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Positivity and Self Adjointness of the $P(\phi)_2$ Hamiltonian

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Abstract. We give a new proof that the locally correct Hamiltonian $H(g)$ is self adjoint, and that the vacuum energy $E(g) = \inf \text{spectrum} H(g)$ satisfies $-O(D) \leqq E(g)$, where $0 \leqq g \leqq 1$ and $D = \text{diam.supp.} g$.

An existence theorem has been proved for boson quantum fields with polynomial self interactions in two space time dimensions, and many basic properties of these quantum field models have been established. A self contained account of this theory is presented in [1]. A principal step in the construction of the field theory is to show that the Hamiltonian (energy operator) for a bounded space time region is bounded from below and self adjoint. The original proof of semiboundedness was given by Nelson [2] for the ϕ^4 theory. It was extended by Glimm [3] to a different type of space cutoff and to a positive polynomial $P(\phi)$ interaction. The authors [4] obtained a volume independent bound on the vacuum energy per unit volume. The original proof of self adjointness was given by the authors [5] for the ϕ^4 theory and by Rosen [6] for the $P(\phi)$ theory. Subsequent simplifications have been given [7–12]. In this note we present an easy proof of self adjointness and of the volume independent lower bound. The previous simplifications did not yield the volume independent lower bound. See [1] for notation.

Let

$$H(\kappa) = H_{o\,V} + H_I(\kappa) = H_{0,V} + \int :P(\phi_{\kappa,V})(x): g(x)\,dx\,,$$

where g is a function with compact support and $0 \leqq g \leqq 1$. We use the fact that the operators $H(\kappa)$, $H_{o,V}$, $H_I(\kappa)$ are each self adjoint and bounded from below.

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Theorem 1. *$H(\kappa)$ is bounded from below, uniformly in κ, and V for $V \geqq 1$. Moreover $\exp(-tH(\kappa))$ converges in norm as $\kappa \to \infty$ and $V \to \infty$, uniformly in t, for t bounded away from zero and infinity.*

The proof of this result depends on the fact that certain perturbation expansions for $H(\kappa)$ converge in the limit of high energy. See also [11]. We use the Duhamel formula to generate such an expansion:

$$e^{-tH(\kappa)} = e^{-tH(\kappa')} - \int_0^t e^{-sH(\kappa')}(H(\kappa) - H(\kappa'))\, e^{-(t-s)H(\kappa)} ds. \tag{1}$$

To simplify the formulas below, we define

$$H(\sigma, s) = \begin{cases} H(\kappa') & \text{for} \quad \sigma < s \\ H(\kappa) & \text{for} \quad \sigma > s \end{cases}$$

and write the integrand in (1) as

$$\left(e^{-\int_0^t H(\sigma,s)d\sigma} \delta H(s)\right)_+ \tag{2}$$

where the subscript $+$ denotes a time ordering and $\delta H(s)$ inserts $\delta H = H(\kappa) - H(\kappa')$ at the time s. The formula (1) follows from differentiating $F(t, s) = \exp(-sH(\kappa')) \exp(-(t-s)(H(\kappa)))$ with respect to s and integrating the result from 0 to t. Since $F(t, s)$ is strongly continuous in s for $0 \leqq s \leqq t$, and it is norm differentiable with the bounded and norm continuous derivative (2) in $0 < s < t$, the equality (1) is valid whenever the integral in (1) exists. To prove Theorem 1, we use an iterated Duhamel formula, and we bound the resulting integrands. The bounds on the integrands are independent of the Duhamel formula and justify its use.

Let $p = \deg P$ and $\kappa_j = \exp(j^{2/p})$. By undoing the Wick ordering in H_I, we have the cutoff dependent bound

$$-O(j) = -O(\log \kappa_j)^{p/2} \leqq H_I(\kappa_j). \tag{3}$$

Let

$$h_j = \begin{cases} H(\kappa_j) & \text{if} \quad \kappa_j \leqq \kappa \\ H(\kappa) & \phantom{\text{if}} \quad \kappa_j \geqq \kappa \end{cases}$$

and let $\delta h_j = H(\kappa) - H(\kappa_j)$. Iterating (1) we obtain

$$\exp(-tH(\kappa)) = \sum_{n=0}^{\infty} (-1)^n \int ds \left(e^{-\int_0^t H(\sigma,s)d\sigma} \prod_{i=1}^{n} \delta h_i(s_i)\right)_+ \tag{4}$$

where the time integration extends over the domain

$$0 \leqq s_1 \leqq s_2 \leqq \cdots \leqq s_n \leqq t \tag{5}$$

and $H(\sigma, s) = h_j$ if $s_{j-1} < \sigma < s_j$. The series terminates for $\kappa_n \geqq \kappa$.

