Commun. math. Phys. 17, 109—126 (1970)

Extensions of the Taub and NUT Spaces and Extensions of their Tangent Bundles

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Received February 10, 1970

Abstract. A system of extensions of the Taub space and the NUT space with the topology due to Misner is constructed having the property: for each incomplete geodesic in these space-times, there is one and only one extension from the system into which the geodesic smoothly continues. Next, the notion of hypermanifold is introduced which is a generalization of tangent bundle of a space-time, and an untrivial hypermanifold is constructed that contains the tangent bundles of the Taub and NUT spaces as proper sub-manifolds, and within which almost all geodesics are complete. Locally, the hypermanifolds do not yield anything new, but they provide much broader choice of global properties than any four-dimensional space-time manifold.

1. Introduction

A significant feature of Einstein's theory of gravitation is that the local characteristics of a space-time, measurable in a small neighbourhood of a point, are closely related to the properties of the solution as a whole, and that these global, topological properties may be untrivial, in fact very impressive, and sometimes quite complicated. This is of invaluable importance for such a global theory as cosmology is, where the general relativity provides a language even to formulate problems, to say nothing about their solutions.

On the other hand, the choice of topologies as actually implied by the theory in particular solutions is sometimes restricted enough, so that closed time- or light-like lines violating the last rests of causality in physics cannot be avoided [1].

With the progress in mathematical tools, the interest of physicists in this field increases. We mention the papers of Penrose [2], Hawking [3–5], and Geroch [6], where the famous singularity theorems have been stated and proved: if some more or less verifiable conditions are fulfilled, then a kind of singularity of the given space-time is inevitable. These conditions are highly general in that no special space symmetry and no explicit state equation of matter are assumed. The singular space-time is defined as follows: 1. The space-time is not extendable, or, there is no space-time including the original one as its proper sub-manifold. 2. There are time- or light-like geodesics that cannot be extended to an arbitrary length of their affine parameter.

It is not difficult to see that the definition includes what is commonly understood under the singularity, namely the unbounded curvature or mass density at some points: in fact, these points must be cut out from the space-time or else this would not be a differentiable manifold, and, then, all geodesics approaching the points remain incomplete. Nevertheless, the definition includes quite different cases as well. One can imagine that the incompleteness of a relatively small number of geodesics would be innocuous, if, at all, of physical interest [1]. Another example is the so-called Misner's singularity, which is exhibited by the Taub-NUT space: the curvature is regular everywhere and still some geodesics are incomplete and maximal within a compact[1] region of the space. The aim of the present paper is to investigate this singularity in some detail, in a hope that this may add to a better understanding of the nature of singular spaces.

In 1951, Taub proposed a homogeneous, non-isotropic, expanding, empty model of the Universe [7], hereafter called the Taub space. The model can possess a topology $S^3 \times (z_1, z_2)$, where we denote the three-dimensional spherical hypersurface by S^3 and the open interval of reals, $z_1 < z < z_2$, by (z_1, z_2). The closed hypersurfaces $z = \text{const}$ are space-like minimal invariant sub-spaces of the three-parameter group of motions of the space. For $z = z_1$, $z = z_2$, the metric becomes singular. Independently, another solution of Einstein's equations without matter allowing a three-parameter isometry group, usually referred to as NUT space, was found by Newman, Tamburino and Unti [8]. The metric is static and displays singularities of two kinds. Misner has shown one of them to be removable, if the topology of the minimal invariant sub-spaces is chosen to be S^3. (Another topology of these three-dimensional hypersurfaces has recently been proposed by Bonnor [10]. The singularity is maintained and can be interpreted as a rotation axis.) Then, there are closed time-like lines, of course, and the topology of the whole space looks like $S^3 \times (z_2, \infty)$, the second singularity being at z_2. Now, the two spaces, Taub and NUT, prove to fit one another analytically along the light-like three-dimensional hypersurface $z = z_2$, in a similar way as the inner and outer Schwarzschild solutions extend one another beyond the hypersurface $r = 2m$. There are two different possibilities how to sew the spaces together, the two famous Taub-NUT spaces coming thus into being and beginning to provide "counterexamples to almost everything" [11].

In fact, the Taub and NUT spaces are two-parameter families of space-times. We denote the parameters by l and m, $l \geqq 0$, $m > 0$. The

[1] Id est, no points are cut out.

space-times corresponding to the value $l=0$ are, respectively, the inner and outer Schwarzschild solution. Nevertheless, for our study, we choose the coordinates z, ζ, Θ, and φ, which can only be introduced, if $l>0$. The line element is given by

$$ds^2=(2l)^2\left[U^{-1}dz^2-U(d\zeta+\cos\Theta\, d\varphi)^2-\frac{4z^2+1}{4}(d\Theta^2+\sin^2\Theta\, d\varphi^2)\right]. \tag{1}$$

where

$$U=\frac{4(z-z_1)(z_2-z)}{4z^2+1},$$

and $z_1<z_2$ are fixed reals, $z_1=-l/(8m^2)$, $z_2=m/l$. The metric (1) has signature -2 and can be obtained from that in [12], Eq. (26) on reversing signature and performing the transformation

$$t=2lz,\quad \psi=\zeta,\quad \Theta=\Theta,\quad \varphi=\varphi,\quad t_i=2lz_i\ (i=1,2).$$

The regions, where the metric is non-singular, are $z_2<z<\infty$, $z_1<z<z_2$, and $-\infty<z<z_1$, the corresponding manifolds being denoted by M_1, M_2, and M_3, respectively. The hypersurfaces $z=\text{const}$ are topologically S^3, and the coordinates ζ, Θ, and φ are introduced similarly as ψ, Θ, and φ in [9]. ζ has the period 4π, Θ and φ behave like the usual spherical coordinates. The manifolds M_1 and M_3 are equivalent to NUT space and M_2 to Taub space.

We briefly re-collect some well-known information about geodesics in these spaces [12] as written in the coordinates z, ζ, Θ, and φ.

