The Energy Momentum Spectrum and Vacuum Expectation Values in Quantum Field Theory, II

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Abstract. We prove that the $\mathscr{P}(\varphi)_2$ quantum field theory satisfies the spectral condition. The space time translation $a = (x, t)$ is implemented by the unitary group $U(a) = \exp(itH - ixP)$, and the joint spectrum of the energy operator H and the momentum operator P is contained in the forward cone. We also obtain bounds on certain vacuum expectation values of products of field operators. Our proofs involve an analysis of the limit $V \to \infty$ for approximate theories in a periodic box of volume V. Assuming the existence of a uniform mass gap, we are able to establish all the Wightman axioms with the exception of the Lorentz invariance of the vacuum.

1. Introduction

We study a boson quantum field φ with a polynomial self interaction $\mathscr{P}(\varphi)$ in two dimensional space time. This theory provides an example of all the Haag-Kastler axioms and many of the Wightman axioms for quantum field theory. In this paper we prove that the energy-momentum spectrum lies in the forward cone. In addition, we prove bounds on the vacuum expectation values of products of the differentiated field operators $\partial_t \varphi(x, t) = \varphi_t$ and $\partial_x \varphi(x, t) = \varphi_x$. Three of the Wightman axioms remain open problems for the $\mathscr{P}(\varphi)_2$ theory. They are the invariance of the vacuum under Lorentz rotations, the uniqueness of the vacuum and the existence of vacuum expectation values of products of the field $\varphi(x, t)$ (without differentiation). Assuming the existence of a mass gap, we verify the latter two of these three missing axioms in Section 4.

The $\mathscr{P}(\varphi)_2$ theory is obtained as a limit of cutoff or approximate field theories. We have previously considered two different space cutoff

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procedures: a "box" cutoff and a "coupling constant" cutoff. In the "coupling constant" cutoff, the coupling constant λ is replaced by a space dependent coupling constant $\lambda g(x)$. For instance, $g(x)$ might equal the characteristic function of a bounded interval. In the "box" cutoff, the Hamiltonian is translation invariant, but we replace the configuration space R^1 by the periodic interval $[-\frac{1}{2}V, \frac{1}{2}V]$ of length V.

In [3] we studied the infinite volume limit of the theory with a coupling constant cutoff. We showed that the limit $g \to 1$ yields a locally Fock vacuum. We now use the same techniques to show that the limit $V \to \infty$, of box cutoff theories, yields a locally Fock vacuum. The algebra of observables $\mathfrak{A}$ and the Poincaré automorphisms $\sigma_{\{a, A\}}$ constructed from the two infinite volume limits, $g \to 1$ or $V \to \infty$, are identical. Thus we obtain from the limit $V \to \infty$ a theory incorporating the results previously established for the limit $g \to 1$. In addition we have new properties for the limit $V \to \infty$. The spectral condition $0 \leq H$, $0 \leq H^2 - P^2$ was previously established in the theory with a box cutoff, $V < \infty$, as were the bounds on φ_t and φ_x [4]. In the present paper, we transfer these results to the limit $V \to \infty$.

We describe the approximate theories in more detail. We follow the notation of our Les Houches lectures [5]. The time zero fields in infinite space R^1 satisfy the canonical commutation relations

$$[\varphi(f_1), \pi(f_2)] = i \langle f_1, f_2 \rangle .$$

These field operators act on the Fock space $\mathcal{F}$, which is the symmetric tensor algebra over $\mathcal{L}_2(R^1)$. For each bounded open interval $\mathcal{B} \subset R^1$, we define the algebra $\mathfrak{A}(\mathcal{B})$ of local observables as the von Neumann algebra generated by the operators

$$\exp(i\varphi(f_1) + i\pi(f_2)), \quad f_i \in \mathfrak{C}_0^\infty(\mathcal{B}) .$$

We also use the C^*-algebra $\mathfrak{A}_0(\mathcal{B})$ generated by these operators; $\mathfrak{A}_0(\mathcal{B})$ is weakly dense in $\mathfrak{A}(\mathcal{B})$. Let

$$\mathfrak{A} = \text{norm closure} \left(\bigcup_{\mathcal{B}} \mathfrak{A}(\mathcal{B}) \right),$$

$$\mathfrak{A}_0 = \text{norm closure} \left(\bigcup_{\mathcal{B}} \mathfrak{A}_0(\mathcal{B}) \right).$$

The Hamiltonian $H(g)$ with the coupling constant cutoff g is defined by

$$H(g) = H_0 + \int :\mathcal{P}(\varphi(x)): g(x)\, dx - E_g .$$

Here $\mathcal{P}$ is a positive polynomial, g is a positive function with compact support, and E_g is chosen so that inf spectrum $H(g) = 0$.

In the periodic box approximation, the field φ_V satisfies (1.1), namely $[\varphi_V(f_1), \pi_V(f_2)] = i\langle f_1, f_2 \rangle$, provided $f_i \in \mathscr{L}_2([-\frac{1}{2}V, \frac{1}{2}V])$. The periodic Fock space $\mathscr{F}_V$ is the symmetric tensor algebra over $\mathscr{L}_2([-V/2, V/2])$. The Hamiltonian H_V for the volume V is

$$H_V = H_{0,V} + \int_{-\frac{1}{2}V}^{\frac{1}{2}V} :\mathscr{P}(\varphi_V(x)): dx - E_V,$$

where E_V is chosen so that $\inf \operatorname{spectrum} H_V = 0$.

There is a natural identification

$$\mathscr{F} \cong \mathscr{F}_V \otimes \mathscr{F}_{\sim V}$$

induced in momentum space by identifying $\ell_2((2\pi/V)Z)$ with the functions in $\mathscr{L}_2(R)$ that are piecewise constant on intervals of length $2\pi/V$ centered about the points in $(2\pi/V)Z \equiv Z_V$. Thus we may regard φ_V and $\mathfrak{A}(\mathscr{B}, V)$ as acting on $\mathscr{F}$.

For any volume V, with $\mathscr{B} \subset [-V/2, V/2]$, there is an isomorphism ϱ_V of $\mathfrak{A}_0(\mathscr{B})$ onto $\mathfrak{A}_0(\mathscr{B}, V)$ generated by the correspondence

$$\varphi(f) \to \varphi_V(f); \qquad \pi(f) \to \pi_V(f).$$

We have not determined whether ϱ_V is ultraweakly continuous, but a natural proof awaits a boson version of the Powers-Størmer result [6]. See [8–9].

Lemma 1.1. *Let* $A \in \mathfrak{A}_0(\mathscr{B})$. *Then* $\|\varrho_V(A)\| = \|A\|$ *and as* $V \to \infty$, *st.* $\lim \varrho_V(A) = A$.

Proof. Isomorphisms between C^*-algebras preserve norm; hence $\|\varrho_V(A)\| = \|A\|$. To prove the second statement, we may take A a generator of $\mathfrak{A}_0(\mathscr{B})$, as in (1.2). By the semigroup convergence theorem, it is sufficient to prove convergence of the fields, $\varphi_V(f) \to \varphi(f)$ and $\pi_V(f) \to \pi(f)$, strongly on a core for $\varphi(f)$ and $\pi(f)$. However this convergence is well known, and can be shown by standard calculations, as in [5].

2. The Locally Fock Property

Each state ω on the C^*-algebra $\mathfrak{A}$ gives rise to a $*$-representation ϱ_ω of $\mathfrak{A}$ by bounded operators on a Hilbert space $\mathscr{H}_\omega$ (the GNS construction). We say ω is *locally Fock* if for each bounded open interval $\mathscr{B}$, the restriction $\omega \upharpoonright \mathfrak{A}(\mathscr{B})$ is normal (ultraweakly continuous). In this case $\varrho_\omega \upharpoonright \mathfrak{A}(\mathscr{B})$ is unitarily implemented; there is a unitary transformation $U_\mathscr{B}$ of Fock space into $\mathscr{H}_\omega$ such that

$$\varrho_\omega(A) = U_\mathscr{B} A U_\mathscr{B}^*, \qquad A \in \mathfrak{A}(\mathscr{B}).$$

This follows from the fact that $\mathfrak{A}(\mathscr{B})$ is a type III factor, and alternately it follows from the more easily verified fact that each $\mathfrak{A}(\mathscr{B})$ contains a type I_∞ factor. We note that a normal state on $\mathfrak{A}_0(\mathscr{B})$ has a unique extension to a normal state on its weak closure, $\mathfrak{A}(\mathscr{B})$, and so a locally Fock state on $\mathfrak{A}_0$ yields a locally Fock state on $\mathfrak{A}$.

The Hamiltonian H_V has an eigenvector Ω_V satisfying $H_V\Omega_V = 0$, $\|\Omega_V\| = 1$. Since Ω_V is unique up to a phase, Ω_V is an eigenvector for the momentum operator P_V for the volume V. Reasoning by continuity in the coupling constant and using the fact that P_V has pure discrete spectrum, we see that the eigenvalue is zero, so $P_V\Omega_V = 0$. We define the state ω_V on the algebra $\mathfrak{B}(\mathscr{F})$ of all bounded operators on $\mathscr{F}$ by

$$\omega_V(A) = \langle \Omega_V, A\Omega_V \rangle \, . \tag{2.1}$$

This state is invariant under the periodic space translation group with generator P_V. Likewise we define the state $\tilde{\omega}_V = \omega_V \circ \varrho_V$ on $\mathfrak{A}_0(\mathscr{B})$. As $V \to \infty$, the states $\tilde{\omega}_V$ are defined eventually on a dense subalgebra of $\mathfrak{A}_0$. Thus by standard compactness arguments, there is a w^*-convergent subnet $\{\tilde{\omega}_{V_\nu}\}$, tending to a state ω defined on all of $\mathfrak{A}_0$.

