Commun. math. Phys. 22, 104–114 (1971)

Representations of Canonical Anticommutation Relations and Implementability of Canonical Transformations

JOSEPH SLAWNY
Institut für Theoretische Physik, Universität Göttingen, Göttingen, Germany

Received March 15, 1971

Abstract. It is proved that irreducible representations of CAR are determined by the groups of implementable automorphisms of the corresponding C^*-algebra. This is done by a study of implementable canonical transformations. Some results in the same directions for factor representations are given.

1. Introduction

Let $\mathfrak{A}$ be a C^*-algebra and let $\mathscr{A}$ be the group of all its automorphisms. $\mathscr{A}$ acts in a natural way in the set of all representations of $\mathfrak{A}$ and for a representation a of $\mathfrak{A}$ let $\mathscr{A}_a$ denote the isotropy subgroup of a, that is, $\mathscr{A}_a$ is the group of all $\tau \in \mathscr{A}$ such that $a \circ \tau$ is equivalent to a.

The mapping $a \mapsto \mathscr{A}_a$ gives a classification of representations. We study here this mapping for irreducible representations of a uniformly hyperfinite (UHF) algebra of Glimm, [2], and prove that in this case it is one-to-one, that is, if $\mathscr{A}_a = \mathscr{A}_b$ then a is equivalent to b.

This is complementary to what is found in [4] by Powers where it is, in particular, proved that $\mathscr{A}$ acts on the set (of the equivalence classes) of irreducible representations of the UHF algebra in a transitive way.

In investigations of physical systems the UHF algebra appears as the C^*-algebra of canonical anticommutation relations (CAR), [7, 5], or as the algebra used for a description of quantum lattice systems. In the case of CAR the C^*-algebra has additional structure, namely, there is given a linear subspace $\mathscr{R}$ which generates $\mathfrak{A}$, which is invariant with respect to involution and on which the norm of $\mathfrak{A}$ is of hilbertian type. The special automorphisms of $\mathfrak{A}$ which leave $\mathscr{R}$ invariant are called canonical, or Bogoliubov, transformations. In this way every canonical transformation gives rise to a unitary operator on $\mathscr{R}$ and conversely: every unitary operator on $\mathscr{R}$ which commutes with involution extends to an automorphism of $\mathfrak{A}$. The group of all canonical transformations is denoted by $\mathscr{K}$ and $\mathscr{K} \cap \mathscr{A}_a$ by $\mathscr{K}_a$.

We study first the mapping $a \mapsto \mathscr{K}_a$, which is more interesting than $a \mapsto \mathscr{A}_a$ as $\mathscr{K}$ is much smaller than $\mathscr{A}$, but for which the results are not so full.

In Section 3 we define the representation a_0 and prove that if $\mathscr{K}_a = \mathscr{K}_{a_0}$ and a is irreducible than a is equivalent to a_0. This is done by picking out a compact subgroup of $\mathscr{K}_{a_0}$ such that in the representation space there exist a unique one-dimensional subspace invariant with respect to this subgroup. By a similar method we proved in [9] corresponding result for the Fock representation.

That from $\mathscr{A}_a = \mathscr{A}_{a_0}$ and irreducibility of a follows equivalence of a to a_0 can be proved a little more simply: Section 2.10 can be shortened and the canonical anticommutation relations need not be mentioned. The stronger theorem is proved here for an eventual fuller investigation of $a \mapsto \mathscr{K}_a$.

Section 4 contains theorems about $a \mapsto \mathscr{K}_a$ for the irreducible case together with generalizations to factor representations. In Section 5, using these theorems and the Powers' theorem about the transitivity of the action of $\mathscr{A}$, we prove that $a \mapsto \mathscr{A}_a$ is one-to-one.

Section 3 is a slight modification of § 3, Chapter III of the Thesis [8]. The rest, apart from the generalizations to factor representations of Section 4.3, can also be found there. Other references are given in suitable places in the following.

2. Resume on Representations of CAR

2.1. Let $\mathscr{R}$ be a complex Hilbert space with involution, that is, on $\mathscr{R}$ is defined, denoted by $x \mapsto x^*$, an antiunitary mapping such that $(x^*)^* = x$. By representation of CAR over $\mathscr{R}$ in the Hilbert space $\mathscr{H}$ we mean a linear mapping $x \mapsto a(x)$ from $\mathscr{R}$ to bounded operators on $\mathscr{H}$ such that

$$a(x^*) = a(x)^*, \quad \text{and} \quad a(x)\,a(y) + a(y)\,a(x) = 2(x, y);$$

here $(,)$ is the symmetric bilinear form defined by $(x, y) = \frac{1}{2}(x^* \mid y)$; we note the inequality $\|a(x)\| \leqq \|x\|$.

