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Stable Potentials, II

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Abstract. A family of stable one-dimensional potentials is shown not to be decomposable into the sum of a non-negative function and a function of non-negative type. This settles in the negative a question raised by Ruelle.

§ 1

In his book on Statistical Mechanics [1] Ruelle raised the question whether every (measurable) stable potential on $\boldsymbol{R}^\nu$ can be decomposed into the sum of a continuous function of non-negative type and a (measurable) non-negative function. In our previous paper on this subject [2] we generalized this problem somewhat and gave examples of stable potentials on $\boldsymbol{Z}_{2k+1}$ ($k \geqq 2$), the group of integers modulo $2k+1$, which are not capable of such a decomposition. The physically relevant case is that concerned with the group $\boldsymbol{R}^\nu$ (for $\nu = 3$). In the present paper we carry the analysis one step closer to this case. In § 2 we give a two parameter family of potentials $\{\varphi_{t,d}\}$ ($0 < t \leqq 1$, $-\infty < d < \infty$) with the following property: There is a critical value $d_0 = d_0(t)$ such that $\varphi_{t,d}$ is stable for $d \geqq d_0$ and unstable for $d < d_0$. In § 3 we consider the particular case $t = 1$ and $d_0(1)$, i.e. a potential that in a sense is critically stable, and show that it cannot be decomposed in the manner suggested by Ruelle. In § 4 we list some unsolved problems suggested by this paper.

§ 2

Let $\varphi(x)$ be a real valued even function of the real variable x. With a given φ and any positive integer n, we associate the function

$$\Phi_n = \Phi_n(x_1, x_2, \ldots, x_n) = \sum_{1 \leqq i, j \leqq n} \varphi(x_i - x_j)\,. \tag{1}$$

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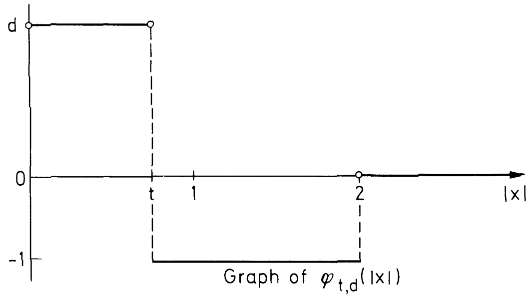


Fig. 1

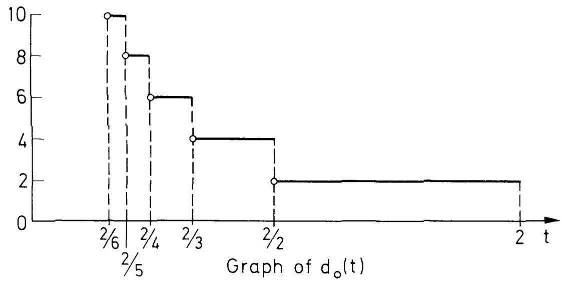


Fig. 2

We say that φ is *stable* if for all positive integers n and all $x_1, x_2, \ldots x_n \in \boldsymbol{R}$

$$\Phi_n(x_1, x_2, \ldots, x_n) \geqq 0 . \tag{2}$$

We now define the family of potentials to be considered. For any positive $t \leqq 1$ and any real d we let

$$\varphi_{t,d}(x) = \begin{cases} d, & \text{for} \quad |x| \leqq t, \\ -1, & \text{for} \quad t < |x| < 2, \\ 0, & \text{for} \quad 2 \leqq |x| \end{cases} \tag{3}$$

(see Fig. 1). Furthermore, let

$$d_0 = d_0(t) = 2(k-1) \quad \text{for} \quad \frac{2}{k} \leqq t < \frac{2}{k-1} \qquad (k = 2, 3, 4, \ldots) \tag{4}$$

(see Fig. 2).