Theorem 2.1. *Any w^*-limit point ω of the sequence $\{\tilde{\omega}_V\}$ is locally Fock.*

Our proof is based on the methods of [3], and we begin by outlining those ideas. In [3] we established the locally Fock property for a vacuum state defined through the limit $g \to 1$. The Hamiltonian $H(g)$ of (1.3) has a ground state eigenvector Ω_g, $H(g)\Omega_g = 0$, $\|\Omega_g\| = 1$. We let $g_n(x) = g(x/n)$, hold g fixed and let $n \to \infty$. The approximate vacuum state ω_n is defined on $\mathfrak{A}$ as the average over spatial translations (by a distance up to $0(n)$) of the states $A \to \langle \Omega_{g_n}, A\Omega_{g_n} \rangle$. The spatial average ensures translation invariance of a limit point ω of the sequence $\{\omega_n \restriction \mathfrak{A}\}$. A norm limit of normal states is normal, and $\omega_n \restriction \mathfrak{A}(\mathscr{B})$ is normal, so the essential step in proving the locally Fock property is to show that the sequence $\{\omega_n \restriction \mathfrak{A}(\mathscr{B})\}$ lies in a norm compact set of states.

Let $\mathscr{C}$ be a bounded interval of length $|\mathscr{C}| \geqq 1$. For $0 \leqq \tau < \frac{1}{4}$, we define the local number operator

$$N_{\tau,\mathscr{C}} = \int a^*(p)\, w_{\tau,\mathscr{C}}(p, q)\, a(q)\, dp\, dq \tag{2.2}$$

where

$$w_{\tau,\mathscr{C}}(p, q) = (2\pi)^{-1} \int E_{\mathscr{C}}(p - l)\, \mu(l)^\tau E_{\mathscr{C}}(l - q)\, dl \tag{2.3}$$

and $E_{\mathscr{C}}$ is the Fourier transform of the characteristic function of $\mathscr{C}$. Note that $N_{0,\mathscr{C}}$ measures the number of particles in $\mathscr{C}$ and that $N_{0,\mathscr{C}} \leqq \text{const.}\, N_{\tau,\mathscr{C}}$.

The locally Fock and norm compact properties are a consequence of the estimate

$$\omega_n(N_{\tau,\mathscr{C}}) \leqq \text{const.}\, |\mathscr{C}| \, . \tag{2.4}$$

We use (2.4) to define a truncated vacuum state ω_n^ε on $\mathfrak{B}(\mathscr{F})$,

$$\omega_n^\varepsilon(A) = \omega_n(RAR),$$

where R is the projection onto vectors in $\mathscr{F}$ with at most $M(1+j^2)$ particles localized in the space interval $\mathscr{X}_j = [j, j+1]$, $j = 0, \pm 1, \ldots$. Hence R projects onto vectors with at most $0(d^3)$ particles at a distance d from the origin. For $M = M(\varepsilon)$ sufficiently large, the estimate

$$\|\omega_n^\varepsilon - \omega_n\| \leqq \varepsilon \tag{2.5}$$

follows from (2.4), see [3].

There is no uniform bound on the number of particles in the wave functions defining ω_n^ε, and in order to achieve this result, we introduce a new approximation, $\omega_n^{\varepsilon\delta}$, all of whose wave functions are supported in a sufficiently large interval $\mathscr{C} = \mathscr{C}(\mathscr{B}, \delta, \varepsilon)$ containing $\mathscr{B}$. We take $\mathscr{C}$ to be a union of the $\mathscr{X}_j$'s. In the approximation $\omega_n^{\varepsilon\delta}$ of ω_n^ε, the error $\omega_n^\varepsilon - \omega_n^{\varepsilon\delta}$ is small only on the local subalgebra $\mathfrak{A}(\mathscr{B})$:

$$\|(\omega_n^{\varepsilon\delta} - \omega_n^\varepsilon) \restriction \mathfrak{A}(\mathscr{B})\| < \delta,$$

$$\|(\omega_n^{\varepsilon\delta} - \omega_n) \restriction \mathfrak{A}(\mathscr{B})\| < \varepsilon + \delta. \tag{2.6}$$

To define $\omega_n^{\varepsilon\delta}$, we note that ω_n^ε is a normal state, and hence a sum of vector states. Let $\langle \theta, \cdot\, \theta \rangle$ be a vector state contributing to ω_n^ε and let θ_{ij} be the component of θ with i particles in $\mathscr{C}$, j particles in $\sim\mathscr{C}$. In configuration space variables, $\theta_{ij} = \theta_{ij}(x_1 \ldots, x_i, y_1, \ldots, y_j)$, $x_\alpha \in \mathscr{C}$, $y_\beta \in \sim\mathscr{C}$. We now regard θ_{ij} as an i particle state in $\mathscr{F}$, depending on parameters $y_1, \ldots, y_j$. We thereby eliminate particles outside of $\mathscr{C}$ from θ and from ω_n^ε. The contribution of $\langle \theta, A\theta \rangle$ to $\omega_n^{\varepsilon\delta}(A)$ is

$$\sum_{j, i_1 i_2} \int \langle \theta_{i_1 j}, A\theta_{i_2 j} \rangle \, dy_1 \ldots dy_j. \tag{2.7}$$

where A acts on $\theta_{i_1 j}$ and $\theta_{i_2 j}$ as i_1 and i_2 particle states. We achieve an equivalent definition of $\omega_n^{\varepsilon\delta}$ by eliminating from A its action on particles outside $\mathscr{C}$. We have $\mathscr{F}^\mathscr{C} \subset \mathscr{F}$, $\mathscr{F}^{\sim\mathscr{C}} \subset \mathscr{F}$ and

$$\mathscr{F} \cong \mathscr{F}^\mathscr{C} \otimes \mathscr{F}^{\sim\mathscr{C}}, \tag{2.8}$$

where $\mathscr{F}^\mathscr{C}$ is the Fock space over $\mathscr{L}_2(\mathscr{C})$. We use a superscript $\mathscr{C}$ to denote the natural imbedding

$$\mathscr{L}_2(\mathscr{C}) \subset \mathscr{L}_2(R^1), \qquad \mathscr{F}^\mathscr{C} \subset \mathscr{F},$$

obtained by extending $f \in \mathscr{L}_2(\mathscr{C})$ to a function identically zero on $\sim\mathscr{C}$. (This is distinct from the periodic imbedding $\mathscr{F}_V \subset \mathscr{F}$.) Let $Q^\mathscr{C}$ be the projection of $\mathscr{F}$ onto $\mathscr{F}^\mathscr{C}$. Then $Q^\mathscr{C} A Q^\mathscr{C}$ acts on $\mathscr{F}^\mathscr{C}$, and $Q^\mathscr{C} A Q^\mathscr{C} \otimes I$ acts on $\mathscr{F}^\mathscr{C} \otimes \mathscr{F}^{\sim\mathscr{C}}$. If A^δ is the corresponding operator on $\mathscr{F}$, under the

isomorphism (2.8), then

$$\omega_n^{\varepsilon\delta}(A) = \omega_n^{\varepsilon}(A^{\delta}) = \omega_n^{\varepsilon}(Q^{\mathscr{C}} A Q^{\mathscr{C}} \otimes I).$$

In other words, $\omega_n^{\varepsilon\delta}$ is the state which results from first projecting ω_n^{ε} onto a state on $\mathfrak{B}(\mathscr{F}^{\mathscr{C}})$ by an integration over variables localized outside $\mathscr{C}$ and then tensoring the result with the no particle (bare vacuum) state on $\mathfrak{B}(\mathscr{F}^{\sim\mathscr{C}})$.

The algebra $\mathfrak{A}(\mathscr{B})$, however, is not contained in $\mathfrak{B}(\mathscr{F}^{\mathscr{C}})$ and the proof of (2.6) involves an estimate on the difference between the two notions of localization. The algebra $\mathfrak{B}(\mathscr{F}^{\mathscr{C}})$ is localized with respect to the support of the configuration space wave functions in $\mathscr{F}$, while the algebra $\mathfrak{A}(\mathscr{B})$ is defined by localization of the time zero fields $\pi(x)$ and $\varphi(x)$. Due to the energy factors $\mu(p)^{\pm\frac{1}{2}}$ which occur in the definition of $\pi(x)$ and $\varphi(x)$, the configuration space localization of the fields π or φ involves convolution of the wave functions by one of the kernels

$$k^{\pm}(x) = (2\pi)^{-1} \int \mu(p)^{\pm\frac{1}{2}} e^{-ipx} dp. \tag{2.9}$$

The kernels $k^{\pm}(x)$ are $\mathbb{C}^{\infty}$ except at $x = 0$, and outside of a neighborhood of $x = 0$, they satisfy

$$|k^{\pm}(x)| \leqq 0(1) e^{-m|x|}. \tag{2.10}$$

Hence the difference between the localization of $\varphi(x)$ and $\pi(x)$ and the localization of $\mathfrak{B}(\mathscr{F}^{\mathscr{C}})$ involves an exponentially small tail. A lengthy analysis of this effect yields (2.6), see [3].