The representation is defined uniquely by its restriction to the real Hilbert space of all vectors of $\mathscr{R}$ satisfying: $x^* = x$; this restriction is the starting point of [7].

Our notation is taken from the Appendix 1 of [3]. It is connected with the "a^+" formulation as follows.

Let H be a complex Hilbert space and let a^+ be a linear mapping from H to the bounded operators on $\mathscr{H}$ such that if $a(f) := (a^+(f))^*$ then

$$a(f)\,a^+(g) + a^+(g)\,a(f) = (f \mid g)$$

and

$$a^+(f)\,a^+(g)+a^+(g)\,a^+(f)=0, \forall f, g\in H\,.$$

Then the transition to our notation is through the following definitions: $\mathscr{R}=H\oplus\bar{H}$, where $\bar{H}$ is the Hilbert space conjugated to H, $(f\oplus\bar{g})^*:=g\oplus\bar{f}$ and $a(f\oplus\bar{g}):=a^+(f)+a(g)$. The "$a^+$" description is used in Section 3.10.

2.2. Let $\mathscr{K}$ denote the group of all the unitary operators on $\mathscr{R}$ that commute with the involution. The elements of $\mathscr{K}$ will be called canonical transformations.

In the "a^+" description, every canonical transformations is given by a pair (A, B), A-linear and B-antilinear operators in H, such that

$$f\mapsto a^+(Af)+a(Bf)$$

is again a representation of the CAR over H.

We will say that $k\in\mathscr{K}$ is implementable in the representation a of CAR over $\mathscr{R}$ if there exists a unitary operator U on $\mathscr{H}_a$ such that $a(kx)=U a(x)\,U^{-1}, \forall x\in\mathscr{R}$. If a is irreducible U is unique up to a factor of modulus one. $\mathscr{K}_a$ will denote the subgroup of $\mathscr{K}$ of all the canonical transformations implementable in the representation a.

The representation a will be said to be even if the canonical transformation $x\mapsto -x$ is implementable in a.

2.3. Let, for $i\in\mathbb{N}$, a_i be an even representation of CAR over $\mathscr{R}_i$ in the Hilbert space $\mathscr{H}_i$ and let E_i be a unitary operator on $\mathscr{H}_i$ implementing $x\mapsto -x$ such that $E_i^2=I_i$. Then in the infinite tensor product space $\mathscr{H}=\bigoplus_{i\in\mathbb{N}}(\mathscr{H}_i, h_i)$, where h_i is a normalized vector in $\mathscr{H}_i$, there exists a representation a of CAR over $\mathscr{R}=\bigoplus_{i\in\mathbb{N}}\mathscr{R}_i$, such that

$$a(x)=a_1(x)\otimes I_2\otimes\ldots, \quad \text{for} \quad x\in\mathscr{R}_1\,,$$

and

$$a(x)=E_1\otimes\cdots\otimes E_{i-1}\otimes a_i(x)\otimes I_{i+1}\otimes\ldots, \quad \text{for} \quad x\in\mathscr{R}_i, i>1\,.$$

a will be called the crossed product of a_i and will be denoted by $\bigotimes_{i\in\mathbb{N}}(a_i, h_i)$. If all a_i are irreducible then their crossed product is also irreducible. We have the following property of the crossed product: if σ is a permutation of $\mathbb{N}$ such that $\sigma(i)\neq i$ only for a finite number of $i\in\mathbb{N}$ then $\bigotimes_{i\in\mathbb{N}}(a_{\sigma(i)}, h_{\sigma(i)})$ *is equivalent to* $\bigotimes_{i\in\mathbb{N}}(a_i, h_i)$.

Let us also remark that if a is even and irreducible then the operator implementing $x\mapsto -x$ can be always chosen to be unipotent and that the construction of the crossed product is obviousely also possible when we have a finite sequence of representations and all, with possible exception of the last, are even.

2.4. Let a be an even representation with E implementing $x \mapsto -x$ and let U_k implements $k \in \mathcal{K}_a$. We will say that k is even (resp., odd) if U_k commutes (resp., anticommutes) with E; $\mathcal{K}_a^+$ will denote the set of all even elements of $\mathcal{K}_a$ and $\mathcal{K}_a^-$ the set of odd elements. If a is irreducible then $\mathcal{K}_a^+ \cup \mathcal{K}_a^- = \mathcal{K}_a$ and $\mathcal{K}_a^+$ is a normal subgroup of $\mathcal{K}_a$ of index two.