The first integrals of the geodesic equation are

$$p_1=U\sin\Theta\cos\varphi\dot\zeta-\frac{4z^2+1}{4}\sin\varphi\dot\Theta+\left(U-\frac{4z^2+1}{4}\right)\sin\Theta\cos\Theta\cos\varphi\dot\varphi, \tag{2}$$

$$p_2=U\sin\Theta\sin\varphi\dot\zeta+\frac{4z^2+1}{4}\cos\varphi\dot\Theta+\left(U-\frac{4z^2+1}{4}\right)\sin\Theta\cos\Theta\sin\varphi\dot\varphi, \tag{3}$$

$$p_3=U\cos\Theta\dot\zeta+\left(\frac{4z^2+1}{4}\sin^2\Theta+U\cos^2\Theta\right)\dot\varphi, \tag{4}$$

$$p_{||}=U(\dot\zeta+\cos\Theta\dot\varphi), \tag{5}$$

$$\frac{\varkappa}{4l^2}=U^{-1}\dot z^2-U(\dot\zeta+\cos\Theta\dot\varphi)^2-\frac{4z^2+1}{4}(\dot\Theta^2+\sin^2\Theta\dot\varphi^2), \tag{6}$$

where $\varkappa = \pm 1, 0$ according to the kind of geodesic. From (2)–(5), it follows

$$p_1 \sin\Theta \cos\varphi + p_2 \sin\Theta \sin\varphi + p_3 \cos\Theta = p_{\|}\,, \tag{7}$$

the integrals being not independent. If $p = \sqrt{p_1^2 + p_2^2 + p_3^2} \neq 0$, Eq. (7) is non-trivial and has the following parametric solution

$$\begin{aligned} \sin\Theta \cos\varphi &= A_{11} \sin\alpha \cos\psi + A_{21} \sin\alpha \sin\psi + A_{31} \cos\alpha\,, \\ \sin\Theta \sin\varphi &= A_{12} \sin\alpha \cos\psi + A_{22} \sin\alpha \sin\psi + A_{32} \cos\alpha\,, \\ \cos\Theta &= A_{13} \sin\alpha \cos\psi + A_{23} \sin\alpha \sin\psi + A_{33} \cos\alpha\,, \end{aligned} \tag{8}$$

where A_{ij} is an arbitrary fixed orthogonal matrix having $\operatorname{Det} A_{ij} = 1$ and satisfying the relations $p_i = p \cdot A_{3i}$, α is the constant, $0 \leqq \alpha \leqq \pi$, determined by $\cos\alpha = p_{\|} \cdot p^{-1}$, and ψ is the parameter. Simple calculations give

$$\begin{aligned} \dot\Theta^2 + \sin^2\Theta \dot\varphi^2 &= \sin^2\alpha \dot\psi^2\,, \\ \dot\psi^2 &= \frac{4p}{4z^2+1}\,. \end{aligned} \tag{9}$$

Then, (5) and (6) imply

$$\dot z = \sqrt{p_{\|}^2 + \left(\frac{\varkappa}{4l^2} + \frac{4p_{\perp}^2}{4z^2+1}\right) U}\,, \tag{10}$$

where $p_{\perp} = p \sin\alpha$, while from (4) and (5) we have

$$\dot\zeta = \frac{p_{\|}}{U} - \frac{4p}{4z^2+1} \cdot \frac{A_{33}\cos\Theta - \cos\alpha \cos^2\Theta}{\sin^2\Theta}\,. \tag{11}$$

The formulae (2)–(11) will be of use later.

In section 2, various extensions of M_1, M_2, and M_3 are described and their relation to incomplete geodesics is examined. In addition to the two Taub-NUT extensions, a new one, denoted P_i, is found for each M_i, so that every incomplete maximal geodesic of M_i looses its maximality within one and only one of the three extensions. Thus, a family of incomplete geodesics is associated with each extension, and these families turn out to be identical with the three classes of geodesics according as $p_{\|} < 0$, $p_{\|} > 0$, and $p_{\|} = 0$.

In Section 3, extensions of the tangent bundle of M_i are defined and shown to allow more geodesics to be complete than any extension of M_i does, in such a way that one of these extensions is constructed and some of its properties including the behaviour of geodesics are investigated.

2. Complete System of Extensions

As it is well-known [9, 11, 12], each space-time M_i has two different Taub-NUT extensions. Some basic information concerning these follows.

The metric is given by

$$ds^2 = -(2l)^2 \left[2(d\xi + \cos\Theta\, d\varphi)\, dz + U(d\xi + \cos\Theta\, d\varphi)^2 + \frac{4z^2+1}{4}(d\Theta^2 + \sin^2\Theta\, d\varphi^2) \right] \tag{12}$$

for the space which we denote by T_1, and by

$$ds^2 = -(2l)^2 \left[-2(d\eta + \cos\Theta\, d\varphi) dz + U(d\eta + \cos\Theta\, d\varphi)^2 + \frac{4z^2+1}{4}(d\Theta^2 + \sin^2\Theta\, d\varphi^2) \right] \tag{13}$$

for the space denoted by T_2. ξ and η are periodic coordinates with the period 4π, related to ζ and to one another, in the regions M_1, M_2, and M_3 of T_1 or T_2, by

$$\zeta = \xi - z + \frac{1}{4(z_2 - z_1)} [(4z_1^2 + 1) \lg|z - z_1| - (4z_2^2 + 1) \lg|z - z_2|], \tag{14}$$

$$\eta = \xi - 2z + \frac{1}{2(z_2 - z_1)} [(4z_1^2 + 1) \lg|z - z_1| - (4z_2^2 + 1) \lg|z - z_2|]. \tag{15}$$

Note that the coordinate lines $\xi = \text{const}$, $\Theta, \varphi = \text{const}$ are light-like geodesics with $p = p_{||} = 1$, and the lines $\eta = \text{const}$, $\Theta, \varphi = \text{const}$ are light-like geodesics with $-p = p_{||} = -1$.

Let us denote the boundary of a set N in T_1 by $\partial_1 N$, in T_2 by $\partial_2 N$. $\partial_1 M_1$ and $\partial_2 M_1$ are regular three-dimensional closed hypersurfaces $z = z_2$, homeomorphic to S^3, with metric

$$-(2l)^2 \frac{4z_2^2+1}{4}(d\Theta^2 + \sin^2\Theta\, d\varphi^2)$$

in the coordinates ξ, Θ, φ, and η, Θ, φ, respectively, the curves $\Theta, \varphi = \text{const}$ being closed and light-like.

Theorem 1. *For each geodesic γ in M_1, along which the integral $p_{||} > 0$ ($p_{||} < 0$), and only for these geodesics, there is just one point p_γ on $\partial_1 M_1$ ($\partial_2 M_1$) such that $p_\gamma \in \partial_1 \{\gamma\}$ ($p_\gamma \in \partial_2 \{\gamma\}$). ($\{\gamma\}$ is the set of points lying on γ).*

If we rewrite the relations (9)–(11) in the coordinates z, ξ, Θ, and φ (z, η, Θ, and φ), the proof will be quite analogous to that of Theorem 3 as given in the Appendix. We do not write it explicitly.

