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Attempt of an Axiomatic Foundation of Quantum Mechanics and More General Theories VI*

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Abstract. This contribution continues the series of papers [2, 4, 5, 12] treated by Ludwig and collaborators. It is based on the generalized frame given in [6]; there Ludwig has set up an "infinite" axiomatic scheme as extension of the "finite" system [4, 5]. The results of [12] are then proved for a "locally finite" case; they lead to an extended representation theorem.

I. Introduction

In his paper "Notes on Axioms for Quantum Mechanics" [10] MacLaren has set up the final axiom:

(C) The set of all atoms of every finite sublattice of the orthomodular lattice G of decision effects is compact in the norm topology.

This axiom guarantees that the division ring which is constructed over G is the real, complex or quaternionic numbers [14].

We base here on Ludwig's general axiomatic scheme ([6], III.) restricted by [6], III. § 18 condition $\mathbf{V}_3$ *("locally-finite" case!)* which is a generalization of the *"finite"* system given in [4, 5]. Within this frame the purpose of this paper is

1. to prove a slightly weaker form **(C̲)** (Lemma 8 in part III), of statement **(C)**,

2. to show that **(C̲)** is sufficient to exclude discontinuous and disconnected division rings,

3. to give further topological properties of the lattice G.

II. Preliminaries

In the following largely is used the punctuation and terminology of [12]. We give a somewhat modified summary of Ludwig's conclusions [6] which are different from those used in [12].

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The starting position is a set $\underline{K}$ of all physical ensembles v and a set $\underline{L}$ of all physical effects f. Then $(\underline{K}, \underline{L})$ is a dual pair according to

Axiom 1a ([6], III. Satz 2.5). *There exists a mapping μ of $\underline{K} \times \underline{L}$ into $\boldsymbol{R}_+$ satisfying:*

(α) $0 \leqq \mu(v,f) \leqq 1$ *for all* $(v,f) \in \underline{K} \times \underline{L}$.

(β) $\mu(v_1,f) = \mu(v_2,f)$ *for all* $f \in \underline{L}$ *and* $v_1, v_2 \in \underline{K}$ *implies* $v_1 = v_2$.

(γ) $\mu(v,f_1) = \mu(v,f_2)$ *for all* $v \in \underline{K}$ *and* $f_1, f_2 \in \underline{L}$ *implies* $f_1 = f_2$.

(δ) *There exists* $o \in \underline{L}$ *such that* $\mu(v,0) = 0$ *for all* $v \in \underline{K}$.

(ε) *For each* $v \in \underline{K}$ *there exists* $f \in \underline{L}$ *such that* $\mu(v,f) = 1$.

Axiom 1b ([6], III., Axiome 3a, b). *$\underline{K}$ and $\underline{L}$ are countable sets.*

Let $\underline{B}$ (resp. $\underline{D}$) be the set of all functions $x(f) = \sum_{i=1}^{n} a_i \mu(v_i, f)$ on $\underline{L}$ with $v_i \in \underline{K}$ $\left(\text{resp. } y(v) = \sum_{i=1}^{n} a_i \mu(v, f_i) \text{ on } \underline{K} \text{ with } f_i \in \underline{L}\right)$, a_i real, n finite integer. Hence $\underline{B}$ and $\underline{D}$ are real linear spaces and we may identify $\underline{K}$ (resp. $\underline{L}$) as subsets of $\underline{B}$ (resp. $\underline{D}$). By natural extension we can define $\mu(x, y)$ over all $\underline{B} \times \underline{D}$. Let R be the set of all linear functions y on $\underline{B}$ such that $\mu(v, y) < \infty$ for all $v \in \underline{K}$. Then Ludwig has shown:

Proposition 1 ([6], III. Satz 3.5). *There is a subspace B' with $\underline{D} \subsetneqq B' \subsetneqq R$, so that*

a) *B' is a Banach-space by the norm* $\|y\| := \sup(|\mu(v, y)| : v \in \underline{K})$.

b) *B' is the dual of the closure B of $\underline{B}$ with respect to the norm* $\|x\| = \sup(|\mu(x, y)| : \|y\| \leqq 1, y \in B')$.

c) $x \in B$ *with* $\mu(x,f) = 0$ *for all* $f \in \underline{L}$ *implies* $x = 0$.

d) $\|v\| = 1$ *for all* $v \in \underline{K}$, $\|f\| = 1$ *for all* $f \in \underline{L}$

Now let $\hat{L}$ be the $\sigma(B', B)$-closed convex hull of $\underline{L}$ and let K be the norm-closed convex hull of $\underline{K}$.