Theorem. *For $d < d_0$, $\varphi_{t,d}$ is unstable. For $d \geqq d_0$, $\varphi_{t,d}$ is stable.*

Proof. We first let $d < d_0$. For any n let us choose $x_1 = \alpha$, $x_2 = 2\alpha, \ldots, x_n = n\alpha$, where α satisfies

$$\frac{2}{k} \leqq t < \alpha < \frac{2}{k-1} \tag{5}$$

and k is related to d_0 according to (4). Then

$$\begin{aligned}\Phi_n &= nd - 2(n-1) - 2(n-2) - \cdots - 2(n-k+1) \\ &= k(k-1) - (d_0 - d)n .\end{aligned} \tag{6}$$

This is negative for sufficiently large n, so $\varphi_{t,d}$ is unstable and the first part of the theorem is established[1].

To prove the second part we first remark that $\varphi_{t,d}(x)$ is a non-decreasing function of both t and d and therefore it is enough to show stability for

$$t = \frac{2}{k}, \qquad d = d_0 = 2(k-1). \tag{7}$$

In the following we assume (7) and write simply φ for $\varphi_{t,d}$. We have

$$\Phi_1 = d_0 > 0 . \tag{8}$$

We now proceed by induction: We assume that for some $n \geqq 2$

$$\Phi_{n-1}(x_1, x_2, \ldots, x_{n-1}) \geqq 0 \tag{9}$$

for all $x_1, x_2, \ldots, x_{n-1} \in \boldsymbol{R}$ and we show that then (2) holds.

Let then $x_1, x_2, \ldots, x_n$ be chosen arbitrarily and let

$$\Psi_i = \Phi_n(x_1, x_2, \ldots, x_n) - \Phi_{n-1}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \qquad (i = 1, 2, \ldots, n). \tag{10}$$

If for some i, $1 \leqq i \leqq n$, $\Psi_i \geqq 0$ then it follows from the induction hypothesis that $\Phi_n \geqq 0$.

We therefore assume that for $i = 1, 2, \ldots, n$

$$\Psi_i = \varphi(0) + 2 \sum_{\substack{j=1 \\ j \neq i}}^{n} \varphi(x_i - x_j) < 0 \tag{11}$$

and show that this leads to a contradiction.

Let n_i denote the number of those integers r, $r \neq i$, $1 \leqq r \leqq n$, for which $|x_r - x_i| \leqq t = 2/k$. Similarly let f_i be the number of r for which $t < |x_r - x_i| < 2$. Then

$$\Psi_i = (1 + 2n_i)\, 2(k-1) - 2f_i . \tag{12}$$

Now let $\bar{n} = \max_r(n_r)$ and let s be the smallest integer r such that $n_r = \bar{n}$. From (11) and (12) we get

$$(1 + 2\bar{n})(k-1) < f_s . \tag{13}$$

We note that no interval of length t contains more than $\bar{n} + 1$ of the x_i, for if it did then for some i it would be the case that $n_i > \bar{n}$ which contra-

[1] We are indebted to Mr. John Gaisser for a different argument for $t = 1$.

dicts the definition of $\bar{n}$. Moreover, if such an interval lies wholly below x_s it can not even contain as many of the x_i as $\bar{n}+1$ for if it did then for some i it would be the case that $x_i < x_s$ and $n_i = \bar{n}$ which contradicts the definition of s. Consider now the two intervals $(x_s - 2, x_s - t)$ and $(x_s + t, x_s + 2)$ and decompose each into $k-1$ disjoint subintervals of length t each. By the above remark we see then that the total number f_s of integers i such that x_i falls into these two intervals satisfies

$$f_s \leqq (2\bar{n}+1)(k-1). \tag{14}$$

This contradicts (13), and thus the proof of the theorem is complete.