Theorem 2. *For each geodesic γ in M_1 with $p_{||} > 0$ ($p_{||} < 0$), there is a unique smooth extension beyond the boundary $\partial_1 M_1$ ($\partial_2 M_1$) into the region M_2.*

Proof. Each Taub-NUT space is a locally regular pseudo-Riemannian manifold. Every point and direction determine a unique geodesic. The desired extension is the geodesic passing through the point p_γ, whose existence is assured by Theorem 1, in the direction defined at the point by γ.

Analogic considerations for remaining regions and boundaries yield similar results.

The Theorems 1 and 2 assert that the geodesics in M_i may be divided into three families according as $p_{||} > 0$, $p_{||} < 0$, and $p_{||} = 0$. For each of the first two families, an extension of M_i exists, within which all incomplete maximal geodesics of the family, and only these, may be extended. Now, we try to construct an extension for each M_i having this property with respect to the family characterized by $p_{||} = 0$. The extensions we shall arrive at are the minimal ones with the desired property; they display many unusual features.

Forgetting of the metric we have a differentiable manifold from T_1, on which the three submanifolds with boundaries, P'_1, P'_2, P'_3, can be defined by means of coordinates z, ξ, Θ, φ (for the definition of a manifold with boundary, see [14], p. 30 and ff.):

$$P'_1:\ z_2 \leqq z < \infty, \quad 0 \leqq \xi < 4\pi, \quad 0 \leqq \Theta < \pi, \quad 0 \leqq \varphi < 2\pi,$$
$$P'_2:\ z_1 \leqq z \leqq z_2, \quad 0 \leqq \xi < 4\pi, \quad 0 \leqq \Theta < \pi, \quad 0 \leqq \varphi < 2\pi,$$
$$P'_3:\ -\infty < z \leqq z_1, \quad 0 \leqq \xi < 4\pi, \quad 0 \leqq \Theta < \pi, \quad 0 \leqq \varphi < 2\pi.$$

In this way, the topological and differentiable structure of P'_i is uniquely determined. Next, we change the notation on P'_i, and write ζ in place of ξ, so that the metric can be introduced by Eq. (1) in all points of P'_i, where (1) makes sense. Denote the resulting spaces by P_i. Clearly, P_i are not any submanifolds of T_1, but it is not difficult to show that M_i is an open submanifold of P_i, $i = 1, 2, 3$. If we introduce the symbols ΔM_1, ΔM_2, and ΔM_3 for the closed submanifolds of P_1, P_2, and P_3 defined by the relations $z = z_2$; $z = z_1$, $z = z_2$; $z = z_1$, respectively, then the set ΔM_i is the boundary of the open set M_i in the topological space P_i, and $P_i = M_i \cup \Delta M_i$, for each i[2].

Now, we state two theorems concerning the space P_1; their proofs can be found in the Appendix. In P_2 and P_3, analogous theorems hold and their proofs are quite similar; I drop them.

[2] Since the appearence of the paper [16], it is not so unusual to consider a space-time with a boundary, on which, moreover, no metric in the common sense is defined. On the other hand, the boundary ΔM_i of the space P_i does not fall under the notion of g-boundary.

Theorem 3. *For each incomplete maximal geodesic γ in M_1 characterized by $p_{\|} = 0$, and only for these geodesics, $\{\gamma\}$ has just one limit point on ΔM_1.*

Accordingly, each geodesic γ of this sort approaches a unique point p_γ on ΔM_1. Adding this point to γ, we could define this to be an extension of γ in P_1. It is a problematic construction, because there is no metric defined in the points of ΔM_1. But it is unique, and, moreover, it has the following interesting property:

Theorem 4. *Let γ be a geodesic on P_1 that cuts ΔM_1 in a point p_γ. Then, there is a unique geodesic $\bar{\gamma}$ on P_1 which cuts ΔM_1 in the same point p_γ and which matches γ at least C^1-smoothly in p_γ*[3].

This theorem suggests that each geodesic segment incomplete in M_1 can be smoothly extended by another segment of this sort, both being joined together by a point of ΔM_1. While the geodesics extended in this way on P_1 and P_3 are space-like, so that their loops are not extraordinary strange, on P_2 we have some interesting phenomena.

Example 1. The curve in P_2 given by

$$z = \frac{z_1 + z_2}{2} + \frac{z_2 - z_1}{2} \sin \Theta, \qquad \zeta = \zeta_0, \ \varphi = \varphi_0$$

is a light-like, smooth, closed geodesic with $p_1 = -p \sin \varphi_0, p_2 = p \cos \varphi_0$, and $p_3 = p_{\|} = 0$, z and Θ being periodic with the period 2π. For $\Theta = \pm\pi/2$, the curve reaches the boundary points.

Example 2. Along the time-like geodesic with $p_{\|} = p_1 = p_2 = 0$, $p = p_3 = \sqrt{3}/4l$, we can set $\zeta = \zeta_0$, $\Theta = \pi/2$, and we obtain the relation

$$\frac{dz}{d\varphi} = \frac{2\sqrt{3}}{3} \sqrt{(z - z_1)(z_2 - z)(z^2 + 1)},$$

whose solution is given by

$$z = \frac{A z_1 + B z_2 + (A z_1 - B z_2) \operatorname{cn}\left[2\sqrt{\frac{AB}{3}}(\varphi - \varphi_0)\right]}{A + B + (A - B) \operatorname{cn}\left[2\sqrt{\frac{AB}{3}}(\varphi - \varphi_0)\right]},$$

where $A = \sqrt{1 + z_2^2}$, $B = \sqrt{1 + z_1^2}$, and $\operatorname{cn}(x)$ is Jacobi's elliptic cosinus (see, e.g., [15], p. 491) with

$$k = \sqrt{\frac{1}{2} - \frac{1}{2} \frac{1 + z_1 z_2}{AB}}.$$

[3] In fact, by taking higher derivatives, we could see that the curves match C^∞-smoothly one another.

The function $z(\varphi)$ has the period $\frac{2\sqrt{3}\cdot K(k)}{\sqrt{AB}}$, where $K(k)$ is the complete elliptic integral, φ has the period 2π. If $\frac{K(k)}{\pi}\cdot\sqrt{\frac{3}{AB}}$ is a rational, we have a smooth closed loop, in general intersecting itself in some points, and if the ratio is irrational, we have a curve dense in the two-dimensional surface $\Theta=\pi/2$, $\zeta=\zeta_0$.

We shall see in the next section that this behaviour of geodesics on P_i is, in a sense, similar to that on hypermanifold and could be regarded as a certain limiting case of the latter for $p_{||}\to 0$.

3. Taub-NUT Hypermanifold

None of the three extensions of the Taub and NUT space described in the preceding section has been geodesically complete. The question is natural, whether there is a broader, perhaps even geodesically complete, extension.