Proposition 2 (see [6], III. §§ 3, 4).

a) *Properties* (α) *to* (δ) *of Axiom 1a hold for* $K \times \hat{L}$.

b) *As a consequence of Axiom 1b, B and D are separable sets.*

c) *In every norm-bounded set of B (resp. B') the topologies $\sigma(B, \underline{L})$ (resp. $\sigma(B', \underline{K})$) may be characterized by norms and there holds $\sigma(B, D) = \sigma(B, \underline{L})$ (resp. $\sigma(B', B) = \sigma(B', \underline{K})$).*

d) *$\hat{L}$ is a $\sigma(B', B)$-compact set and D is $\sigma(B', B)$-dense in B'.*

Axiom 2$^+$ ([6], III. §§ 5, 7; Axiome 4a, b and 4bz).

a) *To each couple $f_1, f_2 \in L := \sigma(B', B)$-**clos**$(\underline{L})$ there exists $f_3 \in L$ due to the following conditions: $\mu(v, f_i) - \eta \leqq \mu(v, f_3)$, $i = 1, 2$, for any $\eta > 0$ and $\mu(v, f_3) = 0$ for $v \in K_0(f_1) \cap K_0(f_2)$.*

b) *For $f \in \hat{\hat{L}} := \sigma(B', B)$-**clos**$(y \in B' : y = \lambda f, \lambda \geqq 0, f \in \hat{L})$ and a maximal effect $e \in \hat{L}$, $K_0(f) \supsetneqq K_0(e)$ implies $K_1(f) \subsetneqq K_1(e)$.*

c) *For every maximal effect $e \in \hat{L}$, $e \neq 0$ follows $K_1(e) \neq \emptyset$.*

Axiom 3 ([6], III. § 8 Axiom 5) *stays unchanged* [12].

The following proposition is a consequence of the last two axioms.

Proposition 3 (see [6], III. § 6).

a) *For every effect* $f \in \hat{L}$ *exists a maximal effect* $e \geqq f$ *with* $K_0(e) = K_0(f)$, *called decision-effect.*

b) *The set* G *of all decision-effects* $e \in \hat{L}$ *is a complete and orthocomplemented lattice.*

c) $\hat{W}' := (K_1(l) : l \subseteqq \hat{L})$ *is the lattice of all extremal subsets* $C \subseteqq K$ *ordered by inclusion.*

d) *For every extremal subset* $C \subseteqq K$ *(definition see* [5, 6, 12]*) exists* $e \in G$ *with* $C = K_1(e)$. *The mapping* $e \leftrightarrow K_1(e)$ *is a lattice orthoisomorphism of* G *onto* $\hat{W}'$.

Postulate (A). ($\mathbf{V_1}$ in [6], III. § 3). *The convex hull of* $K \cup (-K)$ is *norm-closed.*

Remark. (A) implies Theorems 1 and 2 of [12].

Axiom 2^+ c is equivalent to

Theorem 3⁺. *G is the set of all exposed points of* $\hat{L}$.

Theorem 4⁺. $\sum_{i=1}^{\infty} e_i \leqq 1$ ($\sigma(B', B)$-limit), $e_i \in G$ *implies* $\sum_{i=1}^{\infty} e_i = \bigvee_{i=1}^{\infty} e_i$ *and* $e_i \perp e_k$ *for* $i \neq k$.

Theorem 5. *The lattice operations join* ($\vee$), *meet* ($\wedge$) *and orthocomplementation* ($'$) *are* $\sigma(B', B)$*-continuous*).

Definition. If D, E are subsets of K then $d(D, E) : \inf(d(v, w) : v \in D, w \in E)$ and $d(v, w) := \|v - w\| = \sup(|\mu(v, y) - \mu(w, y)| : y \in B', \|y\| \leqq 1)$. The next axiom we will give in two equivalent forms I, II.

Axiom 4⁺ ([6], III. § 11, Axiom 6).

(I) *If* $x \leqq a \in G$ *and* $d(K_1(a \wedge (x \vee b)), K_1(y)) \neq o$ *for* $y \leqq (b - a \wedge b) \in G$, $b \in G$, *then* $a \wedge (x \vee b) = x \vee (a \wedge b)$.

