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Independence of Local Algebras in Quantum Field Theory

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Abstract. It is shown that local C^*-algebras $\mathfrak{A}(O_1)$ and $\mathfrak{A}(O_2)$ associated with spacelike separated regions O_1 and O_2 in the Minkowski space are independent. The proof is accomplished by a theorem concerning the structure of the C^*-algebra generated by $\mathfrak{A}(O_1)$ and $\mathfrak{A}(O_2)$.

I. Introduction

Let $\mathfrak{A}$, $\mathfrak{A}_1$, $\mathfrak{A}_2$ be C^*-algebras with $\mathfrak{A}_1$ and $\mathfrak{A}_2$ contained in $\mathfrak{A}$. Picking a state φ_1 of $\mathfrak{A}_1$ and a state φ_2 of $\mathfrak{A}_2$ one may ask whether there exists a state φ of $\mathfrak{A}$ whose restriction to $\mathfrak{A}_i$ equals $\varphi_i (i=1,2)$. If this is the case for any choice of the pair φ_1, φ_2 then we shall say that the algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are "statistically independent".

In a Quantum Field Theory let $\mathfrak{A}(O)$ denote the algebra of observables which are associated with the region O of the Minkowski space. We use the symbol $O_1 \times O_2$ to denote that two regions O_1, O_2 lie totally spacelike to each other. In [1] Haag and Kastler raised the question of whether two algebras $\mathfrak{A}(O_1)$ and $\mathfrak{A}(O_2)$ are statistically independent when $O_1 \times O_2$.

If $O_1 + x \times O_2$ for $x \in \mathcal{N}$, $\mathcal{N}$ being a suitably chosen neighbourhood of the origin, we write $O_1 \divideontimes O_2$. Starting from standard assumptions of Quantum Field Theory, Schlieder [2] derived the following

Proposition (Schlieder). *Suppose $O_1 \divideontimes O_2$. If $x \in \mathfrak{A}(O_1)$ and $y \in \mathfrak{A}(O_2)$ are non-vanishing elements, then $xy \neq 0$.*

Schlieder also pointed out that the property $xy \neq 0$ for non-vanishing pairs of elements of two commuting algebras $\mathfrak{A}_1, \mathfrak{A}_2$ is a necessary condition for statistical independence. We shall show here that this property is also a sufficient condition. One has

Theorem 1. *Let $\mathfrak{A}, \mathfrak{A}_1, \mathfrak{A}_2$ be C^*-algebras with unit elements and let $\mathfrak{A}_i \subset \mathfrak{A}$.*

Suppose

(C): $\mathfrak{A}_1$ and $\mathfrak{A}_2$ commute elementwise.

Then $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are statistically independent if and only if they have the property (S): If x and y are non-vanishing elements of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ respectively, then $xy \neq 0$.

In addition, we shall show

Proposition 1. *Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be statistically independent, $\mathfrak{A}_1$ and $\mathfrak{A}_2$ commuting, $\mathfrak{A}_i \subset \mathfrak{A}$. If φ_1 is a pure state over $\mathfrak{A}_1$ and φ_2 is a pure state over $\mathfrak{A}_2$, then there exists an extension φ of φ_1 and φ_2 which is a pure state over $\mathfrak{A}$.*

II.

In this section and in the following one, we shall prove some lemmas and another theorem which will finally yield the proofs of Theorem 1 and Proposition 1. The first essential step is the demonstration of the following

Lemma 1. *Let $\mathfrak{A}, \mathfrak{A}_1, \mathfrak{A}_2$ be as in Theorem 1, satisfying (C) and (S). Suppose $\sum_{i=1}^{n} x_i y_i = 0$ with $x_i \in \mathfrak{A}_1$, $y_i \in \mathfrak{A}_2$. Then, unless all $x_i = 0$ or all $y_i = 0$, neither the $\{x_i, i = 1, \ldots n\}$ nor the $\{y_i, i = 1, \ldots n\}$ can be linearly independent.*