We have seen that the Taub-NUT extensions might be constructed in the following way: all light-like geodesics with $p=p_{||}=1$ are incomplete and can be shown to form a three parameter congruence, say, C_1. They may, therefore, serve as coordinate lines. In such a coordinate system, the metric is not singular at $z=z_1$, $z=z_2$ and has a unique analytic extension, namely T_1. Similarly, for $-p=p_{||}=-1$, the congruence C_2 is obtained which yield the extension T_2. This implies that the extensions of the Taub or NUT spaces including extensions of all geodesics of the congruence C_i must contain at least the boundary of that space in T_i. In particular, the extension of M_1 in which the geodesics of C_1 and C_2 should all be a little longer than in M_1 would have to contain M_1 and both boundaries $\partial_1 M_1$ and $\partial_2 M_1$. Such an extension, however, does not exist. It is not difficult to see that every neighbourhood of a point $z_2, \xi_2, \Theta_2, \varphi_2$ on $\partial_1 M_1$ in M_1 contains a point lying in any neighbourhood of any point $z_2, \eta_2, \Theta_2, \varphi_2$ on $\partial_2 M_1$ we can choose. Then, the space would not be Hausdorff[4].

In [16], an attempt has been made to discard the requirement that space-times be Hausdorff, and, indeed, a more complete non-Hausdorff manifold has been constructed. But this generalization is too drastic, e.g., it would allow for manifolds with geodesics having more than one continuation.

There is another, weaker, generalization, which does not exhibit these pathological features. We know that the boundary $\partial_1 M_1$ is cut by those

[4] What could be done would be to choose another topology for $\partial_i M_1$, for example, to identify all points of $\partial_1 M_1$ differing only in ξ with one another and with those of $\partial_2 M_1$ differing from them only in η, so that we would have a cusp there.

geodesics only, for which $p_{||} > 0$. Because of (11), the component $\dot{\zeta}$ of their tangent vector approaches $+\infty$ at $\partial_1 M_1$. Similarly, $\partial_2 M_1$ is cut only by geodesics, whose $\dot{\zeta}$ tends to $-\infty$ at the boundary. In a space (of more dimensions than four, of course) where $\dot{\zeta}$ could be introduced as an independent differentiable coordinate in addition to z, ζ, Θ, and φ, both boundaries would possibly be topologically distinguishable: the points "near them" would have, then, "very different" coordinate $\dot{\zeta}$.

Thus, we are led to the notion of tangent bundle (for exact theory, see, e.g., [13, 14, 17]). Tangent bundle TM of a space-time M is an eight-dimensional differentiable manifold, whose points are pairs consisting of 1) a point of M 2) a tangent vector to M at the point. The preceding considerations suggest that there should be a well-defined boundary of TM_1 stroken by geodesics with $p_{||} \neq 0$, and that this boundary is likely to form, together with TM_1, a Hausdorff space. Accordingly, we might expect that there is some relatively regular extension of tangent bundle of the Taub or NUT space, within which all geodesic with $p_{||} \neq 0$ could be extended.

If we have a manifold M with differentiable coordinates, say, $x^1, x^2, \ldots, x^n$, in some open region on it, we can describe the tangent vectors at the points of the region by means of the local coordinate systems as induced there by these coordinates (see, e.g., [13], Chap. 4). Denote the corresponding components of a vector $\dot{x}^1, \dot{x}^2, \ldots, \dot{x}^n$. In such a way, we have the coordinates $x^1, x^2, \ldots, x^n, \dot{x}^1, \dot{x}^2, \ldots, x^n$ on the TM, which may be shown to be differentiable, and which we shall call the canonical coordinates. Then, the coordinate transformation

$$y^i = f^i(x^1, x^2, \ldots, x^n), \qquad i = 1, 2, \ldots, n\,,$$

on M induces the transformation

$$y^i = f^i(x^1, x^2, \ldots, x^n), \qquad \dot{y}^i = \sum_{j=1}^{n} \frac{\partial f^i}{\partial x^j}\,\dot{x}^j \tag{16}$$

on TM.

Now, choose the coordinates $z, \xi, \Theta, \varphi, \dot{z}, \dot{\xi}, \dot{\Theta}$, and $\dot{\varphi}$ on TT_1, $z, \eta, \Theta, \varphi, \dot{z}, \dot{\eta}, \dot{\Theta}$, and $\dot{\varphi}$ on TT_2, and cut out the two six-dimensional hypersurfaces $z = z_1, \dot{z} = 0$ and $z = z_2, \dot{z} = 0$ from both TT_1 and TT_2. The two eight-dimensional differentiable manifolds obtained in this way will be denoted by T_1' and T_2', respectively. Next, glue T_1' and T_2' together to form a space T' in the following manner: T_1' and T_2' will be sub-spaces of T' and each point of T' will lie either in T_1' or in T_2' or in both. The regions, where T_1' and T_2' will cover one another, let be TM_1, TM_2, and TM_3, and the transformation between the coordinates ξ and η of the same point let be given by (15). We must show that such a glueing up is possible, i.e., that T' is a differentiable, Hausdorff, manifold.

There is a countable basis $B(T_i')$ of the topology of T_i' $(i=1,2)$, because T_i' is a manifold [13, 14]. The sets $B(T_1')$ and $B(T_2')$ can have a non-zero intersection, as systems of sub-sets of T'. Their union is a countable basis of the topology of T'. Then, the functions $z, \xi, \eta, \Theta, \varphi, \dot{z}, \dot{\xi}, \dot{\eta}, \dot{\Theta}, \dot{\varphi}$ can provide the local diffeomorphisms of T' in E^8. T' is Hausdorff, as we can see from the following considerations. Let us take a pair of points p, q, and distinguish the following cases:

1) The points differ at least in one of the coordinates $z, \xi, \eta, \Theta, \varphi, \dot{z}, \dot{\xi}, \dot{\eta}, \dot{\Theta}, \dot{\varphi}$. More precisely, if we denote this coordinate X, then we can write $|X(p)-X(q)|>0$. Choose $\varepsilon=\frac{1}{3}|X(p)-X(q)|$ and define the ε-neighbourhood U_p of the point p by the usual inequalities of the type $z(p)-\varepsilon<z<z(p)+\varepsilon$, etc., and, in the same manner, the ε-neighbourhood U_q of q. Then, U_p and U_q are open, have no points in common and $p\in U_p$, $q\in U_q$.