Finally we prove that the sequence $\omega_n^{\varepsilon\delta}$ lies in a norm compact set of states. We use (2.4) with $\mathscr{C}$ replaced by $\mathscr{X}_j$ and the facts that $N_{\tau,\mathscr{X}_j}$ commutes with R and that bounded functions of $N_{\tau,\mathscr{X}_j}$ belong to $\mathfrak{B}(\mathscr{F}^{\mathscr{C}})$ to show that

$$\sum_{\mathscr{X}_j \subset \mathscr{C}} \omega_n^{\varepsilon\delta}(N_{\tau,\mathscr{X}_j}) \leqq \text{const.} |\mathscr{C}|.$$

This estimate and the fact that

$$\sum_{\mathscr{X}_j \subset \mathscr{C}} N_{\tau,\mathscr{X}_j}$$

has a compact resolvent on $\mathscr{F}^{\mathscr{C}}$ is sufficient to establish norm compactness of the states $\omega_n^{\varepsilon\delta}$, as states on $\mathfrak{B}(\mathscr{F})$. The norm compactness of the sequence $\{\omega_n \restriction \mathfrak{A}(\mathscr{B})\}$ then follows.

We now use this construction for the periodic field φ_V and the approximate vacuum state ω_V of (2.1). There are three steps in the argument: the approximations $\omega_V \to \omega_V^{\varepsilon}$ and $\omega_V^{\varepsilon} \to \omega_V^{\varepsilon\delta}$ and the existence of a norm compact set of states on $\mathfrak{B}(\mathscr{F})$ containing the family of states $\{\omega_V^{\varepsilon\delta} : V \geqq \text{const.}\}$. The vacuum Ω_V belongs to $\mathscr{F}_V$ [2] and the first two steps, with V held fixed, take place entirely within $\mathscr{F}_V$. In the third step,

$V \to \infty$, and we must work on the full Fock space $\mathscr{F}$. For an interval $\mathscr{C} \subset [-V/2, V/2]$ we have the Fock space $\mathscr{F}_V^{\mathscr{C}}$ of elements of $\mathscr{F}_V$ which are localized in $\mathscr{C}$. This notion of localization is defined by restricting the momentum space wave functions to the lattice Z_V and then inverting the Fourier series to obtain periodic configuration space wave functions with support in $\mathscr{C}$. In terms of the identification $\mathscr{F}_V \subset \mathscr{F}$, this notion of localization is V dependent. Elements of $\mathscr{F}_V^{\mathscr{C}}$ and $\mathscr{F}_V$ yields functions in $\mathscr{F}$ with piecewise constant Fourier transforms. Such functions never have compact support in the sense of $\mathscr{F}$ localization and cannot belong to $\mathscr{F}^{\mathscr{C}}$. We replace (2.8) with the representation $\mathscr{F}_V \cong \mathscr{F}_V^{\mathscr{C}} \otimes \mathscr{F}_V^{\sim \mathscr{C}}$, when working with a fixed V.

For $0 \leqq \tau < \frac{1}{4}$ and $\mathscr{C} \subset [-V/2, V/2]$, we define

$$N_{\tau, V, \mathscr{C}} = (2\pi/V) \sum_{p, q \in Z_V} a_V(p)^* \, w_{\tau, V, \mathscr{C}}(p, q) \, a_V(q)$$
$$= \int a(p)^* \, w_{\tau, V, \mathscr{C}}(p, q) \, a(q) \, dp \, dq \qquad (2.11)$$

where

$$w_{\tau, V, \mathscr{C}}(p, q) = (2\pi)^{-1} \int dl \, E_{\mathscr{C}}(p_V - l_V) \, \mu(l_V)^{\tau} \, E_{\mathscr{C}}(l_V - q_V), \qquad (2.12)$$

where p_V is the lattice point nearest p and where

$$a_V(q) = (V/2\pi)^{\frac{1}{2}} \int\limits_{-\pi/V}^{\pi/V} a(q + r) \, dr \, .$$

The estimates of [3, Section 3] show that $N_{\tau, V, \mathscr{C}}$ is an operator, and as in [3], we have the estimate

$$\omega_V(N_{\tau, V, \mathscr{C}}) \leqq \text{const.} |\mathscr{C}| \, , \qquad (2.13)$$

which replaces (2.4).

For convenience, we suppose that $V/2$ is an integer, so that $[-V/2, V/2]$ is a union of $\mathscr{X}_j$'s. Let R_V be the projection onto vectors in $\mathscr{F}_V$ which have at most $M(1 + j^2)$ particles localized in the interval $\mathscr{X}_j$, $\mathscr{X}_j \subset [-V/2, V/2]$. For $M = M(\varepsilon)$ sufficiently large (but independent of V),

$$\omega_V(I - R_V) \leqq \frac{1}{3}\varepsilon^2 \, ,$$

as a consequence of (2.13). We define

$$\omega_V^\varepsilon(A) = \omega_V(R_V A R_V) \, ,$$

and then

$$\|\omega_V^\varepsilon - \omega_V\| \leqq \varepsilon \, . \qquad (2.14)$$

As before, we choose $\mathscr{C} = \mathscr{C}(\mathscr{B}, \delta, \varepsilon)$ sufficiently large, and we require that V is large enough so that $\mathscr{C} \subset [-V/2, V/2]$. We repeat our construction above, to obtain the state $\omega_V^{\varepsilon\delta}$, by regarding variables $y \in \sim \mathscr{C}$ as parameters and integrating them, as in (2.7). Let $Q_V^{\mathscr{C}}$ be the projection

of $\mathscr{F}_V$ onto $\mathscr{F}_V^{\mathscr{C}}$. Then

$$\omega_V^{\varepsilon\delta}(A) = \omega_V^{\varepsilon}(Q_V^{\mathscr{C}} A Q_V^{\mathscr{C}} \otimes I) = \omega_V^{\varepsilon}(A^{\delta}).$$

Our basic estimates replacing (2.6) are

$$\|(\omega_V^{\varepsilon} - \omega_V^{\varepsilon\delta}) \restriction \mathfrak{A}(\mathscr{B}, V)\| < \delta, \tag{2.15}$$

$$\|(\omega_V - \omega_V^{\varepsilon\delta}) \restriction \mathfrak{A}(\mathscr{B}, V)\| < \varepsilon + \delta. \tag{2.16}$$

In order to establish (2.15), we must compare the notion of localization in $\mathfrak{A}(\mathscr{B}, V)$ with the notion of localization in $\mathfrak{B}(Q_V^{\mathscr{C}}\mathscr{F}_V)$. This is analogous to the case $V = \infty$, $Q_\infty^{\mathscr{C}} = Q^{\mathscr{C}}$, that led to (2.6). The energy factors $\mu(p_V)^{\pm\frac{1}{2}}$ occur in π_V and φ_V. For example with

$$k_V^{\pm}(x) = V^{-1} \sum_{k \in Z_V} \mu(p_V)^{\pm\frac{1}{2}} e^{-ip_V x}$$

$$= \sum_{n=-\infty}^{\infty} k^{\pm}(x + nV), \tag{2.17}$$

we have

$$\varphi_V(x) = (4\pi)^{-\frac{1}{2}} \int \{a(p)^* + a(-p)\} e^{-ip_V x} \mu(p_V)^{-\frac{1}{2}} dp$$

$$= \int_{-V/2}^{V/2} k_V^-(x - y)\, \hat{\varphi}_V(y)\, dy$$

where the (Newton-Wigner) field $\hat{\varphi}_V(y)$ creates and annihilates a wave function, localized at y in the sense of $\mathscr{F}_V$ localization. By (2.10) and (2.17),

$$|k_V^{\pm}(x)| \leqq \text{const.}\, e^{-m|x|}, \qquad \gamma < |x| < V/2,$$

for any $\gamma > 0$. Furthermore $k_V^{\pm}(x)$ is $\mathbb{C}^{\infty}$ away from the origin. Using these facts, we follow the proof of (2.6) in order to establish (2.15).

Next we turn to the third step in the proof of Theorem 2.1, namely the limit of the states $\omega_V^{\varepsilon\delta}$ as $V \to \infty$.

Lemma 2.2. *Let ε and δ be fixed. The sequence $\{\omega_V^{\varepsilon\delta} : 1 \leqq V\}$ of states of $\mathfrak{B}(\mathscr{F})$ lies in a norm compact set. Any limit point $\omega^{\varepsilon\delta}$ is normal.*

Proof. Since we consider the norm topology, it is sufficient to work with sequences. Given any subsequence of the $\omega_V^{\varepsilon\delta}$, we must draw from it a subsubsequence that is norm convergent as $V \to \infty$. Since V is not fixed, we must work on the full Fock space $\mathscr{F}$.