We need another lemma to prove this. Let $\mathfrak{B}_i$ be an abelian C^*-subalgebra of $\mathfrak{A}_i$, $i = 1, 2$; let $\mathfrak{B}_i^*$ be its spectrum, that is, the set of all characters of $\mathfrak{B}_i$ with the weak topology [3]. The elements of $\mathfrak{B}_1^*$ and $\mathfrak{B}_2^*$ may be denoted by χ' and χ'' respectively. Since $\mathfrak{B}_1$ and $\mathfrak{B}_2$ commute, they generate an abelian C^*-subalgebra $\mathfrak{B}_{12}$ of $\mathfrak{A}$, $\mathfrak{B}_{12}^*$ denoting its spectrum. A character $\chi \in \mathfrak{B}_{12}^*$, restricted to $\mathfrak{B}_i$, clearly defines an element of $\mathfrak{B}_i^* : \chi | \mathfrak{B}_i \in \mathfrak{B}_i^*$. Now define the subset $\mathscr{M}$ of the topological product $\mathfrak{B}_1^* \times \mathfrak{B}_2^*$ by

$$\mathscr{M} = \{(\chi | \mathfrak{B}_1, \chi | \mathfrak{B}_2) | \chi \in \mathfrak{B}_{12}^*\}.$$

Lemma 2. *If (S) is satisfied, then $\mathscr{M}$ is dense in $\mathfrak{B}_1^* \times \mathfrak{B}_2^*$.*

Proof. Assume the contrary. Then we can find an element (χ_0', χ_0'') and a neighbourhood $U((\chi_0', \chi_0''))$ such that $\mathscr{M} \cap U = \emptyset$. U contains a neighbourhood $U_1(\chi_0') \times U_2(\chi_0'')$. Define continuous functions $f(\chi')$ and $g(\chi'')$ over $\mathfrak{B}_1$ and $\mathfrak{B}_2$ respectively, with $\operatorname{supp} f \subset U_1$, $\operatorname{supp} g \subset U_2$. As is well known, $\mathfrak{B}_i$ is isomorphic to the C^*-algebra of continuous complex functions over $\mathfrak{B}_i^*$ vanishing at infinity; the isomorphism is furnished by the Gelfand transformation ([4], Theorem 1.4.1). Therefore, if f and g do not vanish identically, they are Gelfand transforms of elements

$x \in \mathfrak{B}_1$ and $y \in \mathfrak{B}_2$. Consider $\chi(xy)$ for arbitrary $\chi \in \mathfrak{B}_{12}^*$. Clearly,

$$\chi(xy) = \chi(x)\,\chi(y) = f(\chi|\mathfrak{B}_1)\,g(\chi|\mathfrak{B}_2) = 0$$

because of our assumption $\mathscr{M} \cap U = \emptyset$ and the support properties of f and g. Hence $xy = 0$, $x \neq 0$, $y \neq 0$, which contradicts the property (S).

Proof of Lemma 1. (i) The main task is to prove the lemma for commuting x_i and commuting y_i. Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be abelian C^*-subalgebras of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ containing $\{x_i\}$ and $\{y_i\}$ respectively. $\sum_{i=1}^{n} x_i y_i = 0$ implies

$$\chi\left(\sum_{i=1}^{n} x_i y_i\right) = \sum_{i=1}^{n} \chi(x_i)\,\chi(y_i) = \sum_{i=1}^{n} \chi|\mathfrak{B}_1(x_i)\,\chi|\mathfrak{B}_2(y_i) = 0$$

for all $\chi \in \mathfrak{B}_{12}^*$ and, with the help of Lemma 2,

$$\sum_{i=1}^{n} \chi'(x_i)\,\chi''(y_i) = 0 \quad \text{for all} \quad \chi' \in \mathfrak{B}_1^*,\ \chi'' \in \mathfrak{B}_2^*. \tag{1}$$

Unless all $y_i = 0$, we can find a χ_0'' such that not all $\chi_0''(y_i)$ vanish. With $\gamma_i = \chi_0''(y_i)$ we have

$$\chi'\left(\sum \gamma_i x_i\right) = \sum \chi'(x_i)\,\chi_0''(y_i) = 0 \quad \text{for all} \quad \chi' \in \mathfrak{B}_1^*,$$

and therefore, $\sum \gamma_i x_i = 0$. Due to the symmetry of Eq. (1) with respect to $\{x_i\}$ and $\{y_i\}$, the $\{y_i\}$ are linearly dependent, too.