(II) *Let* C_i, $i = 1, 2$, *be elements of* $\hat{W}'$. *Then* $C_1 \cap C_2 = \emptyset$, $C_3 \subseteqq C_1 \vee C_2$ *(i.e., extremal hull of* C_1, C_2), $C_1 \perp C_3$ *(i.e.,* $d(C_1, C_3) = 2$*) and* $d(C_1 \vee C_3, C_2) \neq 0$ *imply* $C_3 = 0$.

Remark. Obviously (II) is a slightly generalization of the former Axiom 4 [5, 12] because $d(C_1 \vee C_3, C_2) \neq 0$ implies $(C_1 \vee C_3) \cap C_2 = \emptyset$.

Definition. An element $C \in \hat{W}'$ (i.e., an extremal subset of $K \subseteqq B$) is *finite* if the closed linear span $\overline{\mathbf{lin}}\, C =: M(C)$ of C in B has finite dimension. For such finite extremal sets Theorems 6, 7, and 11 of [12] hold (Theorems 8, 9, 10 [12] hold for all $C \in W'$). An element $e \in G$ is *finite* if $K_1(e) \in \hat{W}'$ is finite. Now let G be *atomic* (i.e., every extremal subset $C \subseteqq K$

contains an extreme point); we then say that an element e of G (resp., G itself) has *finite lattice dimension* if e (resp., each element of G) is a finite sum of atoms or, equivalently, is a finite join of mutually orthogonal atoms. By an *ascending chain* between 0 and $e \in G$ we mean a set $(e_i \in G)$ with $0 < e_1 < e_2 < \cdots < e$.

Postulate (B). $\mathbf{V_3}$ in [6], III. § 18): **"locally-finite"** condition! *Every element $C \in \hat{W}'$ is the join of an ascending chain of finite elements of $\hat{W}'$.*

III. Some Consequences of Axiom 4^+ and Postulate (B)

Definition. We write $(b, a)\,M$ if $a \wedge (x \vee b) = x \vee (a \wedge b)$ for all $x \leqq a$.

Lemma 1. *$d(K_1(a), K_1(b - a \wedge b)) \neq o$ for $a, b \in G$ implies $(b, a)\,M$.*

Proof. $K_1(a) \supseteqq K_1(a \wedge (x \vee b))$ for all $x \in G$; hence $0 \neq d(K_1(a), K_1(b - a \wedge b)) \leqq d(K_1(a \wedge (x \vee b)), K_1(b - a \wedge b))$ and Axiom 4^+ gives $(b, a)\,M$. □

Lemma 2. *If for $a, b \in G$, $K_1(a)$ or $K_1(b)$ is finite then holds $(b, a)\,M$.*

Proof. Finite extremal subsets of K are compact. Therefore one of the sets $D := K_1(a)$ or $E := K_1(b - a \wedge b) \subsetneqq K_1(b)$ is compact. We assume E to be compact. Since the metric $d: B \times B \to \boldsymbol{R}$ is continuous also $d(D, .): E \to \boldsymbol{R}$ is a continuous mapping of the compact set E into $\boldsymbol{R}$; hence there is $z \in E$ with $d(D, z) = d(D, E)$. Supposing $d(D, z) = 0$ would mean that z is a touching point of the extremal set D. As a closed set however D must contain z contrary to $D \cap E = \emptyset$. So $d(D, E) \neq 0$. Lemma 1 then then finishes the proof. □

Definition. $a \in G$ is a *modular element* if a) $G_a := (x \in G : x \leqq a)$ is a modular lattice and b) for all $b \in G$ holds $(b, a)\,M$.

With the help of Lemma 2 we find

Lemma 3. *Every finite $a \in G$ is a modular element.*

Especially follows $(p, a)\,M$ for every atom $p \in G$ and for all $a \in G$ which is equivalent to the **"covering condition"** [7]: **COV**: *If p is an atom and $p \nleqq a \in G$ then $a \lessdot a \vee p$* (i.e., if $a \leqq c \leqq a \vee p$ then $c = a$ or $c = a \vee p$).