For irreducible representation we have the following criterium of evenness: let h be an eigenvector of E, then k is even iff $U_k h$ is eigenvector of E with the same eigenvalue as h.

2.5. Let a_i, $i \in \mathbb{N}$, be even and irreducible and $a := \bigoplus_{i \in \mathbb{N}} (a_i, h_i)$. Then if $k_i \in \mathcal{K}_{a_i}^+$ with U_i implementing k_i such that $U_i h_i = h_i$ then $k := \bigoplus_{i \in \mathbb{N}} k_i$ is implementable in a by $U = \bigotimes_{i \in \mathbb{N}} U_i$.

For general theorems about implementability in infinite crossed product see [6, 8].

2.6. There exists a C^*-algebra $\mathfrak{A}$ and injection $i: \mathcal{R} \to \mathfrak{A}$ such that

i) $i(\mathcal{R})$ generates $\mathfrak{A}$,

ii) $i(x^*) = i(x)^*$ and $i(x)\, i(y) + i(y)\, i(x) = 2(x, y), \forall x, y \in \mathcal{R}$,

iii) for every representation a of CAR over $\mathcal{R}$ there exists a unique representation $\tilde{a}$ of $\mathfrak{A}$ such that $\tilde{a} \circ i = a$.

The pair $(\mathfrak{A}, i)$ is unique up to a natural isomorphism.

From uniqueness, every canonical transformation extends to an automorphism of $\mathfrak{A}$. We will often identify $\mathcal{K}$ with the corresponding subgroup of the group $\mathcal{A}$ of all the automorphisms of $\mathfrak{A}$. Similarly, we will often not distinguish between a and $\tilde{a}$.

$\mathcal{A}_a$ will denote the group of all elements of $\mathcal{A}$ which are implementable in the representation a; $\mathcal{K}_a \subset \mathcal{A}_a$. We have:

$$\mathcal{A}_{a \circ \tau} = \tau^{-1} \mathcal{A}_a \tau, \quad \text{for} \quad \tau \in \mathcal{A}$$

with analogous identity for $\mathcal{K}_a$.

On $\mathfrak{A}$ there exists a complex conjugation $x \mapsto \bar{x}$ which extends the mapping $x \mapsto x^*$ on $\mathcal{R}$. To every representation a there corresponds the conjugated representation $\bar{a}$ in the Hilbert space $\overline{\mathcal{H}_a}$ such that $\bar{a}(x)\, \bar{h} = \overline{a(x^*)\, h}$. We have: $\mathcal{K}_{\bar{a}} = \mathcal{K}_a$.

2.7. If $\mathcal{R}$ is of a finite and even dimension then irreducible representation of CAR is unique up to equivalence.

If a is any representation of CAR over $\mathcal{R}$ then a is equivalent to $a' \otimes I$, with a' irreducible.

Because of the uniqueness, every canonical transformation is in the finite dimensional case implementable and the evenness of a canonical transformation does not depend on the choice of the irreducible representation.

2.8. We draw attention to the following abuses of notation: we do not distinguish between $\mathcal{R}_j$ and the corresponding subspace of $\bigoplus_{i \in I} \mathcal{R}_i$, or between operators on $\mathcal{H}$ and the corresponding operators on $\mathcal{H} \otimes \mathcal{H}'$; the same refers to the direct products of groups. We also identify $\mathcal{H}$ with $\bigotimes_i \mathcal{H}_i$ if there is a natural isomorphism, in our context, between them. E.t.c.

3. Representation with a Large Compact Group of Implementable Canonical Transformations

From now on, the Hilbert space $\mathcal{R}$ over which representations of CAR are defined is assumed to be separable. Therefore all irreducible representations over $\mathcal{R}$ act on separable Hilbert spaces and the same can be assumed, after possible passage to quasi-equivalent representation, about the factor representations.

3.1. $\mathcal{K}$ equiped with the strong operator topology is a topological group and this topology we have in mind speaking about, for instance, compact subgroups of $\mathcal{K}$.

Lemma. *Let a be irreducible and let G be a compact subgroup of $\mathcal{K}_a$. Then there exists a finite-dimensional G-invariant subspace of $\mathcal{H}_a$.*

Remark. As for an irreducible representation the operators implementing canonical transformations are unique up to a factor of modulus one, we can speak unambigously about subspaces of the representation space invariant with respect to canonical transformations.