2) Let the following equations hold

$$\begin{aligned} &z(p)=z(q)=z_2, && \dot{z}(p)=\dot{z}(q)=A,\\ &\xi(p)=\xi_2,\ \eta(q)=\eta_2, && \dot{\xi}(p)=\dot{\xi}_2,\ \dot{\eta}(q)=\dot{\eta}_2,\\ &\Theta(p)=\Theta(q), && \dot{\Theta}(p)=\dot{\Theta}(q),\\ &\varphi(p)=\varphi(q), && \dot{\varphi}(p)=\dot{\varphi}(q). \end{aligned}$$

Then, there is $\varepsilon>0$ satisfying

i) $(A-\varepsilon)(A+\varepsilon)>0$,

ii) within the interval $(z_2-\varepsilon, z_2+\varepsilon)$, U is a monotone function of z, and an open neighbourhood U_p of p such that the following relations are obeyed by the coordinates z, $\dot{z}$ and $\dot{\xi}$ of each point in U_p:

$$z_2-\varepsilon<z<z_2+\varepsilon,\qquad A-\varepsilon<\dot{z}<A+\varepsilon,\qquad \dot{\xi}_2-\varepsilon<\dot{\xi}<\dot{\xi}_2+\varepsilon.$$

Similarly, there is an open neighbourhood U_q of q where

$$z_2-\varepsilon<z<z_2+\varepsilon,\qquad A-\varepsilon<\dot{z}<A+\varepsilon,\qquad \dot{\eta}_2-\varepsilon<\dot{\eta}<\dot{\eta}_2+\varepsilon.$$

By means of (15) and (16) we find

$$\dot{\eta}=\dot{\xi}+2U^{-1}\dot{z}, \tag{17}$$

from which the following inequalities can be derived for the coordinate $\dot{\eta}$ of the points in $U_p\cap U_q$:

$$\dot{\eta}_2-\varepsilon<\dot{\eta}<\dot{\eta}_2+\varepsilon,$$

$$z>z_2:\ \dot{\eta}<-\frac{A}{2\varepsilon}\,\frac{4(z_2+\varepsilon)^2+1}{z_2-z_1+\varepsilon}+\frac{4(z_2+\varepsilon)^2+1}{2(z_2-z_1+\varepsilon)}+\dot{\xi}_2+\varepsilon,$$

$$z<z_2:\ \dot{\eta}>\frac{A}{2\varepsilon}\,\frac{4(z_2-\varepsilon)^2+1}{z_2-z_1-\varepsilon}-\frac{4(z_2-\varepsilon)^2+1}{2(z_2-z_1-\varepsilon)}+\dot{\xi}_2-\varepsilon,$$

under the assumption that $A > 0$, and

$$\dot{\eta}_2 - \varepsilon < \dot{\eta} < \dot{\eta}_2 + \varepsilon ,$$

$$z > z_2: \dot{\eta} > \frac{|A|}{2\varepsilon} \frac{4(z_2+\varepsilon)^2+1}{z_2 - z_1 + \varepsilon} - \frac{4(z_2+\varepsilon)^2+1}{2(z_2-z_1+\varepsilon)} + \dot{\xi}_2 - \varepsilon ,$$

$$z < z_2: \dot{\eta} < -\frac{|A|}{2\varepsilon} \frac{4(z_2-\varepsilon)^2+1}{z_2 - z_1 - \varepsilon} + \frac{4(z_2-\varepsilon)^2+1}{2(z_2-z_1-\varepsilon)} + \dot{\xi}_2 + \varepsilon ,$$

under the assumption $A < 0$. Infer that $\varepsilon < z_2 - z_1$, because U is not monotone in the whole interval (z_1, z_2). Thus, clearly, there is always $\varepsilon > 0$ small enough[5] to ensure that $U_p \cap U_q = \emptyset$.

3) The same equations holds as in 2) with the exception that $z(p) = z(q) = z_1$. The case is completely analogous to 2) and need not, therefore, be explicitly analyzed here.

Thus, T' is shown to be Hausdorff, and, consequently, a differentiable manifold. T' is, however, much richer in properties; it is a certain generalization of tangent bundle of a space-time.

Definition 1. Hypermanifold H is an eight-dimensional differentiable manifold, on which a set system S is given. The elements of S are called simple sets and satisfy the following axioms:

1) S is an open covering of H.

2) If $M \in S$, $N \in S$, then $M \cap N \in S$.

3) For each $M \in S$, $M \neq 0$, there is a four-dimensional pseudo-Riemannian differentiable manifold, $\pi(M)$, such that $M \subset T\pi(M)$.

4) If $\pi_M : T\pi(M) \to \pi(M)$ denotes the natural tangent bundle projection of $T\pi(M)$[6], then $\pi_M(M) = \pi(M)$.

5) If $M \in S$, $N \in S$, $N \subset M$, $N \neq 0$, and $\pi_M | N$ denotes the restriction of the map π_M to the set N, then $\pi_N = \pi_M | N$, and $\pi(N)$ is a pseudo-Riemannian submanifold of the manifold $\pi(M)$.

[5] Now, cutting out the points with $z = z_2$, $\dot{z} = 0$ or $z = z_1$, $\dot{z} = 0$ can be explained. Suppose, we should have taken TT_1 and TT_2 instead of T_1' and T_2', and perhaps still TP_1, TP_2, and TP_3, and made an attempt to glue them all together along the regions TM_1, TM_2 and TM_3. Then, we should have had one more sub-case: $A = 0$, and the Eq. (17) as well as the derived inequalities immediately suggest that, in this sub-case, we should not have found the desired ε, for which the two ε-neighbourhoods U_p and U_q would be disjunct. Therefore, if we wanted to maintain the Hausdorff property, we should have had to identify all points with $z = z_2$, $\dot{z} = 0$ or $z = z_2$, $\dot{z} = 0$ differing in the coordinates ξ, ζ, η, $\dot{\xi}$, $\dot{\zeta}$, $\dot{\eta}$ only. Thus, the three six-dimensional hypersurfaces $z = z_2$, $\dot{z} = 0$ or $z = z_1$, $\dot{z} = 0$ would shrink into a four-dimensional one, with coordinates Θ, φ, $\dot{\Theta}$, and $\dot{\varphi}$ on it. Then, however, there would be a cusp there, and the space would not be a differentiable manifold.

[6] As already mentioned, every point of the tangent bundle TM of a manifold M is a pair (p, u_p), where $p \in M$ and u_p is a tangent vector to M at p. The map $\pi : TM \to M$ defined by $\pi(p, u_p) = p$ is a distinguished map of the tangent bundle and is called its natural projection.

Definition 2. Two hypermanifolds H and H' with the systems S and S' of simple sets, respectively, are equivalent, if there is a map $\Psi: H \to H'$ having the following properties:

1) Ψ is a diffeomorphism with respect to the manifold structure of H and H'.