Lemma 4. *The set $A(G)$ of all atoms of G is join-dense in G.*

Proof [12]. Theorem 11 implies the existence of an extreme point $v_1 = K_1(p_1) =: C_1$, $p_1 \in A(G)$ in every finite extremal set $C \subsetneqq K$. By Krein-Milman [3] C is the closed convex hull of its extreme points; so, if $C \neq C_1$ there is $v_2 \in K$ with $v_2 = K_1(p_2) \neq K_1(p_1)$, $p_2 \in A(G)$. With the condition **COV** then follows $C \supseteqq C_2 := K_1(p_1) \vee K_1(p_2) = K_1(p_1 \vee p_2) > C_1 = K_1(p_1)$.

Since $\mathbf{dim}\, M(C) < \infty$ and $\mathbf{dim}\, M(C_{i+1}) \geqq \mathbf{dim}\, M(C_i) + 1$ for $C_i \subset C_{i+1} \subset \cdots \subseteqq C$, successive applying of **COV** ends after finitely many steps and gives by induction $C_i = \bigvee_{\nu=1}^{\nu=i} K_1(p_\nu)$ and $C = \bigvee_{i=1}^{n_c} C_i = \bigvee_{\nu=1}^{n_c} K_1(p_\nu)$ for every finite $C \in \hat{W}'$. Postulate **(B)** then finishes the proof. □

As a consequence of this proof we find

Corollary 1. *G is atomic and the modular elements are join-dense in G, i.e., $1 \in G$ is the join of modular elements.*

The next lemma secures that we need not distinguish between the terms "*finite*" and "*finite lattice dimension*".

Lemma 5. *The finite elements of G are just the elements with finite lattice dimension.*

Proof. Let $e \in G$ be finite. Then $C = K_1(e)$ has finite chain length (see proof of Lemma 4) and there are only finitely many mutually orthogonal elements in G_e, i.e., e has finite lattice dimension.

To prove the other direction let $e \in G$ be a finite sum of atoms, i.e., a finite join of mutually orthogonal atoms of G. Postulate **(B)** then guarantees the existence of an ascending chain $0 < e_1 < e_2 < \cdots < e$ of finite elements $e_i < e$. Using **COV** we find a covering chain $0 \lessdot p_1 \lessdot p_1 + p_2 \lessdot \cdots \lessdot \Sigma p_i = e$ which has finite and also maximal length; hence all chains between 0 and e have finite length. Therefore the finite join $e = \bigvee_{i=1}^{n} e_i$ is finite because all e_i are finite. □

Lemma 6. *G is a semimodular (*also called *M-symmetric) lattice.*

For the proof see [10], Lemma 10 and Corollary 16 or [9].

Definition. A lattice is *nearly modular* if it is orthomodular, semimodular and each element is the join of modular elements.

Corollary 1 and Lemma 6 result

Proposition 4. *G is a nearly modular lattice.*

MacLaren [8] has shown

Proposition 5. *G is orthoisomorphic to the direct sum of irreducible, atomic, orthocomplemented sublattices $G_i \subseteqq G$.*

IV. Topological Properties of the Lattice G

In this part we consider in $\hat{W}'$ only the *norm topology* induced from B and in G the *$\sigma(B', B)$-topology* induced from B'.

Lemma 7. *$v_\alpha \Rightarrow v$ (norm convergence) in K, $v_\alpha = K_1(p_\alpha)$, $p_\alpha \in A(G)$ and $p_\alpha \to f$ ($\sigma(B', B)$-convergence) in $\hat{L}$ imply $v \in K_1(f)$.*

Proof. Given $\varepsilon > 0$, $v_\alpha \Rightarrow v$ means: an integer n exists such that for all indices $\alpha \geqq n$ and for all $y \in B'$, $\|y\| \leqq 1$:

$$|\mu(v_\alpha - v, y)| < \varepsilon .$$

$p_\nu \to f$ means: for every $v \in K$ there is an integer $m = m(v)$ such that for all $\nu \geqq m$:

$$|\mu(v, p_\nu - f)| < \varepsilon$$

Therefore for all $\alpha, \nu \geqq \max(n, m(v))$ holds

$$|\mu(v_\alpha, p_\nu - f)| \leqq |\mu(v_\alpha - v, p_\nu)| + |\mu(v_\alpha - v, f)| + |\mu(v, p_\nu - f)| < 3\varepsilon$$

especially for $\alpha = \nu$:

$$|\mu(v_\alpha, p_\alpha - f) = |1 - \mu(v_\alpha, f)| < 3\varepsilon , \quad \text{i.e.,} \quad u(v_\alpha, f) \to 1 .$$