(ii) Now let us consider x_i, y_i which do not all commute with each other, with $\sum_{i=1}^{n} x_i y_i = 0$. Without loss of generality, we may assume that there exists a y_{k_0} such that not all $y_i' = [y_i, y_{k_0}]$ vanish, and we have

$$\sum_{\substack{i=1 \\ i \neq k_0}}^{n} x_i y_i' = 0\,. \tag{2}$$

Trivially, the lemma is true for $n = 1$. Suppose it is proven for $\nu \leq n-1$. Because the sum in (2) contains less than n terms, the $\{x_i, i \neq k_0\}$ and, of course, the $\{x_i, i = 1, \ldots n\}$ are linearly dependent. Let $\gamma_{i_0} \neq 0$, $c_i = \gamma_i/\gamma_{i_0}$, $x_{i_0} = -\sum_{i \neq i_0} c_i x_i$. It follows that $\sum_{i \neq i_0} x_i(y_i - c_i y_{i_0}) = 0$. Then either all $y_i = c_i y_{i_0}$, which gives us already the desired linear dependence of the $\{y_i\}$ or not all $(y_i - c_i y_{i_0}) = 0$; and therefore, since we have less than n terms, we can find non-trivial β_i with

$$\sum_{i \neq i_0} \beta_i y_i + \left(\sum_{i \neq i_0} \beta_i c_i\right) y_{i_0} = \sum_{i \neq i_0} \beta_i (y_i - c_i y_{i_0}) = 0\,.$$

This proves Lemma 1 [5].

Now it is easy to demonstrate

Proposition 2. *Let $\mathfrak{A}, \mathfrak{A}_1, \mathfrak{A}_2$ be as in Lemma 1, satisfying (C) and (S). Suppose $\sum_{i=1}^{n} x_i y_i = 0$, $x_i \in \mathfrak{A}_1$, $y_i \in \mathfrak{A}_2$, not all $x_i = 0$, not all $y_i = 0$. Then there exist non-trivial complex numbers α_{ik} such that*

$$\sum_{i=1}^{n} \alpha_{ik} x_i = 0, \qquad k = 1, \ldots n, \tag{3}$$

$$\sum_{k=1}^{n} \alpha_{ik} y_k = y_i, \qquad i = 1, \ldots n. \tag{4}$$

α_{ik} are called non-trivial if

1) not all α_{ik} vanish,
2) not all $\alpha_{ik} = \delta_{ik} = \begin{cases} 0, & i \neq k \\ 1, & i = k \end{cases}$.

Proposition 2 is so to speak symmetric in $\{x_i\}$ and $\{y_i\}$ because with $\alpha'_{ik} = -\alpha_{ki} + \delta_{ki}$ we have

$$\sum_i \alpha'_{ik} y_i = 0, \quad k = 1, \ldots n; \quad \sum_k \alpha'_{ik} x_k = x_i, \quad i = 1, \ldots n,$$

with non-trivial α'_{ik}.

Proof by induction. $n = 1$ is evident due to assumption (*S*). Let the assertion be proven for $v \leqq n-1$. $v = n$: According to Lemma 1, $\{x_i\}$ are linearly dependent; without loss of generality, let us assume that $x_1 = -\sum_{i=2}^{n} \gamma_i x_i$. This implies $\sum_{i=2}^{n} x_i (y_i - \gamma_i y_1) = 0$. If not all $y_i = \gamma_i y_1$, there exist non-trivial numbers β_{ik} with

$$\sum_{i=2}^{n} \beta_{ik} x_i = 0, \quad k = 2, \ldots n; \quad \sum_{k=2}^{n} \beta_{ik} (y_k - \gamma_k y_1) = y_i - \gamma_i y_1, \quad i = 2, \ldots n,$$

since we assume that the proposition is true for $v \leqq n-1$. If one puts

$$\begin{aligned} \alpha_{11} &= 1, \\ \alpha_{1k} &= 0, \quad k = 2, \ldots n, \\ \alpha_{i1} &= \gamma_i - \sum_{k=2}^{n} \beta_{ik} \gamma_k, \quad i = 2, \ldots n, \\ \alpha_{ik} &= \beta_{ik}, \quad i, k \geqq 2, \end{aligned}$$

one can directly verify that Eqs. (3) and (4) hold. Clearly, α_{ik} are non-trivial because β_{ik} are non-trivial. If $y_i = \gamma_i y_1$ for all $i = 2, \ldots n$, then $\left(x_1 + \sum_{i=2}^{n} \gamma_i x_i\right) y_1 = 0$ and, due to (*S*), $y_1 = 0$. Thus the problem is reduced to the case $v \leqq n-1$; and if $\sum \alpha_{ik} x_i = 0$, $\sum \alpha_{ik} y_k = y_i$ for $i, k \geqq 2$, (3) and (4) hold for $i, k = 1, \ldots n$ with $\alpha_{1k} = \alpha_{i1} = 0$.