2) If $M \in S$, then $\Psi(M) \in S'$.

3) For each $M \in S$ there is a diffeomorphism and isometry $\psi_M: \pi(\Psi(M)) \to \pi(M)$ such that $\psi_M^* | M = \Psi | M$ [7].

Example 3. Given a manifold M, then every open submanifold H of its tangent bundle TM is a hypermanifold; H is simultaneously its own simple set.

Example 4. T' is a hypermanifold; its simple sets are T_1', T_2' and all open subsets of T_1' or T_2'. Note that there is no space-time M having tangent bundle TM such that $T' \subset TM$. This may be seen from the fact that the projections of T_1' and T_2' contain $M_1 \cup \partial_1 M_1$ and $M_1 \cup \partial_2 M_1$, so that the space-time would have to contain $M_1 \cup \partial_1 M_1 \cup \partial_2 M_1$ and this is impossible.

Our construction of the hypermanifold T' is strongly dependent on coordinates: we have chosen the functions z, ξ, Θ, φ on T_1 and z, η, Θ, φ on T_2 and all operations performed further on have been described exclusively by means of these. What we have shown, therefore, is that there is one hypermanifold T' for every choice of z, ξ, Θ, φ on T_1 and z, η, Θ, φ on T_2. Of course, these coordinates as defined have certain invariant properties: they fit the topology and differentiable structure of T_1 and T_2 and the line element is of the form (12) and (13) in them. That is to say, if we have another coordinates $\bar{z}$, $\bar{\xi}$, $\bar{\Theta}$, $\bar{\varphi}$ on T_1 and z', η', Θ', φ' on T_2 of this sort, then the maps $\psi_1: T_1 \to T_1$ and $\psi_2: T_2 \to T_2$ defined by

$$\psi_1(z) = \bar{z}, \quad \psi_1(\xi) = \bar{\xi}, \quad \psi_1(\Theta) = \bar{\Theta}, \quad \psi_1(\varphi) = \bar{\varphi},$$

$$\psi_2(z) = z', \quad \psi_2(\eta) = \eta', \quad \psi_2(\Theta) = \Theta', \quad \psi_2(\varphi) = \varphi'$$

are diffeomorphisms and isometries.

Choosing the coordinates $\bar{z}$, $\bar{\xi}$, $\bar{\Theta}$, $\bar{\varphi}$, $\dot{\bar{z}}$, $\dot{\bar{\xi}}$, $\dot{\bar{\Theta}}$, $\dot{\bar{\varphi}}$ on TT_1, we cut out the points with $\bar{z} = z_1$, $\dot{\bar{z}} = 0$ and $\bar{z} = z_2$, $\dot{\bar{z}} = 0$ and denote the resulting manifold by $\bar{T}_1'$. Similarly, the manifold T_2'' is obtained, if the points with $z' = z_1$, $\dot{z}' = 0$ and $z' = z_2$, $\dot{z}' = 0$ are omitted from TT_2. From $\bar{T}_1'$ and $\bar{T}_2''$, we construct the hypermanifold T'' on identifying each point of $\bar{T}_1'$ of coordinates $\bar{z}, \bar{\xi}, \ldots, \dot{\bar{\varphi}}$, $\bar{z} \neq z_1$, $\bar{z} \neq z_2$, with the point of

[7] Every diffeomorphism $\psi: M \to M'$ of differentiable manifold M onto M' induces a diffeomorphism $\psi^*: TM' \to TM$ (see [13], p. 82ff.).

T_2'', whose coordinates are given by

$$z' = \bar{z}, \quad \Theta' = \bar{\Theta}, \quad \varphi' = \bar{\varphi}, \quad \dot{z}' = \dot{\bar{z}}, \quad \dot{\Theta}' = \dot{\bar{\Theta}}, \quad \dot{\varphi}' = \dot{\bar{\varphi}},$$

$$\eta' = \bar{\xi} - 2\bar{z} + \frac{1}{2(z_2 - z_1)} [(4z_1^2 + 1)\lg|\bar{z} - z_1| - (4z_2^2 + 1)\lg|\bar{z} - z_2|],$$

$$\dot{\eta}' = \dot{\bar{\xi}} + 2U^{-1}(\bar{z})\dot{\bar{z}}.$$

Now, it is not difficult to show that T'' is equivalent to T':

The diffeomorphism ψ_1 induces the diffeomorphism $\psi_1^* : TT_1 \to TT_1$, whose restriction to $\bar{T}_1'$, $\psi_1^* | \bar{T}_1' : \bar{T}_1' \to T_1'$, is a diffeomorphism of $\bar{T}_1'$ onto T_1'. Likewise, we have the diffeomorphism $\psi_2^* | T_2''$ of T_2'' onto T_2'. But $\bar{T}_1'$ and T_2'' are two open submanifolds covering T''; their images, T_1' and T_2', cover T'; each of the maps $\psi_1^* | \bar{T}_1'$ and $\psi_2^* | T_2''$ is one-to-one and differentiable, and it is immediate that they are identical on $\bar{T}_1' \cap T_2''$. Therefore, the map $\Psi : T'' \to T'$ given by $\Psi | \bar{T}_1' = \psi_1^* | \bar{T}_1'$, $\Psi | T_2'' = \psi_2^* | T_2''$ is a well-defined diffeomorphism of T'' onto T'. The two remaining conditions of Definition 2 are obviously satisfied. Thus, the hypermanifold T' is uniquely determined by our construction.

Physical interpretation of hypermanifolds may be based on the points 3) and 4) of the Definition 1, which claim, in fact, that a hypermanifold is, locally, equivalent to a tangent bundle of some space-time. Since physical meaning is attached only to local properties of space-times such as metric, connection, curvature, etc., the fact that a hypermanifold need not be, as a whole, a tangent bundle of any space-time is not of so much importance: the projections $\pi(M)$ of simple sets are pseudo-Riemannian manifolds of usual physical interpretation. What is generalized is only joining together these patches.

On the other hand, tangent bundle is something like the phase space of a relativistic particle. More exactly, it includes the phase space as a proper subspace, because there are also points in it with corresponding tangent vector not unit or time-like. Thus, even a direct physical meaning can be ascribed to the hypermanifold as a whole: it is a phase space of a particle.

As an example, we generalize the notion of geodesic, the path of a free particle, for hypermanifolds:

Definition 3. A curve γ on hypermanifold H is a geodesic, if there is a geodesic γ_M on $\pi(M)$ such that $\gamma(t) = \tilde{\gamma}_M(t)$[8] on M for every simple set M with $M \cap \{\gamma\} \neq \emptyset$.