§ 3

Let $t=1$ so that $d_0(t)=2$ and

$$\varphi_{t,d_0}(x) = \begin{cases} 2, & 0 \leqq |x| \leqq 1, \\ -1, & 1 < |x| < 2, \\ 0, & 2 \leqq |x|. \end{cases} \tag{15}$$

According to the theorem just proved this function is stable (in the following it will be denoted by φ_1). We shall now show that it can not be decomposed into the sum of a non-negative function and a continuous function of non-negative type, i.e. that there exists no continuous function of non-negative type (to be referred to as a Bochner function) ψ such that

$$\psi(x) \leqq \varphi_1(x) \quad \text{for all} \quad x \in \boldsymbol{R}. \tag{16}$$

We assume then (16) and show that it leads to a contradiction.

Every Bochner function has the representation

$$\psi(x) = \int_{\infty}^{\infty} e^{ixu}\, dF(u) \tag{17}$$

where F is bounded and non-decreasing. It has the property

$$|\psi(x)| \leqq \psi(0). \tag{18}$$

Moreover, equality in (18) holds only if $|\psi|$ is periodic [3]. For any n and any choice of the arguments x_i the matrices (M_{ij}) with $M_{ij} = \psi(x_i - x_j)$, $(1 \leqq i, j \leqq n)$ are non-negative definite so that in particular $\det M \geqq 0$. We shall consider such determinantal inequalities for special choices of the x_i.

Let $x_1 = 0$, $x_2 = y$, $x_3 = 2y$ for any y with $\frac{1}{2} \leqq y \leqq 1$. Let $\psi(0) = a$, $\psi(y) = b$, $\psi(2y) = c$. The determinantal inequality then reads

$$\begin{vmatrix} a & b & c \\ b & a & b \\ c & b & a \end{vmatrix} = (a-c)(a^2 + ac - 2b^2) \geqq 0 \tag{19}$$

(18) implies $|c| \leqq a$. Suppose $|c| = a$. Then $|\psi|$ is periodic. Now (16) and (15) imply that

$$\lim_{\alpha\to\infty} \frac{1}{2\alpha} \int_{-\alpha}^{\alpha} \psi(x)\,dx = \frac{1}{l(P)} \int_P \psi(x)\,dx < 0 \tag{20}$$

where P is an appropriately chosen period of $|\psi|$ such that in it $\psi \leqq 0$. But a general theorem on Bochner functions [4] shows that the left hand side of (20) equals the discontinuity $F(0+) - F(0-) \geqq 0$. This is a contradiction. Thus

$$|c| < a \leqq 2 \tag{21}$$

and from (19)

$$b^2 \leqq \tfrac{1}{2}(a^2 + ac) \leqq a + c \tag{22}$$

(16) implies $c \leqq -1$ and so from (21) and (22) we get $b^2 \leqq 1$. We now consider several possibilities.

Suppose $0 \leqq a < 2$. Let φ^* be defined as follows:

$$\varphi^*(x) = \begin{cases} a, & 0 \leqq |x| \leqq \frac{1}{2}, \\ 1, & \frac{1}{2} < |x| \leqq 1, \\ -1, & 1 < |x| < 2, \\ 0, & 2 \leqq |x|. \end{cases} \tag{23}$$

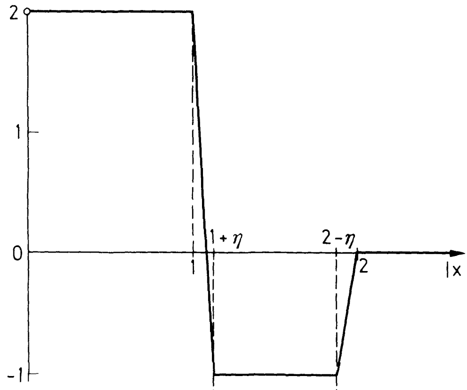
φ^* is unstable since for $x_i = i\alpha$, $1 \leqq i \leqq n$, with $1 < \alpha < 2$ we have $\Phi_n^*(x_1, x_2, \ldots, x_n) = na - 2(n-1) < 0$ for sufficiently large n. On the other hand we have shown that if $\frac{1}{2} \leqq y \leqq 1$, then $b = \psi(y) \leqq 1$ and therefore $\psi(x) \leqq \varphi^*(x)$ for all x. But a Bochner function and hence any function dominating it is stable. This contradiction shows that only the case $a = 2$ needs to be considered.