Let $\tau < \frac{1}{4}$. We show that the resolvent of the operator $N_{\tau, V, \mathscr{C}}$ converges strongly as $V \to \infty$. It is sufficient to prove that on the one particle space the resolvent of $w_{\tau, V, \mathscr{C}}$ converges strongly to the resolvent of $w_{\tau, \mathscr{C}}$. These operators are defined by the kernels (2.12) and (2.3) respectively. From [3, Eq. (3.1.11)] we have the inequality

$$\|w_{\tau, \mathscr{C}}\theta\| \leqq \text{const.}\, \|\mu^{2\tau + \varepsilon}\theta\|.$$

It follows that $w_{\tau,\mathscr{C}}$ (and likewise $w_{\tau,V,\mathscr{C}}$) is defined on the domain $\mathbb{C}_0^\infty$, and we take $w_{\tau,\mathscr{C}}$ to be the Friedrichs extension of $w_{\tau,\mathscr{C}} \upharpoonright \mathbb{C}_0^\infty$.

To compare our $\mathscr{F}_V$ periodic configuration space localization in $w_{\tau,V,\mathscr{C}}$ with the nonperiodic $\mathscr{F}$ configuration space localization, we introduce the isometry

$$U_V = T_V^{-1} X_V T : L_2(R) \to L_2(R) \tag{2.18}$$

where T is the $\mathscr{L}_2(R)$ Fourier integral transform, X_V is the orthogonal projection of $\mathscr{L}_2(R)$ onto functions piecewise constant on lattice intervals in momentum space, and T_V^{-1} is the inverse Fourier series mapping onto $\mathscr{L}_2(-\frac{1}{2}V, \frac{1}{2}V)$, imbedded in $\mathscr{L}_2(R)$ as functions with support in $[-\frac{1}{2}V, \frac{1}{2}V]$.

Let S be the closed operator $\mu^{\tau/2} E_\mathscr{C}$. Since the Friedrichs extension of $w_{\tau,\mathscr{C}} \upharpoonright \mathbb{C}_0^\infty$ is the only positive self adjoint extension of $w_{\tau,\mathscr{C}} \upharpoonright \mathbb{C}_0^\infty$ with domain in $\mathscr{D}(S)$,

$$w_{\tau,\mathscr{C}} = S^* S. \tag{2.19}$$

Similarly, let $\varDelta_V$ be the operator d^2/dx^2 with periodic boundary conditions on $\mathscr{L}_2(-\frac{1}{2}V, \frac{1}{2}V)$, and let

$$\mu_V^{\tau/2} = (-\varDelta_V + m^2)^{\tau/4}, \quad S_V = \mu_V^{\tau/2} E_\mathscr{C},$$

$$\hat{S}_V = U_V^* S_V U_V.$$

Then

$$w_{\tau,V,\mathscr{C}} = \hat{S}_V^* \hat{S}_V = U_V^* (S_V^* S_V) U_V.$$

We note that on $\mathbb{C}_0^\infty \times \mathbb{C}_0^\infty$, the difference $S_V^* S_V - S^* S$ is bounded and

$$\|S_V^* S_V - S^* S\| \leqq 0(V^{-1}).$$

Hence by the resolvent equation, the resolvent $R_V(\zeta) = (S_V^* S_V - \zeta)^{-1}$ of $S_V^* S_V$ converges in norm to the resolvent R of $S^* S = w_{\tau,\mathscr{C}}$. But $U_V \to I$ and $U_V^* \to I$ strongly, so

$$\hat{R}_V(\zeta) = U_V^* R_V(\zeta) U_V = (w_{\tau,V,\mathscr{C}} - \zeta)^{-1}$$

converges strongly as $V \to \infty$ to $R(\zeta) = (w_{\tau,\mathscr{C}} - \zeta)^{-1}$. Moreover, this proof shows that as $V \to \infty$, the resolvent of

$$A_V = \sum_{\mathscr{X}_j \subset \mathscr{C}} N_{\tau,V,\mathscr{X}_j} \tag{2.20}$$

converges strongly to the resolvent of

$$A = \sum_{\mathscr{X}_j \subset \mathscr{C}} N_{\tau,\mathscr{X}_j}. \tag{2.21}$$

Since $A \upharpoonright \mathscr{F}^\mathscr{C}$ has a compact resolvent [3] and since

$$A \cong (A \upharpoonright \mathscr{F}^\mathscr{C}) \otimes I,$$

A has pure discrete spectrum with no accumulation points and with infinite multiplicity. The same conclusions apply to A_V. The usual contour integral formula shows that eigenspaces and eigenvalues converge, as $V \to \infty$.

We restrict the operators to the single particle space, to get more detailed information. On the single particle space the nonzero eigenvalues of A_V or A have finite multiplicity, and when the eigenvalues are numbered in increasing order, counting multiplicity, they converge as $V \to \infty$. Also the eigenvectors converge. We construct a basis for $\mathscr{F}^{\mathscr{C}}$ and $\mathscr{F}_V^{\mathscr{C}}$ using tensor products of one particle eigenfunctions of $A \upharpoonright \mathscr{F}^{\mathscr{C}}$ and $A_V \upharpoonright \mathscr{F}_V^{\mathscr{C}}$. Since the single particle eigenfunctions converge as $V \to \infty$, so do their tensor products. These tensor products are eigenfunctions for A or A_V. Thus we have constructed a basis $\{e_{1,V}, e_{2,V}, \ldots\}$ for $\mathscr{F}_V^{\mathscr{C}}$ with the properties

$$e_j = \lim_{V \to \infty} e_{jV} \text{ exists},$$

$$\mu_j = \lim_{V \to \infty} \mu_{j,V} \text{ exists},$$

$$A_V e_{jV} = \mu_{j,V} e_{jV} \to A e_j = \mu_j e_j,$$

$$0 \leq \mu_1 \leq \mu_2 \leq \cdots \to \infty.$$

From the last statement, it follows that as $j \to \infty$, $\mu_{j,V} \to \infty$, uniformly in V.

The state $\omega_V^{\varepsilon\delta}$ is a normal state on $\mathfrak{B}(\mathscr{F})$, so there is a trace class matrix Λ_V with trace norm $\|\Lambda_V\|_1 = \|\omega_V^{\varepsilon\delta}\| \leq 1$ and such that

$$\omega_V^{\varepsilon\delta}(A) = \mathrm{Tr}_{\mathscr{F}}(\Lambda_V A), \qquad A \in \mathfrak{B}(\mathscr{F}).$$

Now $\omega_V^{\varepsilon\delta}$ has the property

$$\omega_V^{\varepsilon\delta}(A) = \omega_V^{\varepsilon\delta}(Q_V^{\mathscr{C}} A Q_V^{\mathscr{C}}),$$

which implies that $\Lambda_V = Q_V^{\mathscr{C}} \Lambda_V Q_V^{\mathscr{C}}$. Thus $\Lambda_V^{\frac{1}{2}}$ can be expressed by a matrix (ϱ_{ij}^V) in terms of the basis $\{e_{j,V}\}$. Hence by (2.13) and (2.20),

$$\omega_V^{\varepsilon\delta}(A_V) = \sum_{i<j} (\mu_{iV} + \mu_{jV}) |\varrho_{ij}^V|^2$$
$$+ \sum_i \mu_{iV} |\varrho_{ii}^V|^2 \qquad (2.21)$$
$$\leq \text{const}.$$

Passing to a subsequence, the ϱ_{ij}^V's converge as $V \to \infty$. Let

$$\Lambda_V^{\frac{1}{2}} = \varrho^{V,\kappa} + \delta\varrho^V$$

where

$$\varrho^{V,\kappa} = \sum_{i,j \leq \kappa} \varrho_{ij}^V e_{i,V} \otimes e_{j,V}$$

and similarly $\delta\varrho^V$ is the tail of the series. Then using the Hilbert Schmidt norm we have

$$\|\Lambda_V^{\frac{1}{2}} - \Lambda_{V'}^{\frac{1}{2}}\|_2 \leqq \|\varrho^{V,\kappa} - \varrho^{V',\kappa}\|_2 + \|\delta\varrho^{V,\kappa}\|_2 + \|\delta\varrho^{V',\kappa}\|_2 . \qquad (2.22)$$

Because of the bound (2.21),

$$\|\delta\varrho^{V,\kappa}\|_2 + \|\delta\varrho^{V',\kappa}\|_2 \leqq \gamma/2$$

for all V, V', if κ is chosen large enough so that $\mu_{\kappa,V} > \text{const.} \gamma^{-1}$, for all V. We now hold κ fixed. The right side of (2.22) is bounded by

$$\sum_{i,j \leqq \kappa} (|\varrho_{ij}^V - \varrho_{ij}^{V'}| + \text{const.} \|e_{i,V} - e_{i,V'}\|$$

$$+ \text{const.} \|e_{j,V} - e_{j,V'}\|) + \gamma/2 \leqq \gamma$$

for V, V' sufficiently large, in our subsequence. As in [3, Lemma 4.7],

$$|(\omega_V^{\varepsilon\delta} - \omega_{V'}^{\varepsilon\delta})(A)| = |\text{Tr}_{\mathscr{F}}((\Lambda_V - \Lambda_{V'})A)|$$

$$\leqq \|A\| \|\Lambda_V - \Lambda_{V'}\|_1$$

$$\leqq \text{const.} \|A\| \|\Lambda_V^{\frac{1}{2}} - \Lambda_{V'}^{\frac{1}{2}}\|_2$$

$$\leqq o(1) \|A\| ,$$

for V, V' in the subsequence. This completes the proof of the lemma.