Proof of Lemma. Let U_g implements $g \in G$ and let $\mathfrak{A}_0$ denote the $*$-algebra generated by $\{a(x) : x \in \mathcal{R}\}$. If we can show that for $A \in \mathfrak{A}_0$ the mapping $g \mapsto U_g A U_g^{-1}$ is norm continuous (in fact, weak continuity is sufficient) then the proof follows from what was said in [9], Section 1,2. For $A = a(x)$, $U_g A U_g^{-1} = a(gx)$, and the continuity follows from the inequality: $\|a(x)\| \leqq \|x\|$, $\forall x \in \mathcal{R}$. Each element of $\mathfrak{A}_0$ is a finite sum of products $a(x_1) \ldots a(x_n)$ and therefore we have the required continuity.

3.2. Theorem. *Let, for $i \in \mathbb{N}$, a_i be an irreducible representation of* CAR *over $\mathcal{R}_i$ in $\mathcal{H}_i$, let the dimension of $\mathcal{R}_i$ be finite and even and let G_i be such a compact subgroup of $\mathcal{K}_i^+$ that in $\mathcal{H}_i$ there exists, and only one, G_i-invariant one-dimensional subspace. Let $\mathcal{R} := \bigoplus_{i \in \mathbb{N}} \mathcal{R}_i$, $G := \prod_{i \in \mathbb{N}} G_i$ and let a be an irreducible representation of CAR over $\mathcal{R}$ in which G is implementable. Then a is equivalent to $a_0 := \bigotimes_{i \in \mathbb{N}} (a_i, h_i)$, where h_i is a normalized vector generating the one-dimensional G_i-invariant subspace of $\mathcal{H}_i$.*

The proof is given in Sections 3.3—3.9.

3.3. It is easy to see that G is a compact subgroup of $\mathscr{K}_a$. Therefore Lemma 3.1 is applicable and we conclude the existence of a finite-dimensional G-variant subspace $\mathscr{H}'$ of $\mathscr{H}\left(=\bigotimes_{i\in\mathbb{N}}(\mathscr{H}_i, h_i)\right)$.

3.4. Let $\tilde{G}$ denote the subgroup of G of all such g that $g_i = \mathrm{e}$ for $i>n$, n depending on g. Then we can choose U_g, $g\in\tilde{G}$, in such a way that $g\mapsto U_g$ is a representation of $\tilde{G}$ continuous on each G_i, $i\in\mathbb{N}$.

Proof. Let $U_i(g)$ be the operator on $\mathscr{H}_i$ implementing $g\in G_i$ such that $U_i(g)\,h_i = h_i$; $U_i(g)$ is defined uniquely and $g\mapsto U_i(g)$ is a continuous representation of G_i.

From Section 2.7 we have, for each $i\in\mathbb{N}$, the unique decomposition $\mathscr{H}=\mathscr{H}_i\otimes\mathscr{H}_{i'}$ with $a=a_i\otimes I_{i'}$. Because of the evenness of G_i, $U_i(g)\otimes I_{i'}$ implements $(g_1, g_2, \ldots)$ with $g_i = g$ and $g_j = e$ for $j\neq i$. The operators obtained in this way commute, for a proof once more the evenness must be used, and multiplying them we get needed representation of $\tilde{G}$.

3.5. Lemma. *There exists $y\in\mathscr{H}$ and $n\in\mathbb{N}$ such that the subspace generated by y is G_i-invariant for all $i>n$.*

Proof. Let us consider the restriction of the representation of $\tilde{G}$ of Section 3.4 to the finite-dimensional subspace $\mathscr{H}'$ of Section 3.3.

Let G_{i_1} be the first subgroup from $\{G_i : i\in\mathbb{N}\}$ which acts in $\mathscr{H}'$ in a non-trivial way. As we are dealing here with continuous representation of a compact group we can write

$$\mathscr{H}' = \bigoplus_{\alpha}(\mathscr{H}'_\alpha\otimes\mathscr{H}''_\alpha),$$

where the subspaces $\mathscr{H}'_\alpha\otimes\mathscr{H}''_\alpha$ are G_{i_1}-invariant, G_{i_1} acts irreducibly in $\mathscr{H}'_\alpha$ and the representations of G_{i_1} in $\mathscr{H}'_\alpha$ and $\mathscr{H}'_\beta$ are inequivalent for $\alpha\neq\beta$.