It is clear, that every geodesic has a unique extension or that every curve has no more than one end point on hypermanifold – that is to say,

[8] A curve $C : [a, b] \to M$ of class C^k, $k > 1$, on a manifold M determines a unique curve $\tilde{C} : [a, b] \to TM$ of class C^{k-1} on TM such that $\tilde{C}(t)$ is the tangent vector to C at the point $C(t)$, $t \in [a, b]$.

no pathological features of the non-Hausdorff manifolds as mentioned by Geroch in [16] exist here.

The geodesics on T' are made from segments, whose projections are nothing but geodesics on T_1, T_2, M_1, M_2, and M_3, described by the relations (2)–(11).

Example 5. The geodesics with $p_{||}=0$ consist each of just one segment which lies entirely on one of the simple sets TM_i, $i=1,2,3$. They remain incomplete within T', because they approach the points $z=z_1$, $\dot{z}=0$ or $z=z_2$, $\dot{z}=0$, which were cut out from T'.

Example 6. Time-like geodesics with $p_{||}=p>0$ $(\alpha=0)$. The Eqs. (2)–(11) imply, for these values of constants, the following relations:

$$\Theta=\Theta_0, \qquad \varphi=\varphi_0,$$
$$\dot{z}=\sqrt{p^2+U(2l)^{-2}}, \qquad \dot{\zeta}=pU^{-1}$$

which read as transformed into the coordinates z, ξ:

$$\dot{z}=\sqrt{p^2+U(2l)^{-2}}, \qquad \dot{\xi}=-\frac{1}{4l^2}\,\frac{1}{p^2+\sqrt{p^2+U(4l)^{-2}}}. \tag{18}$$

In general, there are two values $\bar{z}_1$ and $\bar{z}_2$ for which $\dot{z}=0$, and p may be chosen such that $\bar{z}_1<z_1$ and $z_2<\bar{z}_2$. Then, we have a unique geodesic segment on T_1 determined by Eqs. (18) and passing through the point $z=\bar{z}_1$, $\xi=\bar{\xi}_1$.

Another time-like geodesic with $p_{||}=-p$ fulfills, in the coordinates z, η, Θ, φ, the following relations

$$\dot{z}=\sqrt{p^2+\frac{U}{4l^2}}, \quad \dot{\eta}=\frac{1}{4l^2}\,\frac{1}{p+\sqrt{p^2+U/(4l^2)}}, \quad \Theta=\Theta_0, \quad \varphi=\varphi_0, \tag{19}$$

and $\dot{z}$ reaches the value zero for $z=\bar{z}_1$ and $z=\bar{z}_2$ again. We have a unique geodesic segment on T_2 passing through the point

$$z=\bar{z}_1, \qquad \eta=\bar{\eta}_1=\bar{\xi}_1-2\bar{z}_1 + \frac{1}{2(z_2-z_1)}\left[(4z_1^2+1)\lg(z_1-\bar{z}_1)+(4z_2^2+1)\lg(z_2-\bar{z}_1)\right]$$

and satisfying (19). The points $z=\bar{z}_1$, $\xi=\bar{\xi}_1$, and $z=\bar{z}_1$, $\eta=\bar{\eta}_1$ are, however, identical according to (15), and the tangent vectors to the two segments at the point are just opposite to one another (because of $\dot{z}=0$ is $\dot{\eta}=\dot{\xi}$ there); therefore, the two segments match smoothly one another. Similar procedure can be repeated at the upper end points of the two segments, where $z=\bar{z}_2$. In general, the points need not be identical, so that we must use its own segment for each end point to go on. The case can obviously occur, when such geodesic as extended step by step in this way fills up the two-dimensional surface $\bar{z}_1 \leqq z \leqq \bar{z}_2$, $\Theta = \Theta_0$, $\varphi = \varphi_0$, so that it is dense there.

These examples show that there are still incomplete geodesics, but their ammount is substantially reduced in comparison with either Taub-NUT space; it is not difficult to see, that all geodesics having $p_{||} \neq 0$ are complete. In this sense, we can say that the introduction of hypermanifolds can help to reduce the singularity. Moreover, the remaining incompleteness is not of the strange nature proper to that of the Taub-NUT spaces: no geodesic of the hypermanifold is maximal and incomplete within a compact region. Indeed, finite are only the geodesics approaching certain points which had to be cut out from the hypermanifold.

On the other hand, on T', we have still worse behaviour of geodesics with respect to the causality principle than on T_1 or T_2, resulting in a breakdown of the global time orientability. But this is due to the fact that we have made too many identifications (for the sake of simplicity): it is not necessary to identify T_1' and T_2' along all three regions TM_1, TM_2, and TM_3. Instead, we could glue together an infinite number of copies of T_1' and T_2', each two neighbouring along one of the regions only, into a ladder-like construction.

Acknowledgement. It is a pleasure to express my profound gratitude to the referee, whose critical remarks helped to clarify the text in many points. I am deeply indebted to Prof. H. Leutwyler and Prof. H. Debrunner for many criticisms and improvements in the first draft of this paper. My special thanks go to Prof. A. Mercier for his invaluable encouragement and support.

Appendix

Proof of the Theorem 3. If there is such a point p_γ, then its coordinates are given by

$$z = z_2, \ \zeta = \zeta_2 = \lim_{z \to z_2} \zeta(z), \ \Theta = \Theta_2 = \lim_{z \to z_2} \Theta(z), \ \varphi = \varphi_2 = \lim_{z \to z_2} \varphi(z), \tag{20}$$

where $\zeta(z)$, $\Theta(z)$, and $\varphi(z)$ are coordinates ζ, Θ, and φ as functions of z along the geodesic γ. On the other hand, if the limits (20) exist, then the point of coordinates z_2, ζ_2, Θ_2, and φ_2 is the desired point p_γ. Therefore, it is sufficient to examine the existence of the limits (20) for all geodesics. Consider the following cases:

1) $p = p_{||} = 0$.

Eqs. (2)–(5) and (10) imply $\varkappa = -1$ and

$$\dot{z} = (2l)^{-1} \sqrt{U}, \qquad \dot{\zeta} = \dot{\Theta} = \dot{\varphi} = 0. \tag{21}$$

Hence $\zeta = \text{const}$, $\Theta = \text{const}$, $\varphi = \text{const}$; the limits (20) exist and are equal to the constants.

2) $p \neq 0$, $p_3 = p_{||} = 0$.