Let ω be a w^*-limit point of the sequence $\{\tilde{\omega}_V : 1 \leqq V\}$. Then there is a subnet $\tilde{\omega}_{V(\alpha)} \to \omega$. By the norm compactness, Lemma 2.2, there is a subsubnet such that the $\omega_V^{\varepsilon\delta}$ have a norm limit $\omega^{\varepsilon\delta}$, for all $\varepsilon, \delta = 1/n$:

$$\omega^{\varepsilon\delta} = \text{norm} \lim_{\beta} \omega_{V(\alpha(\beta))}^{\varepsilon\delta} . \qquad (2.23)$$

Since the states $\omega^{\varepsilon\delta}$ are normal, Theorem 2.1 follows from our next result.

Lemma 2.3. $\|(\omega - \omega^{\varepsilon\delta}) \restriction \mathfrak{A}_0(\mathfrak{B})\| \leqq \varepsilon + \delta.$

Remark. Assuming the result of Lemma 2.3, we use Theorem 2.1 to extend ω to a locally Fock state of $\mathfrak{A}(\mathfrak{B})$. By Kaplansky's density theorem, $\omega - \omega^{\varepsilon\delta}$ has the same norm, when restricted to $\mathfrak{A}_0(\mathfrak{B})$ or to $\mathfrak{A}(\mathfrak{B})$. Hence Lemma 2.3 yields

$$\|(\omega - \omega^{\varepsilon\delta}) \restriction \mathfrak{A}(\mathfrak{B})\| \leqq \varepsilon + \delta . \qquad (2.24)$$

Proof. As states on $\mathfrak{A}_0(\mathfrak{B})$,

$$\omega^{\varepsilon\delta} - \omega = (\omega^{\varepsilon\delta} - \omega^{\varepsilon\delta} \circ \varrho_V) + (\omega^{\varepsilon\delta} - \omega_V^{\varepsilon\delta}) \circ \varrho_V$$
$$+ (\omega_V^{\varepsilon\delta} - \omega_V) \circ \varrho_V + \tilde{\omega}_V - \omega . \qquad (2.25)$$

Since $\omega^{\varepsilon\delta}$ is normal, Lemma 1.1 ensures that the first term is w^* convergent to zero as $V \to \infty$. By hypothesis, the last term is w^* convergent to zero. By (2.23), the second term is norm convergent to zero. Thus

$$(\omega^{\varepsilon\delta} - \omega) \restriction \mathfrak{A}_0(\mathscr{B}) = w^*\lim_{V \to \infty}(\omega_V^{\varepsilon\delta} - \omega_V) \circ \varrho_V \restriction \mathfrak{A}_0(\mathscr{B}). \qquad (2.26)$$

Since ϱ_V is an isomorphism, we have by (2.16),

$$\|(\omega_V^{\varepsilon\delta} - \omega_V) \circ \varrho_V \restriction \mathfrak{A}_0(\mathscr{B})\| = \|(\omega_V^{\varepsilon\delta} - \omega_V) \restriction \mathfrak{A}_0(\mathscr{B}, V)\|$$

$$\leqq \varepsilon + \delta.$$

Since w^*-limits cannot increase norms, the lemma follows from (2.26).

Remark. ω is also a w^*-limit of a *subsequence* of the $\tilde\omega_V$'s. It is sufficient to prove this for $\omega \restriction \mathfrak{A}_0(\mathscr{B})$ and $\tilde\omega_V \restriction \mathfrak{A}_0(\mathscr{B})$. We use (2.25) to write $(\tilde\omega_V - \omega) \restriction \mathfrak{A}_0(\mathscr{B})$ as a sum of four terms. The terms $\omega^{\varepsilon\delta} - \omega$ and $(\omega_V^{\varepsilon\delta} - \omega_V) \circ \varrho_V$ converge to zero as $\varepsilon = \delta = 1/n \to 0$, uniformly in V. We now hold $\varepsilon = \delta = 1/n$ fixed and let $V \to \infty$. The term $\omega^{\varepsilon\delta} - \omega^{\varepsilon\delta} \circ \varrho_V$ converges to zero in this limit. The final term is $(\omega^{\varepsilon\delta} - \omega_V^{\varepsilon\delta}) \circ \varrho_V$. Here $V = V(\alpha)$ belongs to the convergent net. Lemma 2.2 gives us norm compactness, and we can choose a subsequence of the subnet such that $\omega^{\varepsilon\delta} - \omega_V^{\varepsilon\delta} \to 0$. By the diagonal process we can choose one subsequence for all $\varepsilon = \delta = 1/n$. Hence as $V \to \infty$ through this subsequence, $(\tilde\omega_V - \omega) \restriction \mathfrak{A}_0(\mathscr{B})$ is w^*-convergent to zero.

3. The Hamiltonian and Momentum Operators

We show that the space-time translation automorphism $\sigma_a = \sigma_{\tau,\alpha}$ defined previously [2] through the limit $g \to 1$ of the Hamiltonians $H(g)$ agrees with the automorphism defined here through the limit $V \to \infty$ of the Hamiltonians H_V.

Theorem 3.1. *Let* $\sigma_{a,V}$ *be the two parameter automorphism group generated by the operators* H_V *and* P_V. *Then for* $A \in \mathfrak{A}_0(\mathscr{B})$,

$$\sigma_a(A) = s\text{-}\lim_{V \to \infty}\sigma_{a,V}(\varrho_V(A)).$$

We also show that the vacuum state ω of Section 2 is translation invariant,

$$\omega \circ \sigma_a = \omega. \qquad (3.1)$$

If $U_V(a)$ is the unitary group generated by H_V and P_V, and $\tilde\omega_{V_j} \to \omega$, we prove that

$$\omega(C\sigma_a(A)) = \lim_{j \to \infty}\omega_{V_j}(\varrho_{V_j}(C) U_{V_j}(a) \varrho_{V_j}(A)). \qquad (3.2)$$

These results permit us to transfer the spectral condition, previously established for $V < \infty$ [4] to the limit $V \to \infty$. Let $A \to A_{\mathrm{ren}}$ denote the GNS representation defined by ω, and let A_{ren} act on $\mathscr{F}_{\mathrm{ren}} = \mathscr{H}_\omega$.

Theorem 3.2. *There exist commuting self adjoint operators H and P on $\mathscr{F}_{\mathrm{ren}}$ that generate space time translations. Thus for $a = (\tau, \alpha) \in R^2$ and $A \in \mathfrak{A}$,*

$$\sigma_a(A)_{\mathrm{ren}} = U(a) A_{\mathrm{ren}} U(a)^* ,$$

$$U(a) = \exp(i\tau H - i\alpha P) .$$

Furthermore $H\Omega = 0 = P\Omega$, and $0 \leq H$; $0 \leq H^2 - P^2$.

Remark. The inequalities state that the spectrum lies in the forward cone. The second inequality is one of the improvements in the $\mathscr{P}(\varphi)_2$ theory permitted by this paper. Another improvement is that the bounds of [4] on vacuum expectation values of products of φ_t and φ_x can now be transferred to the limit $V \to \infty$, see Theorem 3.3. We define the (non-negative) mass operator M by $M = (H^2 - P^2)^{\frac{1}{2}}$. An important open question is whether M has an isolated, positive eigenvalue.

Proof of Theorem 3.1. To interpolate between the g and V cutoffs, we introduce

$$H(g, V) = H_{0,V} + \int_{-V/2}^{V/2} :\mathscr{P}(\varphi_V(x)): g(x)dx - E_g . \qquad (3.3)$$

We have the strong limits

$$\exp(i\tau H(g)) = \mathrm{st.} \lim_{V \to \infty} \exp(i\tau H(g, V)) , \qquad (3.4)$$

$$\exp(i\alpha P) = \mathrm{st.} \lim_{V \to \infty} \exp(i\alpha P_V) \qquad (3.5)$$

where P generates space translation on $\mathscr{F}$. To prove (3.4), we appeal to the semigroup convergence theorem, and use strong convergence of the generators on a core. $\mathfrak{C}^\infty(H_0)$ is a core for $H(g)$, by Rosen's higher order estimates [7], and $H(g, V) \to H(g)$ strongly on $\mathfrak{C}^\infty(H_0)$ by elementary estimates [5]. For (3.5), we also use the core $\mathfrak{C}^\infty(H_0)$.