Proposition 2 implies the following

Corollary. *Let $\mathfrak{A}$, $\mathfrak{A}_1$, $\mathfrak{A}_2$ be C^*-algebras with unit elements, $\mathfrak{A}_i \subset \mathfrak{A}$. If (C) and (S) are fulfilled, $\mathfrak{A}_1 \vee \mathfrak{A}_2$ is isomorphic to $\mathfrak{A}_1 \odot \mathfrak{A}_2$.*

Here $\mathfrak{A}_1 \vee \mathfrak{A}_2$ denotes the normed involutive subalgebra of $\mathfrak{A}$ generated by $\mathfrak{A}_1$ and $\mathfrak{A}_2$; $\mathfrak{A}_1 \odot \mathfrak{A}_2$ denotes the direct algebraic product of $\mathfrak{A}_1$ and $\mathfrak{A}_2$, that is, the set of all formal finite sums $\sum x_i \otimes y_i$ with

$$\left(\sum_i x_i \otimes y_i\right)\left(\sum_j x'_j \otimes y'_j\right) = \sum_{i,j} x_i x'_j \otimes y_i y'_j; \qquad \left(\sum x_i \otimes y_i\right)^* = \sum x_i^* \otimes y_i^*.$$

($\sum x_i y_i$ and $\sum x_i \otimes y_i$ are always finite sums).

The isomorphism is given by $\Phi(\sum x_i y_i) = \sum x_i \otimes y_i$.

We have to show that $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are algebraically independent [6], that is, if $\{x_i,\ i=1, \ldots n\}$ and $\{y_j,\ j=1, \ldots m\}$ are sets of linearly independent elements of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ respectively, then $\{x_i y_j,\ i=1, \ldots n,\ j=1, \ldots m\}$ is a linearly independent set in $\mathfrak{A}$. Assume the existence of numbers $\varkappa_{ij}$ with $\sum_{i,j} \varkappa_{ij} x_i y_j = 0$. Then $\sum_j x'_j y_j = 0$, with $x'_j = \sum_i \varkappa_{ij} x_i$. Unless all $x'_j = 0$, there are non-trivial α_{jk} such that $\sum_k \alpha_{jk} y_k = y_j$, which contradicts the linear independence of $\{y_j\}$. Hence $x'_j = \sum \varkappa_{ij} x_i = 0, j = 1, \ldots m$, and because of the linear independence of $\{x_i\}$ we get $\varkappa_{ij} = 0$.
As one can check easily, algebraic independence of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ implies that $\mathfrak{A}_1 \vee \mathfrak{A}_2$ and $\mathfrak{A}_1 \odot \mathfrak{A}_2$ are isomorphic (cf. [6]).

III.

The second essential step in proving Theorem 1 is to establish the continuity of the isomorphism Φ of $\mathfrak{A}_1 \vee \mathfrak{A}_2$ and $\mathfrak{A}_1 \odot \mathfrak{A}_2$.

We shall use the following notations:

$\mathfrak{A}_{12} \equiv \overline{\mathfrak{A}_1 \vee \mathfrak{A}_2}$ denotes the norm-closure of $\mathfrak{A}_1 \vee \mathfrak{A}_2$, that is, the C^*-subalgebra of $\mathfrak{A}$ generated by $\mathfrak{A}_1$ and $\mathfrak{A}_2$.

If we define a norm β on $\mathfrak{A}_1 \odot \mathfrak{A}_2$, the completion of $\mathfrak{A}_1 \odot \mathfrak{A}_2$ with respect to this norm is denoted by $\mathfrak{A}_1 \otimes_\beta \mathfrak{A}_2$.