Eq. (7) implies either $\sin\Theta = 0$ or $p_1 \cos\varphi + p_2 \sin\varphi = 0$. In the first subcase, φ makes no sense, but we can set $\varphi = 0$. Then, from (5), $\zeta = \text{const}$, and Eqs. (2) and (3) give $p_1 = p_2 = 0$, but this is not compatible with $p \neq 0$. In the second sub-case, we have ζ, $\varphi = \text{const}$. Thus, the limits (20) for and ζ, φ exist and are equal to the constants. Next, $p_1 = \varepsilon p \sin\varphi_2$, $p_2 = -\varepsilon p \cos\varphi_2$, where $\varepsilon^2 = 1$, and from (2) and (3) it follows

$$\dot{\Theta} = -\varepsilon \frac{4p}{4z^2+1}, \qquad \dot{\varphi} = \dot{\zeta} = 0, \tag{22}$$

which, together with Eq. (10), gives $\varkappa = -1$ and

$$\Theta(z) = \bar{\Theta} - \varepsilon \int_{\bar{z}}^{z} \frac{2pl\,dx}{\sqrt{(a-x)(a+x)(x-z_1)(z_2-x)}},$$

where $a = \sqrt{4p^2l^2 - 1/4}$. The improper integral on the right side converges except when $p = (1/4l)\sqrt{4z_2^2+1}$; then, however, the geodesic is complete.

3) $p_3 \neq p_{||} = 0$.

Eq. (7) implies $\sin\Theta \neq 0$, $\sin\Theta_2 \neq 0$. From Eqs. (8) we have

$$\begin{aligned} \sin\Theta \cos\varphi &= A_{11}\cos\psi + A_{21}\sin\psi, \\ \sin\Theta \sin\varphi &= A_{12}\cos\psi + A_{22}\sin\psi, \\ \cos\Theta &= A_{13}\cos\psi + A_{23}\sin\psi. \end{aligned} \tag{23}$$

Eqs. (9)–(11) give $\varkappa = -1$ and

$$\psi(z) = \bar{\psi} + 2pl \int_{\bar{z}}^{z} \frac{dx}{\sqrt{(a-x)(a+x)(x-z_1)(z_2-x)}},$$

$$\zeta(z) = \bar{\zeta} - 2pl \int_{\bar{z}}^{z} \frac{\cos\Theta}{\sin^2\Theta} \frac{dx}{\sqrt{(a-x)(a+x)(x-z_1)(z_2-x)}}.$$

These integrals converge except when $p = (1/4l)\sqrt{4z_2^2+1}$, but then the proper length diverges, too.

4) $p_{||} \neq 0$, $p_3 \neq p_{||}$.

Eq. (7) forbids Θ and Θ_2 to be 0 or π and, from Eqs. (10) and (11), we immediately see that the corresponding integral for $\zeta(z)$ diverges, while the proper length remains finite.

5) $p_3 = p_{||} \neq 0$.

Eqs. (4) and (5) yield either $\cos\Theta = 1$, or

$$\dot{\zeta} = \frac{p\cos\alpha}{U} - \frac{4p\cos\alpha}{4z^2+1}\,\frac{\cos\Theta}{1+\cos\Theta},$$

and then, from (7), $\cos\Theta \neq -1$, $\cos\Theta_2 \neq -1$. In the first sub-case, we can set $\varphi = 0$, which, together with (2)–(5) gives

$$\dot{\zeta} = \frac{\mathrm{p}}{U}.$$

Clearly, in either case, $\zeta(z)$ must diverge. Q.E.D.

Proof of the Theorem 4. The components $\dot{z}_2, \dot{\zeta}_2, \dot{\Theta}_2, \dot{\varphi}_2$, of the tangent vector to γ at p_γ are given by

$$\dot{z}_2 = \lim_{z \to z_2} \dot{z}(z), \qquad \dot{\zeta}_2 = \lim_{z \to z_2} \dot{\zeta}(z),$$
$$\dot{\Theta}_2 = \lim_{z \to z_2} \dot{\Theta}(z), \qquad \dot{\varphi}_2 = \lim_{z \to z_2} \dot{\varphi}(z).$$

We can use the symbolics and the case division of the foregoing proof.

1) $p = p_{||} = 0$.

Eqs. (21) imply $\dot{z}_2 = \dot{\zeta}_2 = \dot{\Theta}_2 = \dot{\varphi}_2 = 0$, and the geodesic with opposit tangent vector at p_γ is just the same geodesic. We shall see, that it is unique.

2) $p \neq 0$, $p_3 = p_{||} = 0$.

Eqs. (10) and (22) give

$$\dot{z}_2 = \dot{\zeta}_2 = \dot{\varphi}_2 = 0, \qquad \dot{\Theta}_2 = -\varepsilon \frac{4p}{4z_2^2 + 1}.$$

Hence, there are just two geodesics, for a given p, and their tangent vectors are opposit to one another.

3) $p_3 \neq p_{||} = 0$.

In order to have a unique description of γ in a neighbourhood of p_γ, we set $\varphi_2 = 0$. Then, we define a parameter β by

$$p_3 = p \sin\Theta_2 \cos\beta, \qquad -\pi < \beta < -\frac{\pi}{2}, \; -\frac{\pi}{2} < \beta < \frac{\pi}{2}, \frac{\pi}{2} < \beta \leqq \pi.$$

Now, the matrix A_{ij} is uniquely determined. In particular, from (23)

$$A_{11} = \sin\Theta_2 \cos\varphi_2, \qquad A_{21} = -\cos\Theta_2 \cos\varphi_2 \sin\beta - \sin\varphi_2 \cos\beta,$$
$$A_{12} = \sin\Theta_2 \sin\varphi_2, \qquad A_{22} = -\cos\Theta_2 \sin\varphi_2 \sin\beta + \cos\varphi_2 \cos\beta,$$
$$A_{13} = \cos\Theta_2, \qquad A_{23} = \sin\Theta_2 \sin\beta,$$

and we have

$$\dot{z}_2 = 0, \qquad \dot{\zeta}_2 = -\frac{4p}{4z_2^2 + 1} \frac{\cos\Theta_2}{\sin\Theta_2} \cos\beta,$$

$$\dot{\Theta}_2 = -\frac{4p}{4z_2^2 + 1} \sin\beta, \qquad \dot{\varphi}_2 = \frac{4p}{4z_2^2 + 1} \frac{\cos\beta}{\sin\Theta_2}.$$

Hence, the geodesics reaching the point p_γ form a two parameter family, and we see immediately, that for every γ with the parameters p, β there is $\bar{\gamma}$ with the parameters p, $\bar{\beta} = \beta + \pi$ (or $\beta - \pi$, in order that $-\pi < \beta \leqq \pi$), whose tangent vector at p_γ is just the inverse to that of γ, and that $\bar{\gamma}$ is the unique geodesic of this property. Q.E.D.

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