Let $A \in \mathfrak{A}_0(\mathscr{B})$, $\mathscr{B}_{|\tau| + |\alpha|} \subset [-V/2, V/2]$ and let g equal 1 on $\mathscr{B}_{|\tau| + |\alpha|}$. By Lemma 1.1 and (3.4–5),

$$\sigma_a(A) = \lim_{V \to \infty} \exp(i\tau H(g, V)) \exp(-i\alpha P_V) \varrho_V(A)$$

$$\times \exp(i\alpha P_V) \exp(-i\tau H(g, V)) .$$

Thus to complete the proof of Theorem 3.1, we only need to show that H_V and $H(g, V)$ define the same dynamics on $\mathfrak{A}(\mathscr{B}_{|\alpha|}, V)$ for times $|t| \leq |\tau|$. We use the Trotter formula. In the free periodic box dynamics (i.e. with the interaction polynomial $\mathscr{P}$ replaced by zero), the commutator function

is given by the formula

$$\Delta_V(x - x', t - t') = i[\varphi_{0V}(x, t), \varphi_{0V}(x', t')]$$

$$= \sum_{n=-\infty}^{\infty} \Delta(x - x' + nV, t - t').$$

and

$$\varphi_{0V}(x, t) = \int\limits_{|x-y|<V/2} \Delta_V(x - y, t)\,\pi_V(y)\,dy$$

$$+ \int\limits_{|x-y|<V/2} \frac{\partial \Delta_V}{\partial t}(x - y, t)\,\varphi_V(y)\,dy.$$

Since $\Delta(x, t) = \Delta_\infty(x, t)$ vanishes outside the light cone, namely for $|t| < |x|$, it follows that

$$\Delta_V(x, t) = 0 \quad \text{for} \quad |t| < |x| < V/2.$$

Thus the free time translation $\sigma_{t,V}^0$ with generator $H_{0,V}$ defines an automorphism with unit propagation speed. For $A \in \mathfrak{A}(\mathscr{B}_{|\alpha|}, V)$ and $|t| \leq |\tau|$,

$$\sigma_{t,V}^0(A) = \exp(itH_{0,V})\,A\exp(-itH_{0,V}) \in \mathfrak{A}(\mathscr{B}_{|t|+|\alpha|}, V).$$

Furthermore the interaction dynamics $\sigma_{t,V}^I$ with generator $H_I(g, V)$ leaves each $\mathfrak{A}(\mathscr{B}_s, V)$ invariant,

$$\exp(itH_I(g, V))\,\mathfrak{A}(\mathscr{B}_s, V)\,\exp(-itH_I(g, V)) = \mathfrak{A}(\mathscr{B}_s, V)$$

and on $\mathfrak{A}(\mathscr{B}_s, V)$ with $|s| < \tau$, our restriction on g makes $\sigma_{t,V}^I$ independent of g (see [5]). We use the essential self adjointness of $H(g, V)$ and the Trotter product formula to establish the g independence and unit propagation speed of $\sigma_{t,V}$, see [5]. Taking $g \equiv 1$ on $[-\frac{1}{2}V, \frac{1}{2}V]$ and zero elsewhere yields $H(g, V) = H_V + \text{const.}$, which completes the proof of Theorem 3.1.

Proof of (3.1) *and* (3.2). First we note that (3.1) follows from the case $C = I$ of (3.2). In fact since $U_V(a)\Omega_V = \Omega_V$,

$$\omega_{V_j}(U_{V_j}(a)\,\varrho_{V_j}(A)) = \omega_{V_j}(\varrho_{V_j}(A)) \to \omega(A)$$

for $A \in \mathfrak{A}_0(\mathscr{B})$, which, combined with (3.2), yields (3.1). We now establish (3.2). We have

$$\omega(C\sigma_a(A)) - \omega_V(\varrho_V(C)\,\sigma_{a,V}(\varrho_V(A)))$$

$$= (\omega - \omega^{\varepsilon\delta})(C\sigma_a(A))$$

$$+ \omega^{\varepsilon\delta}(C\sigma_a(A) - \varrho_V(C)\,\sigma_{a,V}(\varrho_V(A))) \quad (3.5)$$

$$+ (\omega^{\varepsilon\delta} - \omega_V^{\varepsilon\delta})(\varrho_V(C)\,\sigma_{a,V}(\varrho_V(A)))$$

$$+ (\omega_V^{\varepsilon\delta} - \omega_V)(\varrho_V(C)\,\sigma_{a,V}(\varrho_V(A))).$$

The first and last terms in (3.5) are dominated in magnitude by $(\varepsilon + \delta)\,\|C\|\,\|A\|$, by (2.16), (2.24), and (3.6). We now hold ε and δ fixed. The third term tends to zero as $V \to \infty$ through a subsequence, by Lemma 2.2 and the choice of $\omega^{\varepsilon\delta}$, since

$$\|\varrho_V(C)\,\sigma_{a,V}(\varrho_V(A))\| \leq \|C\|\,\|A\| . \tag{3.6}$$

The second term tends to zero, since $\omega^{\varepsilon\delta}$ is a normal state on $\mathfrak{B}(\mathscr{F})$ and since the operators converge ultrastrongly to zero. This follows from Lemma 1.1, Theorem 3.1 and the fact that the operators are bounded in norm, uniformly in V. This proves (3.2).

Proof of Theorem 3.2. This is proved exactly as in [3, 5], where we construct H, P and $\mathscr{F}_{\text{ren}}$ with $0 \leq H$. From (3.2), we have

$$\text{spectrum}\{H, P\} \subset \liminf \text{spectrum}\{H_{V_j}, P_{V_j}\} ,$$

and the right hand side was bounded in [4].

Theorem 3.3. *Let* $\varphi_v = \partial_x \varphi$ *or* $\partial_t \varphi$. *There is a Schwartz space norm* $|\cdot|_n$ *on* $\mathscr{S}(R^{2n})$ *such that*

$$|\int \langle \Omega, \varphi_{v_1,\text{ren}}(x_1, t_1) \dots \varphi_{v_n,\text{ren}}(x_n, t_n)\Omega \rangle f(x, t)dx\,dt| \leq |f|_n , \tag{3.7}$$

and the above vacuum expectation value is obtained as $V \to \infty$ *through a subsequence from the vacuum expectation values of the box cutoff theories.*

We consider first a finite volume approximation to the space time averaged fields $\varphi(f) = \int \varphi(x,t) f(x,t)dx\,dt$, where $f = \bar{f} \in \mathfrak{C}_0^\infty(R^2)$. This operator is self adjoint and essentially self adjoint on the domain $\mathfrak{C}^\infty(H_0)$. The above integral presentation is valid in the sense of bilinear forms on a dense domain, see [2]. Let $\varphi(f)_{\text{ren}}$ denote the corresponding self adjoint operator on $\mathscr{F}_{\text{ren}}$. Similarly, let $\varphi_V(f)$ be the self adjoint operator determined by the formula

$$\varphi_V(f) = \int e^{iH_Vt}\varphi_V(x)e^{-iH_Vt}f(x,t)dx\,dt . \tag{3.8}$$

Lemma 3.4. *With* $\varphi(f)$, $\varphi_V(f)$ *as above,*

$$\exp(i\varphi(f)) = \operatorname*{st.\,lim}_{V \to \infty} \exp(i\varphi_V(f)) .$$

Proof. It is sufficient to prove convergence of $\varphi_V(f)$ on a core for $\varphi(f)$. Furthermore, as in the proof of Theorem 3.1, we may replace H_V in (3.8) with $H(g, V)$, where $g \equiv 1$ on a sufficiently large interval. We choose the core $\mathfrak{C}^\infty(H(g))$ on which to estimate $\varphi_V(f) - \varphi(f)$.

In order to take advantage of the time integration, we note that for $H \equiv R^{-1} \equiv (H(g) + \text{const.})$,

$$\varphi(f)\theta = R\varphi(f)H\theta + iR\varphi(f_t)\theta .$$

Let $A = \int f(x, t)\,\varphi(x, 0)dx$, $B = i \int (\partial_t f)\,(x, t)\,\varphi(x, 0)dx$. Thus we calculate $(\delta\varphi)\theta = (\varphi_V(f) - \varphi(f))\theta$ for $\theta \in \mathbb{C}^\infty(H(g))$.

$$\delta(\varphi(f))\theta = \delta(R\varphi(f)H)\theta + \delta(iR\varphi(f_t))\theta$$

$$= \int \delta(e^{itH})\,R(AH + B)e^{-iHt}\theta\,dt$$

$$+ \int e^{itH(V)}(\delta R)\,(AH + B)e^{-itH}\theta\,dt$$

$$+ \int e^{itH(V)}R_V\{(\delta A)H + \delta B\}\,e^{-itH}\theta\,dt$$

$$+ \int e^{itH(V)}R_V A_V \delta(e^{-itH})H\theta\,dt$$

$$+ \int e^{itH(V)}R_V A_V e^{-itH(V)}(\delta H)\theta\,dt$$

$$+ \int e^{itH(V)}R_V B_V \delta(e^{-itH})\theta\,dt\,.$$

Here $H(V) = H(g, V) + \text{const.} = R_V^{-1}$,

$$\delta A = A_V - A,\qquad \delta B = B_V - B,$$

etc. The first term converges to zero as $V \to \infty$ since $\delta(e^{itH}) = e^{itH(V)} - e^{itH}$,

$$\text{st.}\lim \delta(e^{itH}) = 0,\tag{3.9}$$

uniformly on compact t intervals, and for $V_i \leqq \infty$,

$$\|R_{V_1}^{\frac{1}{2}} A_{V_2}\| = \|R_{V_1}^{\frac{1}{2}} B_{V_2}\| \leqq |f(\cdot, t)| + |\partial_t f(\cdot, t)|\tag{3.10}$$

where $|f(\cdot, t)|$ denotes a suitable Schwartz space norm independent of V_i. By higher order estimates, $\|\delta R\| \to 0$ as $V \to \infty$ [7], so the second term converges to zero. Elementary N_τ estimates [5] yield

$$\|(\delta A)R^{\frac{1}{2}}\| + \|(\delta B)R^{\frac{1}{2}}\| \leqq 0(V^{-1})\,|f(\cdot\,t)|\,.$$

Convergence of third term follows. The fourth and sixth terms converge on account of (3.9–10). We use the higher order estimate

$$\|N^r R^r\| \leqq M_r,\qquad r = 1, 2, \dots$$

to prove for $r \geqq \frac{1}{2}\deg P$,

$$\|\delta H\theta\| \leqq \|\delta H_{\mathrm{I}}\theta\| + \|\delta H_0\theta\| \leqq \|(\delta H_{\mathrm{I}})\,(N + I)^{-r}(N + I)^r\theta\|$$

$$+ \|(\delta H_0)\,(N + I)^{-1}\,(N + I)\theta\|$$

$$\leqq \text{const.}\|\delta w\|_2\,\|H^r\theta\| + 0(V^{-1})$$

$$\leqq 0(V^{-1})\,,$$

and to establish convergence of the fifth term. Here δw is the kernel $w_V - w$ of δH_{I}. This completes the proof of the lemma.