As G_i commute with G_{i_1}, for $i\neq i_1$, $\mathscr{H}'_\alpha\otimes\mathscr{H}''_\alpha$ are also G_i-invariant and G_i, for $i\neq i_1$, acts in $\mathscr{H}''_\alpha$. By our choise of i_1 there exists such α that $\dim\mathscr{H}''_\alpha<\dim\mathscr{H}'$. Therefore this decomposition repeated at most $\dim\mathscr{H}'$ times leads to the needed vector.

3.6. For any $n\in\mathbb{N}$ there exists a decomposition

$$\mathscr{H}=\mathscr{H}_1\otimes\cdots\otimes\mathscr{H}_n\otimes\mathscr{H}_{n'}$$

and a representation $a_{n'}$ of the CAR over $\bigoplus_{i>n}\mathscr{R}_i$ in $\mathscr{H}_{n'}$ such that a is the crossed product of $a_1, \ldots, a_n, a_{n'}$.

The groups G_i for $i>n$ act in $\mathscr{H}_i$ and $G_{n'} := \{(g_1, g_2, \ldots)\in G : g_i = e$ for $i\leqq n)$ acts in $\mathscr{H}_{n'}$.

Proof. Let $\mathscr{H}=\mathscr{H}_1\otimes\mathscr{H}_{1'}$ be the decomposition of Section 2.7 with $a(x)=a_1(x)\otimes I_{1'}$ for $x\in\mathscr{R}_1$. As a_1 is an irreducible representation over $\mathscr{R}_1$ and $E_1 a(y)$, $y\in\mathscr{R}\ominus\mathscr{R}_1$, commutes with $a(x)$, $x\in\mathscr{R}_1$, we conclude that $E_1 a(y)=I_1\otimes a_{1'}(y)$, which proves the existence of the decomposition for $n=1$. $G_{1'}$ acts in $\mathscr{H}_{1'}$ because the operators implementing $G_{1'}$ commute with a_1, and that G_1 acts in $\mathscr{H}_1$ follows from the irreducibility of a and the evenness of G_1.

The general case follows by induction.

3.7. Lemma. *Let a be an irreducible representation over $\mathscr{R}$ in the Hilbert space $\mathscr{H}$ with a G-invariant vector h. Then a is equivalent to $a_0 = \bigotimes_{i\in\mathbb{N}} (a_i, h_i)$ and the G-invariant vector is unique (up to a scalar factor).*

Proof. Let $\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_{1'}$ be the decomposition of Section 3.6 for $n=1$. Then h can be written as $h = h_1 \otimes h_{1'} + \sum_\alpha v_\alpha \otimes v_{\alpha'}$, with $v_\alpha \perp h_1$. Let $P := \int_{G_1} U_1(g) \otimes I_{1'} dg$, where dg denotes the normalized Haar measure on G_1; P is the projection on the subspace of all G_1-invariant vectors. Then $Ph=h$, $Ph_1 = h_1$ and $Pv_\alpha = 0$ as h_1 is the unique G_1-invariant vector in $\mathscr{H}_1$ and $v_\alpha \perp h_1$. Therefore $h = h_1 \otimes h_{1'}$. Applying this decomposition n times we get: $h = h_1 \otimes \cdots \otimes h_n \otimes h_{n'}$ and suitable decomposition of a. Now it is easy to see that there exists a unique unitary mapping from $\mathscr{H}$ to $\bigotimes_{i\in\mathbb{N}} (\mathscr{H}_i, h_i)$ which maps $a(x_1) \ldots a(x_k) h$ onto $a_0(x_1) \ldots a_0(x_k)(h_1 \otimes h_2 \otimes \cdots)$ and brings a and a_0 into equivalence.

The uniqueness of h follows from the irreducibility of a and from the fact that, as was proved above, each invariant vector defines an isomorphism of a and a_0.

3.8. Let $\mathscr{H} = \mathscr{H}_1 \otimes \cdots \otimes \mathscr{H}_n \otimes \mathscr{H}_{n'}$ be the decomposition of Section 3.6.

Lemma. *The G_i-invariant, $i>n$, vector y of Lemma 3.6 admits factorization as $y = u \otimes h_{n'}$, $u \in \mathscr{H}_1 \otimes \cdots \otimes \mathscr{H}_n$, $h_{n'} \in \mathscr{H}_{n'}$ and $h_{n'}$ is $G_{n'}$ invariant.*

Proof. Let U be the continuous unitary representation of $G_{n'}$ in $\mathscr{H}$ implementing the action of $G_{n'}$ in $\mathscr{R}$ such that $U(g)\,y = y$. Such representation exists and is unique: the existence and the continuity follow from the implementability of G and the continuity of the action of G on $\mathscr{R}$ and the uniqueness follows from the irreducibility of a.