But $v_\alpha \Rightarrow v$ implies $\mu(v_\alpha, f) \to \mu(v, f)$; hence $\mu(v, f) = 1$, i.e., $K_1(f) \neq \emptyset$. □

(C): Lemma 8. *The set $A(e)$ of atoms $p \leqq e$ with finite $e \in G$ is $\sigma(B', B)$-closed, and as subset of $\hat{L}$ also $\sigma(B', B)$-compact.*

Proof (is a slightly modified version of the proof of Theorem 10.3 in [5]). L is compact (Prop. 2). Let $(p_\alpha)_{\alpha \in A} \to f$ be a convergent series in $A(e)$. $\hat{L}_e := (f \in \hat{L} : f \leqq e) = \hat{L}_0 K_0(e)$ is an extremal set and therefore closed, hence $f \in \hat{L}_e$. By $v_\alpha = K_1(p_\alpha)$ we have a series of extreme points in the finite and hence compact $K_1(e)$. In this set we may select a norm-convergent subsequence $v_\beta \Rightarrow v \in K_1(e)$. Taking into account the finite dimension of the modular lattice $G_e := (\underline{e} \in G : \underline{e} \leqq e)$ we can complete every $p_\beta := p_\beta^1$ by mutually orthogonal atoms $p_\beta^i \in A(e)$ to $e = \sum_{i=1}^{n} p_\beta^i$. Because of the compactness of $\hat{L}_e$ and $K_1(e)$, respectively, there is a subsequence $(\nu) \subsetneqq (\beta) \subsetneqq (\alpha)$ such that $p_\nu^i \to f^i$ and $v_\nu^i \Rightarrow v^i$, $i = 1 \ldots n$, with $f^1 =: f$ and $v^1 := v$. Applying Lemma 7 we find $K_1(f^i) \neq 0$. $K_1(f^i)$ is a finite extremal set and contains an extreme point $K_1(q^i)$ with $q^i \in A(e)$, $q^i \leqq f^i$, hence $r^i := f^i - q^i \in \hat{L}_e$ or $f^i = r^i + q^i$. Since $\sum_{i=1}^{n} p_\nu^i = e \to e$, we find $\sum_{i=1}^{n} q^i = e$ and $r^i = 0$ for all $i = 1 \ldots n$. Especially $p_\alpha \to q$, i.e., $A(e)$ is closed. □

Proposition 6. *Every finite sublattice $G_e \subsetneqq G$ is $\sigma(B', B)$-closed and $e_\nu \to \underline{e}$ in G_e implies* $\mathbf{dim}\, G_{e_\nu} \to \mathbf{dim}\, G_{\underline{e}}$.

Proof see [5], Theorem 19.

In analogy to [12] Theorem 16 we find:

Proposition 7. *For every finite $e \in G$ the bijective mapping $(p \in A(e), \sigma(B', B)) \leftrightarrow (K_1(p) \subsetneqq K_1(e), \|\cdot\|_B)$ is an homeomorphism of $A(e)$ onto the set $\mathscr{E}(K_1(e))$ of extreme points of $K_1(e)$.*

Proof. The closed set $A(e) \subseteqq \hat{L}$ is compact. Let $p_\alpha \to p$ be a convergent sequence in $A(e)$. Since also $K_1(e)$ is compact the corresponding sequence $v_\alpha = K_1(p_\alpha)$ has a cluster point v. Let v_β, $(\beta) \subseteqq (\alpha)$, be a subsequence converging to v; then Lemma 7 implies $v_p := K_1(p) \ni v \neq \emptyset$. So v_p is the unique cluster point of (v_α), i.e., $v_\alpha = K_1(p_\alpha) \Rightarrow K_1(p)$. Hence $K_1 : A(e) \to \mathscr{E}(K_1(e))$ is continuous and (by Prop. 3) also bijective. So, having in mind that $A(e)$ is compact and the the norm topology separates in B, we find the inverse mapping also to be continuous. □

Corollary 2. *The set $\mathscr{E}(C)$ of all extreme points of every finite extremal subset of K is compact* (follows immediately with Prop. 3).

The set of all finite decision-effects is an *ideal* J of G [10]. Next we shall show that in J the $\sigma(B', B)$-topology may be represented by a set of linked "*e-norms*" $\|\,.\,\|_e := \sup(|\mu(v,.)| : v \in K_1(e))$ which are seminorms induced from B'.