Let $h_r \in \mathfrak{C}_0^\infty(R^2)$, $1 \leq r \leq m$, let $C \in \mathfrak{A}_0(\mathscr{B})$ and let

$$A_V = \varrho_V(C) \prod_{r=1}^m \exp(i\varphi_V(h_r))$$

$$A = C \prod_{i=1}^m \exp(i\varphi(h_r)).$$

Lemma 3.5. *Let $\tilde{\omega}_{V_j} \to \omega$. Then*

$$\omega(A) = \lim \omega_{V_j}(A_{V_j}).$$

Proof.
$$\omega(A) - \omega_V(A_V) = (\omega - \omega^{\varepsilon\delta})(A) + \omega^{\varepsilon\delta}(A - A_V)$$
$$+ (\omega^{\varepsilon\delta} - \omega_V^{\varepsilon\delta})(A_V) + (\omega_V^{\varepsilon\delta} - \omega_V)(A_V).$$

Note that $\|A\| = \|A_V\| = \|C\|$. We use (2.16) and (2.24) to bound the first and last terms, uniformly in V. We choose ε and δ to be small, but fixed, and then the middle terms converge to zero by (2.23) and Lemma 3.4.

Proof of Theorem 3.3. We have an estimate similar to (3.7), uniform in V, on the vacuum expectation values in the box cutoff theory, $V < \infty$ [4]. Let $f_i \in \mathfrak{C}_0^\infty(R^2)$ and let

$$\theta = \varphi_{v_1,\text{ren}}(f_1) \dots \varphi_{v_n,\text{ren}}(f_n)\Omega,$$

$$\theta_V = \varphi_{V,v_1}(f_1) \dots \varphi_{V,v_n}(f_n)\Omega_V.$$

By induction on n, assume that θ above is defined, $\|\theta\| \leq |f_1 \otimes \dots \otimes f_n|_n$, and for any choice of C, m and h_r in Lemma 3.5, assume that

$$\langle \Omega, A_{\text{ren}}\theta \rangle = \lim_j \langle \Omega_{V_j}, A_{V_j}\theta_{V_j} \rangle.$$

For $n = 0$, take $\theta = \Omega$, $\theta_V = \Omega_V$ and then the induction hypothesis is valid by Lemma 3.5. To establish the hypothesis for $(n+1)$ consider

$$F_V(s) = \langle \Omega_V, A_V \exp(is\varphi_{v,V}(f))\theta_V \rangle$$

with $f \in \mathfrak{C}_0^\infty$. By the induction hypothesis, $F_V(s)$ has a pointwise limit

$$F(s) = \langle \Omega, A_{\text{ren}} \exp(is\varphi_{v,\text{ren}}(f))\theta \rangle.$$

By the cutoff estimates [4], we have bounds

$$\left| \left(\frac{d}{ds}\right)^\alpha F_V(s) \right| \leq \|C\| \, |f \otimes f \otimes \dots \otimes f \otimes f_1 \otimes \dots \otimes f_n|$$

uniform in V, with a Schwartz space norm depending on n and α. It follows that the family $\{D^\alpha F_V\}$ of derivatives is equicontinuous as $V \to \infty$, and hence the derivatives $D^\alpha F_V$ converge uniformly and $F(s) \in \mathfrak{C}^\infty$.

However, the convergence of $F_V'(s)$ gives the induction hypothesis for $(n+1)$. Here we use the fact that vectors $\{A_{\mathrm{ren}}\Omega\}$ are dense in $\mathscr{F}_{\mathrm{ren}}$ as C ranges over $\bigcup_{\mathscr{B}} \mathfrak{A}_0(\mathscr{B})$. Choosing $A = I$, $s = 0$, we have convergence as $V_j \to \infty$ and the bound (3.7) for product wave functions in $\mathfrak{C}_0^\infty$. The general bound (3.7) follows by the nuclear theorem.

4. Consequences of a Uniform Mass Gap

The Hamiltonian H_V is renormalized so that $0 \leq H_V$, $H_V \Omega_V = 0$. Zero is a simple eigenvalue for H_V and $H_V \upharpoonright \mathscr{F}_V$ has a compact resolvent [2]. Since $H_V \upharpoonright (\mathscr{F}_V)^\perp \geq m$, there is a gap between zero and the infimum E_1 of the spectrum of $H_V \upharpoonright \{\Omega_V\}^\perp$. In the limit $V \to \infty$, we expect E_1 to converge to the mass of the lightest particle in the theory, so E_1 is called the mass gap. If $E_1 = E_1(V)$ is bounded uniformly away from zero as $V \to \infty$, we say the theory has a *uniform mass gap*.

Theorem 4.1. *If the $\mathscr{P}(\varphi)_2$ theory has a uniform mass gap, then the vacuum Ω of Section 3 is unique and the vacuum expectation values of products of field operators are tempered distributions. Thus in this case all Wightman axioms are satisfied with the possible exception of unitary operators to implement Lorentz rotations.*

Proof. Let $A \in \mathfrak{A}_0(\mathscr{B})$ and let $c_V = \langle \Omega_V, \varrho_V(A)\Omega_V \rangle$. Then $c_{V_j} \to c = \langle \Omega, A_{\mathrm{ren}}\Omega \rangle$ and so $(A_{\mathrm{ren}} - cI)\Omega \perp \Omega$. Furthermore the set of vectors

$$(A_{\mathrm{ren}} - cI)\Omega\,, \tag{4.1}$$

as A and $\mathscr{B}$ vary, span $\{\Omega\}^\perp$. Thus we prove that Ω is the unique ground state for H if we prove that (4.1) lies in the range of the spectral projection for the interval $[E_1, \infty)$. By (3.2),

$$\langle (A_{\mathrm{ren}} - cI)\Omega, e^{itH}(A_{\mathrm{ren}} - cI)\Omega \rangle$$

$$= \lim_j \langle (\varrho_{V_j}(A) - c_{V_j}I)\Omega_{V_j}, e^{itH_{V_j}} (\varrho_{V_j}(A) - c_{V_j}I)\Omega_{V_j} \rangle\,.$$

Considered as functions of t, we have pointwise convergence and a uniform bound in $\mathscr{L}_\infty$. Thus the functions converge as distributions in t and consequently so do their Fourier transforms. On the right side, the Fourier transforms have support in $[E_1, \infty)$, since

$$(\varrho_{V_j}(A) - c_{V_j}I)\Omega_{V_j} \perp \Omega_{V_j}\,.$$

Thus the left side also has a Fourier transform with support in $[E_1, \infty)$, and we have proved that Ω is unique. Moreover

$$\mathrm{spectrum}\, H \upharpoonright \{\Omega\}^\perp \subset [E_1, \infty)\,. \tag{4.2}$$

Next we bound the vacuum expectation values. Let $\xi \in \mathscr{S}(R^1)$ have a Fourier transform $\tilde{\xi}(E) = \int e^{-itE} \xi(t) dt$ with $\mathrm{suppt} \tilde{\xi} \subset (-\infty, E_1/2)$ and $\tilde{\xi} \equiv 1$ on $(-E_1/3, E_1/3)$. For any $f \in \mathscr{S}(R^2)$, we write

$$f = f_1 + f_2 = \xi * f + (f - \xi * f). \tag{4.3}$$

Since $\tilde{f}_2(p, E)$ vanishes for $|E| < E_1/3$,

$$f_2 = (d/dt)h. \tag{4.4}$$

We assert that

$$\varphi(f)\Omega = \mathrm{const}.\Omega - \pi(h)\Omega, \tag{4.5}$$

where $\varphi(f) = \int \varphi(x, t) f(x, t) dx \, dt$.