As $G_{n'}$ acts in $\mathscr{H}_{n'}$ the same is true about $P := \int_{G_{n'}} dg\, U(g)$, where dg denotes the normalized Haar measure on $G_{n'}$. P is the projection on $G_{n'}$-invariant vectors and, as $Py=y$, $P \neq 0$. Therefore there exists in $\mathscr{H}_{n'}$ a $G_{n'}$-invariant vector. Applying to $G_{n'}$ and $\bigotimes_{i>n} (a_i, h_i)$ Lemma 3.7, we see that the $G_{n'}$ invariant vector in $\mathscr{H}_{n'}$ is unique. Denoting such a vector by $h_{n'}$, writing: $y = u \otimes h_{n'} + \sum_\alpha u_\alpha \otimes u_{\alpha'}$, with $u_{\alpha'} \perp h_{n'}$, and applying to this decomposition P we get, as in Lemma 3.7, that $y = u \otimes h_{n'}$.

3.9. To finish the proof of Theorem 3.2 let us consider h,

$$h := h_1 \otimes \cdots \otimes h_n \otimes h_{n'},$$

where $h_{1'} \dots, h_n$ are the vectors from the definition of a_0 and $h_{n'}$ is the vector of Lemma 3.8. h is G invariant and therefore Lemma 3.7 is applicable. Hence, theorem is proved.

3.10. We are now going to construct a triple (a, h, G) which will be used to define a representation a_0 of Theorem 3.2; in some places the proofs are only indicated. It will be more convenient to do this in the "a^+" notation of Section 2.1.

Let $\mathscr{R} = H \oplus \bar{H}$ and let $\{f_1 \dots, f_4\}$ be an orthonormal basis of H. Let us denote: $a_i^+ := a(f_i)$ and $a_i := a(\bar{f}_i)$. With this notation we have

Proposition. *Let a be an irreducible representation over* $\mathscr{R}$ *and let* Ω *be a normalized vector in* $\mathscr{H}_a$ *such that* $a_i \Omega = 0, i = 1, \dots, 4$. *Let* $h := \frac{1}{\sqrt{2}}(\Omega + a_1^+ \dots a_4^+ \Omega)$ *and let* G *be the group of all these canonical transformations that leave invariant the suspace of* $\mathscr{H}_a$ *generated by* h. *Then* G *is a compact subgroup of* $\mathscr{K}^+$ *and* $\mathbb{C}h$ *is the unique one-dimensional* G*-invariant subspace of* $\mathscr{H}_a$.

Proof. Let E be the operator implementing $x \mapsto -x$ such that $E\Omega = \Omega$. Then $E\Omega' = \Omega'$, where $\Omega' := a_1^+ \dots a_4^+ \Omega$, and also $Eh = h$. If $U(g)$ implements $g \in G$ then $U(g)\,h$ is proportional to h and therefore, see the end of Section 2.4, $g \in \mathscr{K}^+$. This proves that $G \subset \mathscr{K}^+$.

To prove the compactness of G, let us remark that the g-invariance of the subspace generated by h is equivalent to the equalities:

$$(h \mid a(gx_1) \dots a(gx_n)\, h) = (h \mid a(x_1) \dots a(x_n)\, h),$$

for each sequence $x_1, \dots, x_n \in \mathscr{R}, n = 1, 2 \dots$.

As $g \mapsto (h \mid a(gx_1) \dots a(gx_n)\, h)$ are continuous functions on $\mathscr{K}$, the group G is a closed subset of a compact space and therefore compact.

G contains the subgroup $\{(A, 0) : A \in SU(H)\}$, where $SU(H)$ is the group of all the unitary and unimodular operators on H. The only one-dimensional subspaces of $\mathscr{H}_a$ invariant with respect to this subgroup are those generated by vectors of the form: $\alpha\Omega + \beta\Omega'$. For, if $\mathscr{H}_n$ denotes the subspace of $\mathscr{H}_a$ generated by $a_{i_1}^+ \dots a_{i_n}^+ \Omega$ then

$$\mathscr{H}_a = \mathscr{H}_0 \oplus \mathscr{H}_1 \oplus \cdots \oplus \mathscr{H}_4$$

with $\dim \mathscr{H}_n = \binom{4}{n}$, and this decomposition is $SU(H)$-invariant. Moreover, the representation of $SU(H)$ in $\mathscr{H}_n$ are irreducible what follows, for instance, from their equivalence to the representations of $SU(H)$ in the antisymmetrized n-th tensor product of H.