In the representation of Fock space for which the operators $H_I(\kappa)$ are all diagonal, $\exp(-tH_{o,V})$ is an integral operator with a positive kernel. We take absolute values of operator kernels in this representation. We use the Trotter product formula for each factor $\exp(-(s_{j+1}-s_j)h_j)$ in (4) together with the lower bound (3) and the upper bound

$$|\delta h_i| \leqq \kappa_i^{-1/4} + \kappa_i^{1/4}(\delta h_i)^2 = f_i$$

to obtain

$$\begin{aligned}\left\langle \theta, \left(e^{-\int_0^t H(\sigma,s)\,d\sigma} \prod_{i=1}^{n} \delta h_i(s_i)\right)_+ \theta \right\rangle \\ \leqq \|\theta\|^2 e^{O(n)t} \left\| \left(e^{-\int_0^t H_{o,V}\,d\sigma} \prod_{i=1}^{n} f_i(s_i)\right)_+ \right\| .\end{aligned} \tag{6}$$

The time ordered product above is easily seen to be a bounded operator, and it is norm continuous on the open domain $0 < s_1 < \cdots < s_n < t$. To obtain a bound on the closed domain (5), we use the estimate

$$\|f_j(N+I)^{-p}\| \leqq O(\kappa_j^{-1/4}) = 0(\exp(-\tfrac{1}{4}j^{2/p})) . \tag{7}$$

Here N is the number of particles operator, see [1]. The maximum length $s_j - s_{j-1}$ of the time intervals in the time ordered product is at least $t/(n+1)$. We commute np powers of N onto the maximum time interval, and use (7) to show

$$\begin{aligned}\left\| \left(e^{-\int_0^t H_{o,V}\,d\sigma} \prod_{i=1}^{n} f_i(s_i)\right)_+ \right\| &\leqq a^n (n!)^p \|N^{pn} e^{-tH_o/(n+1)}\| \prod_{j=1}^{n} \kappa_j^{-1/4} \\ &\leqq \left(\frac{c}{t}\right)^{pn} (n!)^{2p} \exp(-bn^{1+2/p})\end{aligned}$$

for certain positive constants a, b, c. Hence (6) is bounded uniformly in s and so the integral in (4) exists. This justifies (4) and yields the bound

$$\exp(-tH(\kappa)) \leqq \sum_{n=1}^{\infty} \exp(an \log n - bn^{1+2/p}) < \infty , \tag{8}$$

with new constants a and b. These constants are independent of κ and are bounded uniformly if $V > 1$ and if t is bounded away from zero and ∞. This completes the proof that $H(\kappa)$ is bounded from below. The same estimates yield norm convergence of the semigroup $\exp(-tH(\kappa))$ as $\kappa \to \infty$, and the limit $V \to \infty$ is also easily controlled. It follows from Theorem 1 and the convergence of $H(\kappa)$ on the dense set $C^\infty(H_o)$, that the limit of $\exp(-tH(\kappa))$ is a semigroup $\exp(-tH)$ with self adjoint generator H, see [1].

