order to obtain the spherical functions in another basis we have to introduce a suitable coordinate system. E.g. the coordinate system

$$S_1 = e^{a+i\psi}(-\sin\varphi + i\cos\vartheta\cos\varphi), \qquad S_2 = e^{a+i\psi}(\cos\varphi + i\cos\vartheta\sin\varphi)$$

$$S_3 = -i\sin\vartheta\, e^{a+i\psi} \tag{7}$$

$$-\infty < a < \infty, \qquad 0 \leqq \varphi, \psi < 2\pi, \qquad 0 \leqq \vartheta < \pi$$

leads to the spherical functions in angular momentum basis ($|\Phi\rangle = |j_0\sigma; l\mu\rangle$):

$$f_{l\mu}^{j_0\sigma} = \sqrt{\frac{2l+1}{8\pi^3}}\, e^{a(-1+i\sigma)} D^l_{\mu j_0}(\varphi, \vartheta, \psi)$$

where $D^l_{\mu j_0}$ is the representation of the real three dimensional rotation group.

3. Relation to Gelfand's Homogeneous Functions

Consider now the following parametrization of Σ_0:

$$S_1 = -i(u^2 - v^2), \qquad S_2 = u^2 + v^2, \qquad S_3 = 2iuv\,. \tag{8}$$

It can be easily shown that if u and v transform as spinors of the $SL(2, C)$ group i.e. $\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$, then $\boldsymbol{S}$ transforms as a vector of the three dimensional complex rotation group. Parametrizations (2), (7) can be considered as special cases of (8) and correspond to the following parametrization of spinors:

$$u = \cos\frac{\Theta}{2}\, e^{-i\frac{\Phi}{2}}, \qquad v = \sin\frac{\Theta}{2}\, e^{i\frac{\Phi}{2}}$$

and

$$u = -\sin\frac{\vartheta}{2}\, e^{\frac{a}{2} + i\frac{\psi - \varphi}{2}}, \qquad v = \cos\frac{\vartheta}{2}\, e^{\frac{a}{2} + i\frac{\psi + \varphi}{2}}\,.$$

Spherical functions in terms of u, v in the unitary spinor basis read

$$f_{mm^*}^{jj^*} = \frac{1}{2\sqrt{2}}\, \frac{1}{(2\pi)^2}\, u^{j-m} v^{j+m} u^{*\,-j^*-1+m^*} v^{*\,-j^*-1-m^*}$$

If one considers the linear manifold

$$f(u, v) = \sum_{\mu=-\infty}^{\infty} \int_{-\infty}^{\infty} dv\, a_{mm^*} f_{mm^*}^{jj^*}$$

then under the $SL(2, C)$ group the function $f(u, v)$ transforms as

$$T_g f(u, v) = f(g^{-1}(u, v)) = f(\delta u - \beta v, \ -\gamma u + \alpha v),$$

furthermore, it has the degrees of homogeneity $2j$, $-2j^* - 2$ with respect to u, v and u^*, v^*. Thus, if we fix a basis, say m, m^*, the homogeneous functions investigated by Naimark and Gelfand [5, 6] take the form of the spherical functions (6) defined over the two dimensional complex sphere of zero radius.

Acknowledgement. The author wishes to thank Prof. J. Smorodinsky for helpful discussions.

References

1. Joos, H., Schrader, R.: DESY preprint 68/40, 1968.
2. Huszár, M., Smorodinsky, Y.: Preprint of the JINR E2-4225, 1968; Theor. and Math. Phys. (Moscow) **4**, 328, (1970).
3. — — Preprint of the JINR E2-5020, 1970.
4. Joos, H.: Fortschr. d. Phys. **10**, 65, (1962).
5. Naimark, M. A.: Linear representation of the Lorentz group. New York: Pergamon 1964.
6. Gel'fand, I. M., Gaev, M. I., Vilenkin, N. Ya.: Generalized functions, Vol. V. New York: Academic Press 1966.

M. Huszár
Central Research Institute for Physics
Budapest 114/Hungary
P. O. Box 49