Definition 1. α-norm [7, 8]:

$$\left\| \sum_{i=1}^{n} x_i \otimes y_i \right\|_\alpha = \sup \left\{ \frac{\varphi_1 \otimes \varphi_2 \left[\left(\sum_{i=1}^{m} a_i \otimes b_i \right)^* \left(\sum_{i=1}^{n} x_i \otimes y_i \right)^* \left(\sum_{i=1}^{n} x_i \otimes y_i \right) \left(\sum_{i=1}^{m} a_i \otimes b_i \right) \right]}{\varphi_1 \otimes \varphi_2 \left[\left(\sum_{i=1}^{m} a_i \otimes b_i \right)^* \left(\sum_{i=1}^{m} a_i \otimes b_i \right) \right]} \right\}^{\frac{1}{2}},$$

with $x_i \in \mathfrak{A}_1$, $y_i \in \mathfrak{A}_2$; the supremum is taken over all states φ_1 over $\mathfrak{A}_1$, all states φ_2 over $\mathfrak{A}_2$ and all $a_i \in \mathfrak{A}_1$, $b_i \in \mathfrak{A}_2$. Furthermore,

$$\varphi_1 \otimes \varphi_2 [(\sum a_i \otimes b_i)] = \sum \varphi_1(a_i)\, \varphi_2(b_i)\,.$$

If $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are algebras of operators in a Hilbert space $\mathscr{H}$, $\mathfrak{A}_1 \odot \mathfrak{A}_2$ is an operator algebra in $\mathscr{H} \otimes \mathscr{H}$. In this case, the α-norm is identical with the natural norm in $\mathscr{H} \otimes \mathscr{H}$ (theorem of Wulfsohn [9]).

We want to show that Φ is continuous with respect to the α-norm topology in $\mathfrak{A}_1 \odot \mathfrak{A}_2$. We need some definitions and theorems which can be found in mathematical literature, and which are cited below.

Definition 2 [8]. A norm β of $\mathfrak{A}_1 \odot \mathfrak{A}_2$ is called compatible (with the algebraic structure of $\mathfrak{A}_1 \odot \mathfrak{A}_2$) if the completion of $\mathfrak{A}_1 \odot \mathfrak{A}_2$ with respect to β becomes a C^*-algebra, and if $\|x \otimes y\|_\beta \leqq \|x\|\, \|y\|$.

Definition 3 [10]. A B^*-norm means any norm $\|\ldots\|_\beta$ satisfying $\|u^* u\|_\beta = \|u\|_\beta^2$ for all $u \in \mathfrak{A}_1 \odot \mathfrak{A}_2$.

Proposition (Okayasu) [10]. *Every B^*-norm on $\mathfrak{A}_1 \odot \mathfrak{A}_2$ is compatible.*

Theorem (Takesaki and Okayasu) [8, 10]. *Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be C^*-algebras. Then the set of all B^*-norms on $\mathfrak{A}_1 \odot \mathfrak{A}_2$ is a complete lattice under the ordering "$\leqq$" with the least element $\|\ldots\|_\alpha$.*

Here $\beta_1 \leqq \beta_2$ means $\|u\|_{\beta_1} \leqq \|u\|_{\beta_2}$ for all $u \in \mathfrak{A}_1 \odot \mathfrak{A}_2$.

We define

$$\|\sum x_i \otimes y_i\|_\beta = \|\sum x_i y_i\| \tag{5}$$

and assert

Lemma 3. *The norm β defined in* (5) *is a B^*-norm on $\mathfrak{A}_1 \odot \mathfrak{A}_2$.*

Proof. Because of the isomorphism of $\mathfrak{A}_1 \vee \mathfrak{A}_2$ and $\mathfrak{A}_1 \odot \mathfrak{A}_2$, (5) defines a norm on $\mathfrak{A}_1 \odot \mathfrak{A}_2$; and

$$\begin{aligned}\|(\sum x_i \otimes y_i)^* (\sum x_i \otimes y_i)\|_\beta &= \Big\|\sum_{i,j} x_i^* x_j \otimes y_i^* y_j\Big\|_\beta = \Big\|\sum_{i,j} x_i^* x_j y_i^* y_j\Big\| \\ &= \|(\sum x_i y_i)^* (\sum x_i y_i)\| = \|\sum x_i y_i\|^2 = \|\sum x_i \otimes y_i\|_\beta^2\,,\end{aligned}$$

since $\mathfrak{A}_1 \vee \mathfrak{A}_2$ is contained in a C^*-algebra $\mathfrak{A}_{12}$.

