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A Note on Product Measures and Representations of the Canonical Commutation Relations

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Abstract. There is a well-known theorem which states that a non-zero σ-finite left quasi-invariant measure on a σ-compact locally compact group G must be equivalent to left Haar measure. It is shown in this paper that there is a natural generalization of this fact to the case in which the group G is replaced by a product space, one factor of which is a group. With the aid of this generalization, an easy proof of the following fact, due to H. Araki, is given: the representations of the canonical commutation relations constructed in the usual measure-theoretic manner are ray continuous.

Almost invariably, the most desirable measure on a product space is a product measure. However, in doing integration theory on a product space, one is sometimes confronted with a measure which is *a priori* not even equivalent to a product measure. It is therefore of some interest to find conditions under which a measure on a product space must be equivalent to a product measure. Theorem 1 below gives such a condition which is, roughly speaking, that one of the factors be a group and that the measure in question be quasi-invariant under the action of the group on the product space. Araki proved Theorem 1 for the special case of a Euclidean group [1; Lemma 5.2]. His proof relies on the ray continuity of the representations of the canonical commutation relations (CCRs) constructed in the usual measure-theoretic manner (see [1; Section 1]). It was pointed out by Araki that the converse is also true, i.e., that ray continuity could be deduced from Theorem 1. In fact, the ray continuity is an easy consequence of Theorem 2, which itself depends on Theorem 1.

Suppose that G is a σ-compact locally compact group, that $\mathscr{A}$ is its σ-algebra of Borel sets (i.e., the σ-algebra generated by the open sets), and that λ is a left-invariant Haar measure on $(G, \mathscr{A})$. Suppose further that $\mathscr{B}$ is a σ-algebra of subsets of a non-empty set Z. Let $\mathscr{A} \times \mathscr{B}$ be the product σ-algebra on $G \times Z$. Measurability of subsets of or functions defined on G [resp., $Z, G \times Z$] will always be taken with respect to $\mathscr{A}$ [resp., $\mathscr{B}$, $\mathscr{A} \times \mathscr{B}$]. Setting $x(y, \zeta) = (xy, \zeta)$ for all $x, y \in G$ and all $\zeta \in Z$ defines a left action of G on $G \times Z$. The characteristic function of a subset S of $G \times Z$ will be denoted by 1_S.

Theorem 1. *Suppose that* ν *and* μ *are* σ*-finite measures on* $(G \times Z, \mathscr{A} \times \mathscr{B})$ *and* $(Z, \mathscr{B})$, *resp. Then* ν *is equivalent to* $\lambda \times \mu$ *if and only if*

(i) *for each* $x \in G$ *and* $S \in \mathscr{A} \times \mathscr{B}$, $\nu(S) = 0$ *if and only if* $\nu(xS) = 0$, *and*

(ii) *for each* $B \in \mathscr{B}$, $\mu(B) = 0$ *if and only if* $\nu(G \times B) = 0$.

Proof. With no loss of generality, ν and μ may both be assumed to be finite. Notice that λ is σ-finite since G is, by hypothesis, σ-compact. If ν is equivalent to $\lambda \times \mu$, then clearly (ii) must hold, and by Fubini's theorem, so must (i). Conversely, suppose that (i) and (ii) are satisfied.

Fix, for the moment, a measurable subset S of $G \times Z$. Then

$$S[x] = \{\zeta \in Z : (x, \zeta) \in S\}$$

is measurable for each x in G ([2; p. 141, Theorem A]), and the function $(x, y, \zeta) \to 1_S(x^{-1}y, \zeta)$ is measurable on $G \times G \times Z$ with respect to the product σ-algebra. If Δ is the modular function of G, then

$$\begin{aligned}
&\int_{G \times Z} \int_G 1_S(x^{-1}y, \zeta)\, d\lambda(x)\, d\nu(y, \zeta) \\
&\qquad = \int_{G \times Z} \int_G \Delta(x)\, 1_S(x, \zeta)\, d\lambda(x)\, d\nu(y, \zeta) \\
&\qquad = \int_G \int_{G \times Z} \Delta(x)\, 1_{G \times S[x]}(y, \zeta)\, d\nu(y, \zeta)\, d\lambda(x) \\
&\qquad = \int_G \Delta(x)\, \nu(G \times S[x])\, d\lambda(x)
\end{aligned}$$

by an application of Tonelli's theorem ([2; p. 147, Theorem B]). On the other hand,

$$\int_G \int_{G \times Z} 1_S(x^{-1}y, \zeta)\, d\nu(y, \zeta)\, d\lambda(x) = \int_G \nu(xS)\, d\lambda(x).$$