Let us now consider the canonical transformation $a^+ \mapsto b^+$ defined by:

$$b_1^+ = \frac{1}{\sqrt{2}}(a_1^+ + a_2), \quad b_2^+ = \frac{1}{\sqrt{2}}(a_2^+ - a_1),$$

$$b_3^+ = \frac{1}{\sqrt{2}}(a_3^+ + a_4), \quad b_4^+ = \frac{1}{\sqrt{2}}(a_4^+ - a_3).$$

Direct computation shows that b_i, $i=1,\ldots,4$, applied to $\frac{1}{2}(1-a_1^+a_2^+)(1-a_3^+a_4^+)\Omega$ give zero. Therefore, an operator U implementing this canonical transformation acting on Ω gives a vector proportional to $\frac{1}{2}(1-a_1^+a_2^+)(1-a_3^+a_4^+)\Omega$. Fixing U by demanding equality here we get:

$$U\Omega' = \tfrac{1}{2}(1+a_1^+a_2^+)(1+a_3^+a_4^+)\Omega$$

and therefore

$$(\Omega \mid U\Omega) = (\Omega^* \mid U\Omega') = (\Omega' \mid U\Omega) = (\Omega' \mid U\Omega') = \tfrac{1}{2}.$$

The vector $\alpha\Omega + \beta\Omega'$, $|\alpha|^2+|\beta|^2=1$, generates a one-dimensional U-invariant subspace if, and only if,

$$|(\alpha\Omega+\beta\Omega' \mid U(\alpha\Omega+\beta\Omega')|=1,$$

i.e., $|\alpha\bar{\alpha}+\alpha\bar{\beta}+\beta\bar{\alpha}+\beta\bar{\beta}|=2$. Now, that $\alpha=\beta$ follows from the identities:

$$|\alpha\bar{\alpha}+\beta\bar{\beta}+\alpha\bar{\beta}+\beta\bar{\alpha}| = |\alpha+\beta|^2 \quad \text{and} \quad |\alpha-\beta|^2 = 2(|\alpha|^2+|\beta|^2)-|\alpha+\beta|^2.$$

Therefore only vectors proportional to $\frac{1}{\sqrt{2}}(\Omega+\Omega')$ generate a one-dimensional subspace of $\mathscr{H}_a$ invariant under $SU(H)$ and the canonical transformation just described.

4. The Correspondence $a \mapsto \mathscr{H}_a$

4.1. Theorem. *There exists an irreducible representation a which is determined by $\mathscr{H}_a$, i.e., such that if $\mathscr{H}_b = \mathscr{H}_a$ and b is irreducible then b is equivalent to a.*

Proof. It is enough to show that there exists a representation satisfying the assumptions of Theorem 3.2.

Let us consider the representation $\bigotimes_{i\in\mathbb{N}}(a_i, h_i)$ with (a_i, h_i, G_i) being copies of (a, h, G) of Section 3.10. By Section 2.5, $G_1 \times G_2 \times \cdots$ is implementable in $\bigotimes_{i\in\mathbb{N}}(a_i, h_i)$. Theorem is proved.

4.2. The set of all irreducible representations determined by their groups of implementable canonical transformations has the cardinality of real line. For the proof one can apply to the representation of Theorem 3.2 suitable canonical transformations. We will not give the details.

For the representation of Theorem 3.2, for the Fock representation and for representations obtained from these by applying canonical transformations we have the following situation:

if $\mathscr{K}_b = \mathscr{K}_a$ then b is equivalent either to a or to $\bar{a}$; is this true for every representation of CAR?

4.3. To give a generalization of Theorem 3.2 to factor representations, we let Int $(\mathscr{K}_a)$, for an in general non-irreducible representation a of CAR, denote the subgroup of $\mathscr{K}_a$ of all canonical transformations which are implementable by operators from the von Neumann algebra generated by $\{a(x) : x \in \mathscr{R}\}$.

Theorem. *Let $\mathscr{R}$, G, a_0 be as in Theorem 3.2 and let a be such a factor representation over $\mathscr{R}$ that $G \subset \mathrm{Int}(\mathscr{K}_a)$. Then a is a factor type I representation, quasi-equivalent to a_0.*

Examination of the given proof of Theorem 3.2 shows that it is applicable also in the case of factor representations provided that one has a suitable generalization of Lemma 3.1. Namely, one has to prove that it is possible to choose the operators U_g implementing G in such a way that $g \mapsto U_g$ is a projective measurable representation of G. Such result is proved in our paper [10].