Hence β is compatible, and, according to the theorem of Takesaki and Okayasu, we have

$$\|\sum x_i \otimes y_i\|_\alpha \leqq \|\sum x_i \otimes y_i\|_\beta = \|\sum x_i y_i\|\,. \tag{6}$$

The isomorphism Φ can then be extended to a morphism

$$\bar{\Phi}\colon \mathfrak{A}_{12} = \overline{\mathfrak{A}_1 \vee \mathfrak{A}_2} \to \mathfrak{A}_1 \otimes_\alpha \mathfrak{A}_2\,.$$

Actually, $\bar{\Phi}$ is a homomorphism because it is surjective: for $\bar{\Phi}(\mathfrak{A}_{12})$ is closed ([4], Corollary 1.3.3) and contains $\mathfrak{A}_1 \odot \mathfrak{A}_2$ which is dense in $\mathfrak{A}_1 \otimes_\alpha \mathfrak{A}_2$.

We collect our results formulating

Theorem 2. *Let* $\mathfrak{A}$, $\mathfrak{A}_1$, $\mathfrak{A}_2$ *be C*-algebras with unit elements,* $\mathfrak{A}_i \subset \mathfrak{A}$. *Assume*

(C) $\mathfrak{A}_1$ *and* $\mathfrak{A}_2$ *commute elementwise.*

(S) If x and y are non-vanishing elements of $\mathfrak{A}_1$ *and* $\mathfrak{A}_2$ *respectively, then* $xy \neq 0$.

Then we have

1) *There exists an isomorphism* $\Phi : \mathfrak{A}_1 \vee \mathfrak{A}_2 \to \mathfrak{A}_1 \odot \mathfrak{A}_2$.

2) Φ *is continuous with respect to the* α*-norm on* $\mathfrak{A}_1 \odot \mathfrak{A}_2$ *and can therefore be extended to a homomorphism* $\bar{\Phi} : \mathfrak{A}_{12} \to \mathfrak{A}_1 \otimes_\alpha \mathfrak{A}_2$.

3) *Let* $\mathfrak{M}$ *be any abelian C*-subalgebra of* $\mathfrak{A}_1$. *The restriction of* $\bar{\Phi}$ *to* $\overline{\mathfrak{M} \vee \mathfrak{A}_2}$ *is an isomorphism,* $\bar{\Phi}(\overline{\mathfrak{M} \vee \mathfrak{A}_2}) = \mathfrak{M} \otimes_\alpha \mathfrak{A}_2$.

Parts 1) and 2) are proven. The third part follows from another theorem of Takesaki:

Theorem (Takesaki) [8]. *Let* $\mathfrak{A}_1$ *be an abelian C*-algebra. Then, for any C*-algebra* $\mathfrak{A}_2$, *the* α*-norm is the only compatible norm on* $\mathfrak{A}_1 \odot \mathfrak{A}_2$.

Therefore, since we know that the norm β defined in (5) is compatible, we have for $x_i \in \mathfrak{M}$

$$\|\sum x_i \otimes y_i\|_\alpha = \|\sum x_i \otimes y_i\|_\beta = \|\sum x_i y_i\| ;$$

and this implies that the restriction of $\bar{\Phi}$ to $\overline{\mathfrak{M} \vee \mathfrak{A}_2}$ is an isomorphism of $\overline{\mathfrak{M} \vee \mathfrak{A}_2}$ and $\mathfrak{M} \otimes_\alpha \mathfrak{A}_2$.

This completes the proof of Theorem 2.

IV.

Finally, we shall prove Theorem 1 and Proposition 1. As already mentioned, Schlieder [2] showed that (S) is a necessary condition. (The proof given in [2] is not a quite general one, for one needs the existence of sufficiently many hermitian elements $x \in \mathfrak{A}_1$ and $y \in \mathfrak{A}_2$ with $x^2 = x$, $y^2 = y$; its generalization is given in the appendix.)

