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Spatial Representation of Groups of Automorphisms of von Neumann Algebras with Properly Infinite Commutant

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Abstract

Theorem. Let a topological group G be represented $(a \to \phi_a)$ by $*$-automorphisms of a von Neumann algebra $\boldsymbol{R}$ acting on a separable Hilbert space $\boldsymbol{H}$. Suppose that

(a) G is locally compact and separable,

(b) $\boldsymbol{R}'$ is properly infinite,

(c) for any $T \in \boldsymbol{R}$, $x, y \in \boldsymbol{H}$ the function

$$a \to \langle \phi_a(T)x, y \rangle_{\boldsymbol{H}}$$

is measurable on G. Then there exists a strongly continuous unitary representation of G on $\boldsymbol{H}$, $a \to U_a$, such that for $T \in \boldsymbol{R}$, $a \in G$,

$$\phi_a(T) = U_a T U_a^* .$$

Let $\boldsymbol{R}$ be a von Neumann algebra acting on a separable Hilbert space $\boldsymbol{H}$. Let G be a topological group and the map $a \to \phi_a (a \in G)$ a representation of G by $*$-automorphisms of $\boldsymbol{R}$.

General Problem. When is there a strongly continuous unitary representation, $a \to U_a$, of G on $\boldsymbol{H}$ such that for $a \in G$, $T \in \boldsymbol{R}$, $\phi_a(T) = U_a T U_a^*$? The theorem below gives an affirmative answer for a large class of von Neumann algebras. This result may be of use in Quantum Mechanics. This theorem is a generalization of a theorem of Kallman [2]. The author wishes to express his gratitude to Kallman for suggesting this problem.

Theorem. *In the context of the general problem stated above, suppose that*

(a) *G is locally compact and separable,*

(b) *the commutant of $\boldsymbol{R}$ is a properly infinite von Neumann algebra,*

(c) (*weak measurability*) *for any $T \in \boldsymbol{R}$, $x, y \in \boldsymbol{H}$, the function*

$$a \to \langle \phi_a(T)x, y \rangle_{\boldsymbol{H}}$$

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is measurable on G. Then there exists a strongly continuous unitary representation of G on $\boldsymbol{H}$, $a \to U_a$, *such that for* $T \in \boldsymbol{R}$, $a \in G$,

$$\phi_a(T) = U_a T U_a^* .$$

Proof. Let $L^2(G, \boldsymbol{H})$ be the set of all square summable functions from G to $\boldsymbol{H}$. More precisely a function $f: G \to \boldsymbol{H}$ is in $L^2(G, \boldsymbol{H})$ if (a) f is *weakly measurable* meaning that the scalar functions $a \to \langle f(a), x \rangle_{\boldsymbol{H}}$, $x \in \boldsymbol{H}$ are measurable, and (b)

$$\|f\| = \left(\int_G \|f(a)\|^2 \, da \right)^{\frac{1}{2}} < \infty$$

where da denotes a right invariant Haar measure on G. With this norm and identification of functions differing only on a set of measure zero, $L^2(G, \boldsymbol{H})$ becomes a Hilbert space ([1], p. 142, Example 2). $L^2(G, \boldsymbol{H})$ is separable; a countable dense subset may be obtained as follows: take a countable dense subset $\{x_i\}$ of $\boldsymbol{H}$ and a countable dense subset $\{f_j\}$ of $L^2(G)$ (this exists because G is separable), and set

$P = \{\text{all products } x_i f_j\}$,

$P' = \{\text{all complex rational combinations of elements of } P\}$.

Clearly P' is dense in the closed linear subspace of $L^2(G, \boldsymbol{H})$ generated by P, and one may show easily that $P^{\perp} = \{o\}$. (This is essentially the construction of [1], p. 149, Proposition 7.)