Suppose $b = 1$; then from (22) $c = -1$. In this case we consider the determinantal inequality $\det M \geqq 0$ associated with the four points $x_1 = 0$, $x_2 = y$, $x_3 = 2y$, $x_4 = 3y$. Writing $\psi(3y) = d$ the inequality reduces to

$$\begin{vmatrix} 2 & 1 & -1 & d \\ 1 & 2 & 1 & -1 \\ -1 & 1 & 2 & 1 \\ d & -1 & 1 & 2 \end{vmatrix} = -3(d+2)^2 \geqq 0\,. \tag{24}$$

Hence $d = -2$ and as before, this implies the periodicity of $|\psi|$; a contradiction.

The only remaining case is $a = 2$, $-1 \leqq b < 1$. This means $\psi(y) < 1$ for $\frac{1}{2} \leqq y \leqq 1$. Because of the continuity of ψ this implies $b = \psi(y) \leqq 1 - \varepsilon$



Graph of $\tilde{\varphi}(|x|)$

Fig. 3

for some $\varepsilon > 0$. Now we construct a comparison function which is a slight modification of (23)

$$\varphi^{**}(x) = \begin{cases} 2 & 0 \leqq |x| \leqq \frac{1}{2}, \\ 1-\varepsilon, & \frac{1}{2} < |x| \leqq 1, \\ -1, & 1 < |x| < 2, \\ 0, & 2 \leqq |x|. \end{cases} \tag{25}$$

Taking $x_i = i\alpha$, $1 \leqq i \leqq n$, with $\frac{1}{2} < \alpha \frac{2}{3}$ we get

$$\Phi^{**}(x_1, x_2, \ldots, x_n) = 2n + 2(n-1)(1-\varepsilon) - 2(n-2) - 2(n-3) < 0$$

for sufficiently large n. Thus φ^{**} is unstable and, as before, $\psi(x) \leqq \varphi^{**}(x)$ yields a contradiction.

This establishes that φ_1 is a stable function on $\boldsymbol{R}$, and is not the sum of a non-negative and a Bochner function.

To show that this property of φ_1 is not the consequence of the fact that φ_1 is discontinuous, we have constructed a closely related but continuous function $\tilde{\varphi}$ which still has this property. It is

$$\tilde{\varphi}(x) = \begin{cases} 2, & 0 \leqq |x| \leqq 1, \\ 2 - \frac{3}{\eta}(|x|-1), & 1 \leqq |x| \leqq 1+\eta, \\ -1, & 1+\eta \leqq |x| \leqq 2-\eta, \\ -\frac{1}{\eta}(2-x), & 2-\eta \leqq |x| \leqq 2, \\ 0, & 2 \leqq |x|, \end{cases} \tag{26}$$

where η is a sufficiently small positive number (see Fig. 3). Since $\tilde{\varphi}(x) \geqq \varphi_1(x)$ for all x, $\tilde{\varphi}$ is stable. To show that $\tilde{\varphi}$ is not a sum of a non-negative and a Bochner function we use an argument completely analogous for that used for φ_1.

§ 4

Some open questions follow.

1. Suppose φ is a stable function on $\boldsymbol{R}^3$ such that $\varphi(x) = \varphi(|x|)$. Can φ be decomposed into the sum of a Bochner function and a non-negative function? This is the main open question of physical interest. We conjecture the existence of a counterexample.

2. Fix $t, 0 \leqq t \leqq 1$. Is $\{\lambda \varphi_{t, d_0(t)} : \lambda \geqq 0\}$ an extreme ray of the cone of stable potentials?

3. Determine all the extreme rays of the cone of stable potentials and thus possibly get an integral representation of stable potentials analogous to Bochner's representation of continuous functions of non-negative type.

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