Since $\varphi(f_2) = -\varphi_t(h) = -\pi(h)$, we need only show that $\varphi(f_1)\Omega = \mathrm{const}.\Omega$. However for a bounded function η,

$$\begin{aligned}
\tilde{\eta}(H) \varphi(f_1)\Omega &= \int \eta(t) e^{-itH} \varphi(f_1)\Omega \, dt \\
&= \varphi(\eta * f_1)\Omega \\
&= \varphi(\eta * \xi * f)\Omega = \varphi(\xi * f)\Omega \\
&= \varphi(f_1)\Omega
\end{aligned}$$

if $\tilde{\eta} \equiv 1$ on $\mathrm{suppt}\tilde{\xi}$, $\mathrm{suppt}\tilde{\eta} \subset (-2E_1/3, 2E_1/3)$. For such an η, the range of $\eta(H)$ is one dimensional and spanned by Ω, by (4.2), proving (4.5). The constant in (4.5) can be evaluated as

$$\mathrm{const.} = \int f(x, t) dx \, dt \langle \Omega, \varphi(0, 0)\Omega \rangle. \tag{4.6}$$

The right side is finite, by estimates such as (2.13), and defines a continuous functional on $\mathscr{S}$. Furthermore the mapping $f \to h$ is continuous on $\mathscr{S}$, since f_2 vanishes in a neighborhood of the origin. Thus given any Schwartz norm $|\cdot|_{s_1}$ there is another norm $|\cdot|_{s_2}$ with

$$|h|_{s_1} \leq |f|_{s_2}. \tag{4.7}$$

We use (4.5) to convert the φ_x and $\varphi_t = \pi$ estimates of Theorem 3.3 into φ estimates. Let $\zeta \in \mathfrak{C}_0^\infty$ be a nonnegative function with support in the square $|x| \leq 1$, $|t| \leq 1$ and such that the translates $\zeta_j(x, t) = \zeta(\{x, t\} - j)$ by $j \in Z_2$ form a partition of unity:

$$\sum_{j \in Z_2} \zeta_j = 1.$$

Then $f \to \zeta_j f$ is continuous on $\mathscr{S}$ and for any given norm $|\cdot|_{s_1}$ on $\mathscr{S}$ and any positive integer N, there is a norm $|\cdot|_{s_2}$ on $\mathscr{S}$ such that

$$|\zeta_j f|_{s_1} \leq (|j| + 1)^{-N} |f|_{s_2}, \quad \text{all} \quad j \in Z_2 \tag{4.8}$$

In other words, the series $\sum_j \zeta_j f$ converges in $\mathscr{S}$.

2*

In place of the vacuum Ω in (4.5), we now consider a vector of the form

$$\varphi_{\nu_1}(h_1) \ldots \varphi_{\nu_n}(h_n)\Omega \qquad (4.9)$$

where each φ_{ν_1} is a φ_x or a φ_t. By Theorem 3.3 the vector (4.9) is a continuous function of $h_1 \ldots h_n \in \mathscr{S}(R^2) \times \cdots \times \mathscr{S}(R^2)$. Also the series

$$\sum_{j_i \in Z_2} \varphi_{\nu_1}(\zeta_{j_1} h_1) \ldots \varphi_{\nu_n}(\zeta_{j_n} h_n)\Omega \qquad (4.10)$$

converges in $\mathscr{S}$. We assert that each term belongs to the domain of $\varphi(\zeta_j f)$, for any $f \in \mathscr{S}(R^2)$. In fact with our local unitary equivalence ensured by the locally Fock property, we return to Fock space and recall that for f_1 and $f_2 \in \mathbb{C}_0^\infty(R^2)$, $\varphi(f_1)$, $\varphi(f_2)$ and $\varphi(f_1 + f_2)$ are essentially self adjoint on a common domain. It follows that

$$\mathscr{D}(\varphi(f_1)) \cap \mathscr{D}(\varphi(f_2)) \subset \mathscr{D}(\varphi(f_1 + f_2)). \qquad (4.11)$$

The inclusion (4.11) is then valid on $\mathscr{F}_{\mathrm{ren}}$ also, and we see that $\varphi(f_1 + f_2) = (\varphi(f_1) + \varphi(f_2))^-$. We return to (4.10), and we begin with the fact that $\Omega \in \mathscr{D}(\varphi(\zeta_j f))$. If the space component j^1 of j is large enough, $\varphi(\zeta_j f)$ commutes with the operators $\varphi_{\nu_i}(\zeta_{j_i}, h_i)$. Consequently, these operators map the domain $\mathscr{D}(\varphi(\zeta_j f))$ into itself, proving our assertion in this case.

If $j - j_i$ is time like, then $\varphi(\zeta_j f)$ need not commute, but $\varphi(\zeta_j f)$ is the sum of a space translate that commutes and a gradient. In fact for $h_\alpha(x) = h(x - \alpha)$,

$$\varphi(h) = \varphi(h_\alpha) + \int_0^\alpha (\nabla \varphi)(h_{\alpha'})d\alpha'.$$

A term in (4.10) belongs to the domain of the gradient $\varphi_x(h)$ by Theorem 3.3, and it belongs to the domain of the space translate of $\varphi(\zeta_j, f)$ by local commutativity, as above. By (4.11), the term belongs to $\mathscr{D}(\varphi(\zeta_j f))$ and the assertion is proved in all cases. Combining this argument with (4.5), we have

$$\begin{aligned}
\varphi(\zeta_j f)\, \varphi_{\nu_1}(\zeta_{j_1} h_1) \ldots \varphi_{\nu_n}(\zeta_{j_n} h_n)\Omega &= \mathrm{const.}\varphi_{\nu_1}(\zeta_{j_1} h_1) \ldots \varphi_{\nu_n}(\zeta_{j_n} h_n)\Omega \\
&\quad + \varphi_{\nu_1}(\zeta_{j_1} h_1) \ldots \varphi_{\nu_n}(\zeta_{j_n} h_n)\, \pi(g_1)\Omega \quad (4.12) \\
&\quad + \varphi_x(g_2)\, \varphi_{\nu_1}(\zeta_{j_1} h_1) \ldots \varphi_{\nu_n}(\zeta_{j_n} h_n)\Omega.
\end{aligned}$$

The constant is given by (4.6), independently of j, j_i and h_i. g_1 and g_2 belong to $\mathscr{S}$, and for any norm $|\cdot|_{s_1}$ on $\mathscr{S}$, there is a norm $|\cdot|_{s_2}$ on $\mathscr{S}$ and an integer M such that

$$|g_1|_{s_1} + |g_2|_{s_1} \leqq \left(\sup_i |j - j_i| + 1\right)^M |f|_{s_2} \qquad (4.13)$$

We can now sum the series (4.12) over j and $(j_1, \ldots j_n) \in Z_2 \times \cdots \times Z_2$. The series converges absolutely, and is dominated term by term by a

series of the form

$$\sum_{j,j_i} (|j|+1)^{-N} \left(\sup_i |j-j_i|+1\right)^M \left(\sup_i |j_i|+1\right)^{-N} |h_1|_{s_2} \dots |h_n|_{s_2} |f|_{s_2} \qquad (4.14)$$

In this series, we first choose the norm $|\cdot|_{s_1}$ on g_1 and g_2 required by [4]. This norm determines an M. We then choose $|\cdot|_{s_2}$ by (4.13) and also by (4.8), so that $N > M + 2n$. In this case (4.14) converges, so (4.9) defines a vector in the domain of $\varphi(\zeta_j f)$. Furthermore, we define $\varphi(f)$ to be the closure of $\sum_j \varphi(\zeta_j f)$, and then (4.9) belongs to the domain of $\varphi(f)$ also.

By the nuclear theorem, we can replace the test function $h_1 \otimes \cdots \otimes h_n$ $\in \mathscr{S}(R^2) \otimes \cdots \otimes \mathscr{S}(R^2)$ by a test function $h \in \mathscr{S}(R^{2n})$. We then have

$$\begin{aligned}
\varphi(f)\,(\varphi_{v_1} \dots \varphi_{v_n})\,(h)\Omega = {}& \text{const.}(\varphi_{v_1} \dots \varphi_{v_n})\,(h)\Omega \\
& + (\varphi_{v_1} \dots \varphi_{v_n}\pi)\,(h_1)\Omega \qquad (4.15) \\
& + (\varphi_x\varphi_{v_1} \dots \varphi_{v_n})\,(h_2)\Omega \,.
\end{aligned}$$

The constant is given by (4.6) and h_1 and h_2 depend continuously on f and h in the $\mathscr{S}$-topology. We use (4.15) to bound $\varphi(f_1)\dots\varphi(f_n)\Omega$ by induction on n. This completes the proof.

We conclude with some comments on the status of the missing Wightman axioms. Bounds on the vacuum expectation values of products of fields $\varphi(x,t)$ would follow, for instance, from an estimate

$$N_{\tau,V,\mathscr{C}} \leqq \text{const.}(H_V + I)\,,$$

uniform in V, without consideration of the mass gap. It is possible that a Poincaré-invariant vacuum could be constructed by averaging over the Lorentz automorphism σ_A of homogeneous Lorentz transformations of $\mathfrak{A}$. In the reverse direction, it is possible that a unique vacuum could be found by decomposing into components irreducible under $\mathfrak{A}$, see [1]. In either case, the problem would be to preserve the locally Fock property in the construction. Both constructions preserve the spectral condition. If the vacuum is unique, not just as a Hilbert space vector, but also as a C^* algebra state on $\mathfrak{A}$, then it is Poincaré invariant. If we are given a Poincaré invariant state ω, the automorphisms $\sigma_{\{a,A\}}$ are implemented by a continuous unitary representation $U(a,A)$ of the Poincaré group.

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