So by a second application of Tonelli's theorem

$$\int_G \nu(xS)\, d\lambda(x) = \int_G \Delta(x)\, \nu(G \times S[x])\, d\lambda(x).$$

Since the modular function is everywhere positive, the last equation means that $\nu(xS) = 0$ for λ-a.a. $x \in G$ if and only if $\nu(G \times S[x]) = 0$ for λ-a.a. $x \in G$. Now by (i), $\nu(xS) = 0$ for λ-a.a. $x \in G$ is equivalent to $\nu(S) = 0$. The collection of all those measurable subsets S of $G \times Z$ for which $x \to \mu(S[x])$, is a measurable function on G is a monotone class which includes the measurable rectangles, and is therefore all of $\mathscr{A} \times \mathscr{B}$ [2; p. 27, Theorem A]. From condition (ii), $\nu(G \times S[x]) = 0$ for λ-a.a. $x \in G$ if and only if $\mu(S[x]) = 0$ for λ-a.a. $x \in G$, which, by Fubini's theorem, is equivalent to $\lambda \times \mu(S) = 0$.

Theorem 1 is a generalization of the fact that any left quasi-invariant measure on G is equivalent to λ. In fact, the proofs of this result and of Theorem 1 are similar (see [5; Lemma 3.3]). Say H is a closed subgroup of G. In view of the fact that the left coset space G/H carries a unique left quasi-invariant measure class ([6; Lemma 1.3]), it is hardly surprising that Theorem 1 remains valid when the factor G is replaced by G/H. This extension of the theorem will now be sketched.

Let H be a closed subgroup of G, and let λ_H be its left-invariant Haar measure. Let p: $G \to G/H$ be the natural projection of G onto the left coset space G/H. Let $\mathscr{A}_H$ denote the largest σ-algebra of subsets of G/H making p measurable, and let λ^* be the measure on $(G/H, \mathscr{A}_H)$ defined by p and some finite measure λ' on $(G, \mathscr{A})$ equivalent to λ. Now suppose that μ and ν are finite measures on $(Z, \mathscr{B})$ and $(G/H \times Z, \mathscr{A}_H \times \mathscr{B})$, resp., which satisfy

(iii) for each $x \in G$ and $S \in \mathscr{A}_H \times \mathscr{B}$, $\nu(S) = 0$ if and only if $\nu(xS) = 0$, and

(iv) for each $B \in \mathscr{B}$, $\mu(B) = 0$ if and only if $\nu(G/H \times B) = 0$.

(In (iii), xS means the image of S under the natural left action of x on G/H.) A modification of the argument employed by Mackey in [6; Lemma 1.3] gives a σ-finite measure σ on $(G \times Z, \mathscr{A} \times \mathscr{B})$ which satisfies (i), (ii), and $\sigma(p^{-1}(E)) = \lambda_H(H)\,\nu(E)$ for each E in $\mathscr{A}_H \times \mathscr{B}$. Then σ is equivalent to $\lambda' \times \mu$, and it readily follows that ν must be equivalent to $\lambda^* \times \mu$. This proves the non-trivial part of the following result.

Corollary. *With the above notation, ν and $\lambda^* \times \mu$ are equivalent if and only if ν and μ satisfy* (iii) *and* (iv).