We now establish a volume independent lower bound for $H(\kappa)$. Let $A(x)$ be the configuration space annihilation operator, so $A(x)$ annihilates a particle localized (in the Newton-Wigner sense) at x. We let

$$N_i = \int_{i-1/2}^{i+1/2} A(x)^* A(x)\, dx$$

and

$$N_{\mathrm{loc}} = \sum_{i=-\infty}^{\infty} N_i e^{-m|i|/2}$$

be localized number operators. We assert that for g supported in a fixed interval B,

$$\| f_j (N_{\mathrm{loc}} + I)^{-p} \| \leqq O(\kappa_j^{-1/4}) . \tag{9}$$

Our proof of Theorem 1 is valid with $\varepsilon N_{\mathrm{loc}}$ replacing H_o and (9) replacing (7). Thus

$$0 \leqq \varepsilon N_{\mathrm{loc}} + H_I(\kappa) + O(1)\, \mathrm{I} \tag{10}$$

Summing (10) over translates and using the bound $N \leqq \mathrm{const} H_o$ yields

Theorem 2. *Let* $0 \leqq g \leqq 1$ *and let* $D = \mathrm{diam.supp.} g$. *Then*

$$0 \leqq H(\kappa) + 0(D) ,$$

as $D \to \infty$, *and* $O(D)$ *is uniform in* κ *and* $V \geqq 1$.

Proof. We only need to establish (9), and without loss of generality we may assume that $\mathrm{supp}\, g \subset (-\frac{1}{8}, \frac{1}{8})$. Let $w_j(x_1, \ldots, x_r)$ be the configuration space kernel of a Wick monomial W_j contributing to f_j, $0 < r \leqq 2p$. Let

$$w_{j, i_1, \ldots, i_r}(x) = w_j(x) \prod_{\nu=1}^{r} E_{i_\nu}(x_\nu)$$

where $E_i(x)$ is the characteristic function of the interval $[i - 1/2, i + 1/2]$. We assert that

$$\begin{aligned} \| W_{j, i_1, \ldots, i_r} \theta \| &\leqq \mathrm{const} \| w_{j, i_1, \ldots, i_r} \|_2 \left\| \prod_{\nu=1}^{r} (N_{i_\nu} + I)^{1/2} \theta \right\| \\ &\leqq O(\kappa_j^{-1/4}) \exp(-m|i_1| - \cdots - m|i_r|) \left\| \prod_{\nu=1}^{r} (N_{i_\nu} + I)^{1/2} \theta \right\| . \end{aligned} \tag{11}$$

Since $\|(N_i + I)^{1/2} (N_{\mathrm{loc}} + I)^{-1/2}\| \leqq \exp(m|i|/2)$, summing (11) over the i_ν proves (9). The first inequality in (11) is elementary, since N_i measures the number of particles in $[i - \frac{1}{2}, i + \frac{1}{2}]$. To bound the L_2 norm of

$w_{j,i_1,\ldots,i_r}$, we represent the kernel as

$$w_{j,i_1,\ldots,i_r} = \left(\prod_{\nu=1}^{r} K_{\nu,i_\nu}\right) w_j \tag{12}$$

where K_{ν,i_ν} is an integral operator on the variable x_ν of w_j. Then

$$\|w_{j,i_1,\ldots,i_r}\|_2 \leqq \|w_j\|_2 \prod_{\nu=1}^{r} \|K_{\nu,i_\nu}\|$$

$$\leqq O(\kappa_j^{-1/4}) \prod_{\nu=1}^{r} \|K_{\nu,i_\nu}\| .$$

Let $\mu_x^2 = \left(-\frac{d^2}{dx^2} + m^2\right)$ and let ζ be multiplication by a C^∞ function with support in $(-1/4, 1/4)$, such that for all x and y, $\zeta(x)\,\chi_\kappa(x-y)\,g(y) = \chi_\kappa(x-y)\,g(y)$. Here the ultraviolet cutoff in $H_I(\kappa)$ is introduced by the cutoff field $\phi_\kappa(x) = (\chi_\kappa * \phi)(x)$ and χ_κ is supported in $[-\kappa^{-1}, \kappa^{-1}]$. Then we take

$$K_{\nu,i_\nu} = E_{i_\nu}(x_\nu)\,\mu_{x_\nu}^{-1/2}\,\zeta(x_\nu)\,\mu_{x_\nu}^{1/2} ,$$

and (12) is valid. The operators $\mu_x^{\pm 1/2}$ are given by convolution with a distribution $k_\pm(x)$ that is C^∞ except at $x = 0$, with derivatives $0(1)\exp(-m|x|)$ as $|x| \to \infty$. An easy computation then shows that $\|K_{\nu,i}\| \leqq O(1)e^{-m|i|}$. This completes the proof of Theorem 2.

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