Now let us assume that (*S*) is satisfied; so we can use theorem 2. Let $\tilde{\varphi}$ be any continuous linear functional over $\mathfrak{A}_1 \otimes_\alpha \mathfrak{A}_2$. Then we define a linear functional φ over $\mathfrak{A}_{12}$ by

$$\varphi(u) = \tilde{\varphi}(\bar{\Phi}(u)), \quad u \in \mathfrak{A}_{12}; \quad \text{in short:} \quad \varphi = \tilde{\varphi} \circ \bar{\Phi}. \tag{7}$$

$\bar{\Phi}$ is continuous; therefore, φ is continuous. Clearly, if $\tilde{\varphi}$ is positive, so is φ, since $u \geqq 0$, $u \in \mathfrak{A}_{12}$, implies $\bar{\Phi}(u) \geqq 0$. Put $\tilde{\varphi} = \varphi_1 \otimes \varphi_2$, φ_1 and φ_2 arbitrary states over $\mathfrak{A}_1$ and $\mathfrak{A}_2$ respectively, then

$$\varphi = \varphi_1 \otimes \varphi_2 \circ \bar{\Phi} \tag{8}$$

is the functional over $\mathfrak{A}_{12}$ required by statistical independence:

$$x \in \mathfrak{A}_1 \colon \varphi(x) = \tilde{\varphi}(\bar{\Phi}(x)) = \tilde{\varphi}(x \otimes \mathbf{1}) = \varphi_1(x);$$
$$y \in \mathfrak{A}_2 \colon \varphi(y) = \tilde{\varphi}(\bar{\Phi}(y)) = \tilde{\varphi}(\mathbf{1} \otimes y) = \varphi_2(y).$$

It remains to be checked whether $\varphi_1 \otimes \varphi_2$ is continuous and positive if φ_1 and φ_2 are continuous and positive. The continuity is a direct consequence of the Definition 1 of the α-norm; the positivity follows from an easily provable lemma:

Lemma 4 [6]. *If φ_1 and φ_2 are positive functionals over $\mathfrak{A}_1$ and $\mathfrak{A}_2$ respectively, then $\varphi_1 \otimes \varphi_2$ is a positive functional over $\mathfrak{A}_1 \odot \mathfrak{A}_2$.*

Because of the continuity, $\varphi_1 \otimes \varphi_2$ is also positive over $\mathfrak{A}_1 \otimes_\alpha \mathfrak{A}_2$. This proves the statistical independence of $\mathfrak{A}_1$ and $\mathfrak{A}_2$, since the state φ over $\mathfrak{A}_{12}$ defined in (8) can be extended to a state over $\mathfrak{A}$. We note that

$$\varphi(xy) = \varphi_1(x)\,\varphi_2(y) = \varphi(x)\,\varphi(y), \qquad x \in \mathfrak{A}_1, \qquad y \in \mathfrak{A}_2. \tag{9}$$

Proof of Proposition 1. Let $\mathscr{E}(\mathfrak{A})$ denote the set of states over $\mathfrak{A}$ and $\mathscr{P}(\mathfrak{A})$ the subset of pure states. If φ_1 and φ_2 are pure states, they define irreducible representations π_{φ_1} and π_{φ_2} of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ respectively. The representation π of $\mathfrak{A}_1 \otimes_\alpha \mathfrak{A}_2$, defined by $\varphi_1 \otimes \varphi_2$ is isomorphic to $\pi_{\varphi_1}(\mathfrak{A}_1) \otimes \pi_{\varphi_2}(\mathfrak{A}_2)$, therefore, π is irreducible and $\varphi_1 \otimes \varphi_2 \in \mathscr{P}(\mathfrak{A}_1 \otimes_\alpha \mathfrak{A}_2)$. According to Theorem 2, $\mathfrak{A}_{12}/\mathrm{Ker}\,\bar{\Phi}$ and $\mathfrak{A}_1 \otimes_\alpha \mathfrak{A}_2$ are isomorphic; so $\tilde{\varphi} \to \tilde{\varphi} \circ \bar{\Phi}$ defines an isomorphism Φ' of $\mathscr{E}(\mathfrak{A}_1 \otimes_\alpha \mathfrak{A}_2)$ and $\mathscr{E}(\mathfrak{A}_{12}/\mathrm{Ker}\,\bar{\Phi})$, which transforms pure states into pure states. Therefore, $\varphi = \varphi_1 \otimes \varphi_2 \circ \bar{\Phi}$ is an element of $\mathscr{P}(\mathfrak{A}_{12}/\mathrm{Ker}\,\bar{\Phi})$. (Here we identify $\mathscr{E}(\mathfrak{A}_{12}/\mathrm{Ker}\,\bar{\Phi})$ with the set $\mathscr{E}_0 = \{\chi \mid \chi \in \mathscr{E}(\mathfrak{A}_{12}),\ \chi(\mathrm{Ker}\,\bar{\Phi}) = 0\}$.) Now consider φ as a state over $\mathfrak{A}_{12}$ and suppose that φ majorizes a state $\varphi' \in \mathscr{P}(\mathfrak{A}_{12})$. Since $\varphi(x) = 0$ for all $x \in \mathrm{Ker}\,\bar{\Phi}$, the same holds for φ', which implies $\varphi' \in \mathscr{P}(\mathfrak{A}_{12}/\mathrm{Ker}\,\bar{\Phi})$. But this is a contradiction unless $\varphi' = \varphi$; and therefore, $\varphi \in \mathscr{P}(\mathfrak{A}_{12})$. Any pure state over $\mathfrak{A}_{12}$ can be extended to a pure state over $\mathfrak{A}$; which completes the proof.