Lemma 9. *The $\sigma(B', B)$-topology in $\hat{L}$ is equivalent to the topology of uniforme convergency on all compact subsets of K.*

Proof. $\hat{L}$ is a subset of the unit sphere $S' := (y \in B' : |\mu(x, y)| \leqq 1$ for all $x \in B$ with $\|x\| \leqq 1)$. Since $\|x\| = \sup(|\mu(x, y)| : \|y\| \leqq 1)$ for all $x \in B$, $\|x_\alpha\| \to 0$ implies $y(x_\alpha) = \mu(x_\alpha, y) \to 0$ for all $y \in S'$, especially for all $y \in \hat{L}$. This means: S' and $\hat{L}$ are equicontinuous in 0 ([3], § 15, Nr. 3) and hence uniformly equicontinuous, because B' is a topological vector space. [3], § 21, Nr. 6 (2) now gives the assertion. □

Corollary 3. *$y_\alpha \to y$ in B' implies $\|y_\alpha - y\|_e$ for all finite $K_1(e)$.*

Obviously $\|\,.\,\|_e$ is a seminorm on B'

Lemma 10. *The seminorms $\|\,.\,\|_e$, e finite, are total on $\hat{L}_e$.*

Proof. We have to show: $\|f\|_e = 0$ implies $f = 0$ for $f \leqq e$. $\|f\|_e = 0$ means $\mu(v, f) = 0$ for all $v \in K_1(e)$, i.e., $K_0(f) \supseteqq K_1(e)$. Since $f \leqq e$, also $K_0(f) \supseteqq K_0(e)$; hence $K_0(f) \supseteqq K_1(e) \vee K_0(e) = K$ and therefore $f = 0$. □

We would like to know if $\|\,.\,\|_e$ separates on $\hat{L}_e$. The question is: does $\|f_1 - f_2\|_e = 0$ for $f_1, f_2 \in \hat{L}_e$ already imply $f_1 = f_2$? One sees immediately that this is the case if $f_2 \leqq f_1$. In generality however we can only show

Lemma 11. *If for $f_1, f_2 \in \hat{L}_e$, $\|f_1 - f_2\|_e = 0$ then $K_0(f_1) = K_0(f_2)$ and $K_1(f_1) = K_1(f_2)$.*

Proof. $\|f_1 - f_2\|_e = 0$ means $\mu(v, f_1) = \mu(v, f_2)$ for all $v \in K_1(e)$. Since $f_1, f_2 \leqq e$ we find $K_0(f_1) \cap K_1(e) = K_0(f_2) \cap K_1(e)$ and by orthomodularity

$$K_0(f_1) = K_0(e) \vee (K_0(f_1) \cap K_1(e)) = K_0(e) \vee (K_0(f_2) \cap K_1(e)) = K_0(f_2)\,.$$

The expression $K_1(f_1) = K_1(f_2)$ follows by $K_1(f_i) \subseteqq K_1(e)$, $i = 1, 2$. □

Lemma 12. *For every finite $e \in G$ the semi-metric $d_e(y, z) := \|y - z\|_e$ on B' separates in G_e.*

Proof. We have to show: $d_e(e_1, e_2) = 0$ for $e_1, e_2 \in G_e$ implies $e_1 = e_2$. But $d_e(e_1, e_2) = 0$ means $\mu(v, e_1) = \mu(v, e_2)$ for all $v \in K_1(e)$. Since $e_1, e_2 \leqq e$, Lemma 11 gives $K_0(e_1) = K_0(e_2)$ which is equivalent to $e_1 = e_2$. □

Now we can prove the main theorem:

Proposition 8. *Let $G_e \subsetneqq G$ be a finite sublattice of G. Then the topological space (G_e, d_e) is homeomorphic to the topological space $(G_e, \sigma(B', B))$. Especially, G_e is a d_e-compact set.*

Proof. The identical mapping i: $(G_e, \sigma(B', B)) \to (G_e, d_e)$ is continuous (Corollary 3), G_e is a $\sigma(B', B)$-compact set and the d_e-topology separates in G_e; hence i^{-1} is also continuous. □

Since the convex hull of a compact subset of a finite subspace of B is closed we find, using a theorem of Klee ([3], § 25, Nr. 3 (3)):

Lemma 13. *Every finite $K_1(e) \subsetneqq K$ is the convex hull of its extreme points.*

With this lemma we find in analogy to Proposition 8:

Corollary 4. *The topological space $(G_e, d_{(e)})$ with the semi-metric $d_{(e)}(y, z) := \sup(|\mu(v, y - z)| : v \in \mathscr{E}(K_1(e)))$ is homeomorphic to $(G_e, \sigma(B', B))$.*

V. Final Results

Definition. If $G_e \subsetneqq G$ is an irreducible sublattice with lattice dimension *two* (i.e., e is sum of two atoms) then we say: $\mathfrak{l} := A(e)$ is a *line* in G.