Suppose now that ν is a finite measure on $(G \times Z, \mathscr{A} \times \mathscr{B})$ which satisfies (i). Let $\nu_t(S) = \nu(tS)$ for all t in G and all sets S in $\mathscr{A} \times \mathscr{B}$. For each $t \in G$ and each $\psi \in L^2(G \times Z, \mathscr{A} \times \mathscr{B}, \nu) = L^2(\nu)$, put

$$[V(t)\psi](x, \zeta) = \left[\frac{d\nu_t}{d\nu}(x, \zeta)\right]^{1/2} \psi(tx, \zeta)$$

for all $(x, \zeta) \in G \times Z$. It is readily verified that $t \to V(t)$ is a unitary representation of G on $L^2(\nu)$. It will be shown below that this representation is even strongly continuous.

If μ denotes the finite measure $B \to \nu(G \times B)$ on $(Z, \mathscr{B})$, then ν and $\lambda \times \mu$ must be equivalent (Theorem 1). Let f be the Radon-Nikodym derivative of ν with respect to $\lambda \times \mu$. Then f can be chosen to satisfy $0 < f(x, \zeta) < \infty$ for all points (x, ζ) in $G \times Z$. For any function ψ on $G \times Z$, set $\psi_t(x, \zeta) = \psi(tx, \zeta)$ for all $t, x \in G$ and $\zeta \in Z$. An easy calculation shows that

$$\nu_t(S) = \int_S f_t \, d\lambda \times \mu$$

for each $t \in G$ and all $S \in \mathscr{A} \times \mathscr{B}$. Thus $\dfrac{d\nu_t}{d\lambda \times \mu} = f_t$ ν-a.e., and so $\dfrac{d\nu_t}{d\nu} = \dfrac{f_t}{f}$ ν-a.e. [2; p. 133, Theorem A].

Suppose that ψ_1 and ψ_2 are two functions in $L^2(\nu)$. Since the weak and strong topologies coincide on the unitary operators, it is sufficient to show that the inner product of $V(t)\psi_1$ and ψ_2 in $L^2(\nu)$ depends continuously on t. This inner product is

$$\begin{aligned}\int_{G\times Z} [V(t)\psi_1]\bar{\psi}_2\, d\nu &= \int_{G\times Z} (f_t/f)^{1/2}\psi_{1t}\bar{\psi}_2 f\, d\lambda\times\mu \\ &= \int_{G\times Z} (f^{1/2}\psi_1)_t (f^{1/2}\bar{\psi}_2) d\lambda\times\mu\,.\end{aligned}$$

The functions $(f^{1/2}\psi_1)_t$ and $f^{1/2}\psi_2$ are in $L^2(G\times Z, \mathscr{A}\times\mathscr{B}, \lambda\times\mu)$ $=L^2(\lambda\times\mu)$, and the last integral is their inner product in this Hilbert space. Consequently, it is enough to show that $t\to\psi_t$ is strongly continuous from G to $L^2(\lambda\times\mu)$ for each ψ in $L^2(\lambda\times\mu)$. Now this is certainly the case if $\psi(x,\zeta)=\psi_1(x)\,\psi_2(\zeta)$ for some ψ_1 in $L^2(G,\mathscr{A},\lambda)$ and ψ_2 in $L^2(Z,\mathscr{B},\mu)$ [4; Theorem 30C]. Since the finite linear combinations of such functions are strongly dense in $L^2(\lambda\times\mu)$, the proof of the following theorem is complete.

Theorem 2. *If ν is a finite measure on $(G\times Z, \mathscr{A}\times\mathscr{B})$, then the equation*

$$[V(t)\psi](x,\zeta)=\left[\frac{d\nu_t}{d\nu}(x,\zeta)\right]^{1/2}\psi(tx,\zeta)$$

defines a strongly continuous unitary representation V of G on $L^2(G\times Z, \mathscr{A}\times\mathscr{B}, \nu)$.

Araki proved the special case of Theorem 2 in which G is the real line [1; Lemma 2.3]. The ray continuity of the representations of the CCRs constructed in the usual measure-theoretic manner follows easily from this special case of Theorem 2 (see [1; Theorem 2.4 and its proof]).

In conclusion, it should be noted that G. Hegerfeldt has also given a short proof of the ray continuity [3; Corollary 3.5].

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