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Appendix

Let $\mathfrak{A}_1$, $\mathfrak{A}_2$ be commuting C^*-algebras with unit elements, $\mathfrak{A}_i \subset \mathfrak{A}$, and let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be statistically independent. We want to show that $xy \neq 0$ whenever $x \in \mathfrak{A}_1, y \in \mathfrak{A}_2, x$ and $y \neq 0$.

Assume that we can find non-vanishing elements $x' \in \mathfrak{A}_1$ and $y' \in \mathfrak{A}_2$ with $x'y' = 0$. Then of course $x'^* x' y'^* y' = 0$. Let $\alpha \in \mathrm{Sp}(x'^* x')$, $\alpha \neq 0$ (Sp u denotes the spectrum of u in $\mathfrak{A}_1$). Then for $x = \alpha^{-1}\, x'^*\, x \in \mathfrak{A}_1$,

$y = y'^* y' \in \mathfrak{A}_2$, we have

$$xy = 0, \quad x \neq 0, \quad y \neq 0, \tag{i}$$

$$x^* = x, \quad 1 \in \operatorname{Sp} x, \tag{ii}$$

and therefore,

$$z \equiv (1 - x)^2 \geqq 0, \quad 0 \in \operatorname{Sp} z, \quad \alpha \in \operatorname{Sp}(z + \alpha). \tag{iii}$$

Consider the selfadjoint vector space $\mathscr{D}$ spanned by $\{\mathbf{1}, z\}$ and define $\varphi_1(\mathbf{1}) = 1$, $\varphi_1(z) = 0$. φ_1 is a positive functional on $\mathscr{D}$ because, according to (iii), $\gamma_1 \cdot \mathbf{1} + \gamma_2 z \geqq 0$ implies $\gamma_1/\gamma_2 \geqq 0$ if $\gamma_2 \neq 0$, hence, $\varphi_1(\gamma_1 \mathbf{1} + \gamma_2 z) = \gamma_1 \geqq 0$. As is well known (cf. [4], Lemma 2.10.1), φ_1 can be extended to a state over $\mathfrak{A}_1$, and we have

$$\varphi_1((1 - x)^2) = 0 \tag{iv}$$

and because of $|\varphi_1(u)|^2 \leqq \|\varphi_1\| \, \varphi_1(u^* u)$:

$$\varphi_1(1 - x) = 0 \, . \tag{v}$$

It is clear that we can find a state φ_2 over $\mathfrak{A}_2$ with $\varphi_2(y) \neq 0$.

Since $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are statistically independent, there exists a common extension φ of φ_1 and φ_2. The Schwartz inequality implies

$$|\varphi((1 - x)(1 + y))|^2 \leqq \varphi((1 - x)^2) \, \varphi((1 + y)^2) = \varphi_1((1 - x)^2) \, \varphi_2((1 + y)^2) \, .$$

Hence, according to (iv), $\varphi((1 - x)(1 + y)) = 0$. However,

$$\varphi((1 - x)(1 + y)) = \varphi(1 - x + y) = \varphi_1(1 - x) + \varphi_2(y) = \varphi_2(y) \neq 0$$

according to (i) and (v), which is a contradiction.

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