Proposition 9. *Let $\mathfrak{l} := A(e)$ be a line in an irreducible sublattice $G_i \subsetneqq G$, with* **dim** *$G_i \geqq 4$, and let $s \in \mathfrak{l}$ be a fixed atom. Then there are two lattice operations $\oplus$ and $\odot$ such that the algebraic set $(\mathfrak{l} \backslash s; \oplus, \odot)$ is a locally compact, connected, topological division ring (*also called "*ternary field*"*) D which is isomorphic either to the real, complex or quaternionic numbers. Furtheron G_i is orthoisomorphic to the lattice of closed subspaces of an inner-product space H_i.*

For the proof see [8–10, 13–15].

The next proposition is a result of Amemiya and Araki [1].

Proposition 10. *As a consequence of the orthomodularity of G the inner-product space H_i is complete in the usual norm topology, i.e., H_i is an Hilbert-space.*

With these results Ludwig ([6], III, § 18) has proved the final representation-theorem. Let $\mathfrak{L}_{\mathbf{tr}}(H)$ be the Banach-space of all hermitean operators of the trace-class on an Hilbert-space H and let $\mathfrak{L}_r(H)$ be the Banach-space of all hermitean operators on H then:

Theorem 6. *If* $G \subsetneqq B'$ *is an irreducible lattice with* $\mathbf{dim}\, G \geqq 4$ *then the dual pair* (B, B') *of Banach-spaces is represented by the couple* $(\mathfrak{L}_{tr}(H), \mathfrak{L}_r(H))$ *and*

(a): *the injective mappings* $\psi : B \to \mathfrak{L}_{tr}(H)$ *and* $\varphi : B' \to \mathfrak{L}_r(H)$ *are surjective and preserve norm, order and linearity.*

(b): $\mu(v,f) = \mathbf{Tr}(\psi(v) \,.\, \varphi(f)$ *for* $(v,f) \in (K, \hat{L})$ *and*

(c): $(\psi(K), \varphi(\hat{L}))$ *is a categorical solution of the axiomatic scheme* $((K, \hat{L})$: *Axioms* **1–4**, *postulates* **(A)**, **(B)**).

I am indebted to Prof. G. Ludwig for his stimulating guidance.

Erratum. Corrections to the *proof of Theorem* 20 *on page* 310 *in Commun. math. Phys.* **11** (1969):

Part 1, *line* 6 *to* 9: This statement must be pushed in part 5 of the proof. There the linear extension of $\bar{\chi} : G \to \mathfrak{P}$ is useful because $\bar{\chi}$ preserves linear dependence.

Part 2, *line* 3: $\mu(v, e) = m_v(E) = \mathbf{Tr}\, VE$ for all $E = \bar{\chi}(e) \in \mathfrak{P}$.

Part 2, *line* 11 has to be completed by: This can be done because of the linearity of the mapping $\bar{\psi} : K \to \mathscr{K}$ (Proof: $v = \Sigma \lambda_i v_i$, $\Sigma \lambda_i = 1$, $\lambda_i \geqq 0$ implies $\mathbf{Tr}(VP) = \mu(v, p) = \Sigma \lambda_i \mu(v_i, p) = \mathbf{Tr}((\Sigma \lambda_i V_i) P)$ for all $P \in A(\mathfrak{P})$ with $p = \bar{\chi}^{-1}(P)$ and $V_i = \bar{\psi}(v_i)$, $V = \bar{\psi}(v)$. Hence $V = \Sigma \lambda_i V_i$.)

Part 3, *line* 5 is to complete by: Every $f \in \hat{L}$ has the form $f = \Sigma \lambda_i e_i$, $e_i \in G$. Hence to prove that ψ is an injection we need only all $f = e = \bar{\chi}^{-1}(E) \in G$.

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