For $T \in \boldsymbol{R}$ consider the operator-valued function on G, $a \to \phi_a(T)$. In order that this function define by "multiplication" an operator on $L^2(G, \boldsymbol{H})$ it must be *weakly measurable*, that is the scalar-valued functions

$$a \to \langle \phi_a(T) f(a), \, g(a) \rangle_{\boldsymbol{H}} \quad f, g \in L^2(G, \boldsymbol{H}) \tag{1}$$

must be measurable. It actually suffices to verify this for f and g in some dense subset of $L^2(G, \boldsymbol{H})$ ([1], p. 157, Proposition 1), for example for f and g in P'. By hypothesis (c) the function (1) is measurable for constant functions $f \equiv x$, $g \equiv y$, $x, y \in \boldsymbol{H}$. It is then measurable for the products in P, and then for the linear combinations in P'. So the formula

$$(\hat{T} f)(a) = \phi_a(T) f(a)$$

defines an operator $\hat{T}$ on $L^2(G, \boldsymbol{H})$. By ([1], p. 160, Proposition 2), $\|\hat{T}\| = \|T\|$; therefore the map $T \to \hat{T}$ is an isomorphism of $\boldsymbol{R}$ onto some algebra $\hat{\boldsymbol{R}}$ of operators on $L^2(G, \boldsymbol{H})$.

The map $T \to \hat{T}$ is *normal*, meaning that if T_n is an increasing sequence of positive operators of $\boldsymbol{R}$ with least upper bound T, $T_n \nearrow T$, then also $\hat{T}_n \nearrow \hat{T}$. Sequences are permitted here in place of nets because $\boldsymbol{H}$ and $L^2(G, \boldsymbol{H})$ are separable. From the normality of the map $T \to \hat{T}$ it will follow that $\hat{\boldsymbol{R}}$ is a von Neumann algebra on $L^2(G, \boldsymbol{H})$ ([1], p. 57, Corollary 2). Let $T_n \nearrow T$. Then $\phi_a(T_n) \nearrow \phi_a(T)$, since all $*$-automorphisms are normal; therefore for $a \in G$, $f \in L^2(G, \boldsymbol{H})$

$$\langle \phi_a(T_n) f(a), f(a) \rangle_{\boldsymbol{H}} \nearrow \langle \phi_a(T) f(a), f(a) \rangle_{\boldsymbol{H}} .$$

By the Lebesque Dominated Convergence Theorem,

$$\begin{aligned} \langle \hat{T}_n f, f \rangle_{L^2(G,\boldsymbol{H})} &= \int_G \langle \phi_a(T_n) f(a), f(a) \rangle_{\boldsymbol{H}}\, da \\ &\nearrow \int_G \langle \phi_a(T) f(a), f(a) \rangle_{\boldsymbol{H}}\, da = \langle \hat{T} f, f \rangle_{L^2(G,\boldsymbol{H})} \end{aligned}$$

proving that $\hat{T}_n \nearrow \hat{T}$. The argument of this paragraph, at the suggestion of Kallman, replaces a more complicated argument of the author.

By hypothesis $\boldsymbol{R}'$ is properly infinite. $\boldsymbol{R}'$ therefore contains an infinite collection $\{P_i\}$ of mutually orthogonal, mutually equivalent projections of sum I ([1], p. 319, Corollary 2). $\boldsymbol{R}'$ may be imbedded in $\hat{\boldsymbol{R}}'$ via the operators of multiplication on $L^2(G, H)$ by the constant functions, $a \to T$, $T \in \boldsymbol{R}'$. The image of $\{P_i\}$ is a collection of mutually orthogonal. mutually equivalent projections of sum I in $\hat{\boldsymbol{R}}'$. All of these properties are algebraic, hence preserved by this imbedding. Thus both $\boldsymbol{R}$ and $\hat{\boldsymbol{R}}$ satisfy the hypothesis of Corollary 7 ([1], p. 321), therefore there exists a unitary operator $U: \boldsymbol{H} \to L^2(G, \boldsymbol{H})$ such that $\hat{T} = U\,T\,U^*$, $T \in \boldsymbol{R}$.

On $L^2(G, \boldsymbol{H})$ consider the shift operator $\hat{U}_b (b \in G)$ defined by

$$(\hat{U}_b f)(a) = f(ab) \qquad f \in L^2(G, \boldsymbol{H}) .$$

These give a strongly continuous representation of G on $L^2(G, \boldsymbol{H})$ which furthermore implements the automorphisms of G on $\hat{\boldsymbol{R}}$:

$$\begin{aligned} (\hat{U}_b \hat{T} \hat{U}_b^* f)(a) &= (\hat{T} \hat{U}_b^* f)(ab) = \phi_{ab}(T)(\hat{U}_b^* f)(ab) \\ &= \phi_a(\phi_b(T)) f(a) = (\widehat{\phi_b(T)} f)(a) . \end{aligned}$$

Therefore the operators $U_b = U^* \hat{U}_b U$ give the desired representation of G on $\boldsymbol{H}$.

Bibliography

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