Essentially the same applies to the following generalization of [9]:

Theorem. *Let a be a Fock representation over $\mathscr{R} = H \oplus \bar{H}$ and let b be such a factor representation over $\mathscr{R}$ that $\mathscr{K}_a \subset \mathrm{Int}(\mathscr{K}_b)$. Then b is a factor type I representation, quasi-equivalent either to a or to $\bar{a}$.*

Here by a Fock representation over $\mathscr{R} = H \oplus \bar{H}$ we mean such an irreducible representation a that these exists $\Omega \in \mathscr{K}_a, \Omega \neq 0$, with $a(x)\,\Omega = 0, \forall x \in \bar{H}$.

One might ask here a question analogous to that in the end of Section 4.2.

4.5. The group $\mathrm{Int}(\mathscr{K}_a)$ for the factor type II_1 representation is described in [1]. It appears that in this case $\mathrm{Int}(\mathscr{K}_a)$ is contained in $\mathscr{K}_b$ for each Fock representation b. Hence, one can not expect for general factor representations to have theorems of the type of Section 4.3.

5. The Correspondence $a \mapsto \mathscr{A}_a$

Theorem. *The correspondence $a \mapsto \mathscr{A}_a$ which to every irreducible representation of the C^*-algebra $\mathfrak{A}$ of CAR assignes the group of all implementable automorphisms of $\mathfrak{A}$ is one-to-one.*

Proof. Corollary 3.8 of [4] states, in particular, that if a and b are irreducible representations of $\mathfrak{A}$ then there exists an automorphism τ of $\mathfrak{A}$ such that $b \circ \tau$ is equivalent to a. On the other hand, $\mathscr{A}_{b \circ \tau} = \tau \mathscr{A}_b \tau^{-1}$. Therefore it is enough to prove the following: there exists an irreducible representation a_0 of $\mathfrak{A}$ such that if $\mathscr{A}_a = \mathscr{A}_{a_0}$, and a is irreducible, than a is equivalent to a_0.

For, let $\mathscr{A}_b = \mathscr{A}_{b'}$, let both b and b' be irreducible and let $\tau \in \mathscr{A}$ be such that $b \circ \tau$ is equivalent to a_0. Then $\mathscr{A}_{a_0} = \mathscr{A}_b$ and therefore $\mathscr{A}_{b' \circ \tau} (= \tau \mathscr{A}_{b'} \tau^{-1} = \tau \mathscr{A}_b \tau^{-1})$ is equal to $\mathscr{A}_{a_0}$. Therefore $b' \circ \tau$ is equivalent to $b \circ \tau$ and b' is equivalent to b.

But existence of such a representation a_0 is assured by Theorem 4.1. For, let a_0 be one of the representations of that theorem and let $\mathscr{A}_a = \mathscr{A}_{a_0}$. Then, as $\mathscr{K}_a = \mathscr{K} \cap \mathscr{A}_a$ for every representation a, $\mathscr{K}_a = \mathscr{K}_{a_0}$ and therefore a is equivalent to a_0. Hence, Theorem is proved.

Acknowledgement. The manuscript was prepared at the Institute für Theoretische Physik in Göttingen. I wish to thank Prof. H. J. Borchers for his invitation and for helpful remarks. I am also grateful to the Alexander von Humboldt Stiftung for financial support and to H. A. Schultens for help in the preparation of the manuscript.

References

1. Blattner, R. J.: Pacific J. Math. **8**, 665 (1958).
2. Glimm, J.: Trans. Amer. Math. Soc. **95**, 318 (1960).
3. Haag, R., Kastler, D.: J. Math. Phys. **5**, 848 (1964).
4. Powers, R. T.: Ann. Math. **86**, 138 (1967).
5. Lectures in the Cargese Summer School (1969).
6. Segal, I. E.: Ann. Math. **63**, 160 (1956).
7. Shale, D., Stinespring, W. F.: Ann. Math. **80**, 365 (1964).
8. Slawny, J.: Thesis. The Weizmann Institute of Science, Rehovot (1969).
9. — Commun. math. Phys. **13**, 175 (1969).
10. — On factor representations and the C^*-algebra of canonical commutation relations, Lemma 2.1.

Dr. J. Slawny
Dept. of Physics and Astronomy
Tel Aviv University
Tel Aviv, Ramat Aviv/Israel