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A Class of Field Theories in Two-Dimensional Space-Time with a $U_1 \times U_1$ Symmetry

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Abstract. The derivative coupling of massless pseudoscalar neutral particles with a charged spinor field in two-dimensional space-time is reduced to a self-interacting spinor field and a free pseudoscalar field.

More generally, it is shown that any given local field theory with a conserved vector current and without massless particles can be extended to a local theory with an additional pseudoscalar field and with a $U_1 \times U_1$ symmetry.

I. Introduction

Our interest in the two-dimensional models to be considered arose from a desire to clarify in a simple case the structure of a quantum field theory with a conserved current which does not lead to an ordinary symmetry. We have in mind such objects as the axial-vector currents which exist in the $SU_2 \times SU_2$ invariant Lagrangian models of the interaction of nucleons with massless pseudoscalar mesons. These models are greatly simplified if the Lagrangian contains only neutral mesons which interact with a charged spinor field. This reduces the symmetry to a $U_1 \times U_1$ structure.

As is well known [1], a typical Lagrangian is then given by

$$L = \overline{\psi} i\gamma \partial \psi + 1/2\, \partial^\nu \varphi\, \partial \nu \varphi - m \overline{\psi} e^{-2if\gamma^5 \varphi} \psi .$$

The transformations

$$\psi \to e^{iu} \psi , \qquad \varphi \to \varphi$$

and

$$\psi \to e^{i\gamma^5 v} \psi , \qquad \varphi \to \varphi + v/f$$

give rise to a conserved vector

$$V^\mu = \overline{\psi} \gamma^\mu \psi$$

and a conserved axial vector

$$A^\mu = \overline{\psi} \gamma^5 \gamma^\mu \psi + 1/f\, \partial^\mu \varphi .$$

Further, the Lagrangian can be transformed to ordinary $ps(pv)$ coupling by means of

$$\psi \to e^{if\gamma^5\varphi}\psi .$$

This leads to

$$L = \overline{\psi}(i\gamma\partial - m)\psi + 1/2\,\partial^\nu\varphi\,\partial_\nu\varphi + f\,\overline{\psi}\gamma^5\gamma\partial\varphi\psi .$$

The axial transformation does not involve the new spinor field, its only effect is to change the meson field by an additive constant.

We proceed then to study this Lagrangian in two-dimensional space-time. Actually, we discuss a more general model by adding a term

$$L' = \frac{c\cdot f^2}{2}(\overline{\psi}\gamma^5\gamma^\mu\psi)^2$$

which turns out to be a natural generalization. Two-dimensional space-time has special features which allow a partial solution of these models. In fact, they will be reduced to a self-interacting Fermifield and effectively free mesons. This implies that the conserved axial current has a very simple structure which differs essentially from that in four-dimensional space-time.

It will be apparent that the class of two-dimensional models with these general features is considerably larger than has been indicated so far.

II. Classical Fields

In this section we discuss the Lagrangian

$$L = \overline{\psi}(i\gamma\partial - m)\psi + 1/2\,\partial^\nu\varphi\,\partial_\nu\varphi + f\,\overline{\psi}\gamma^5\gamma\,\partial\varphi\psi + \frac{cf^2}{2}(\overline{\psi}\gamma^5\gamma^\nu\psi)^2 \quad (1)$$

as a classical field theory in two-dimensional space-time.

The conserved currents are

$$V^\mu = \overline{\psi}\gamma^\mu\psi , \quad (2)$$

$$A^\mu = \overline{\psi}\gamma^5\gamma^\mu\psi + 1/f\,\partial^\mu\varphi . \quad (3)$$

The equations of motion:

$$\partial_\mu\,\partial^\mu\varphi = -f\,\partial_\mu(\overline{\psi}\gamma^5\gamma^\mu\psi) = 2imf\,\overline{\psi}\gamma^5\psi , \quad (4)$$

$$(i\gamma\partial - m)\psi + f\gamma^5\gamma\,\partial\varphi\psi + cf^2(\overline{\psi}\gamma^5\gamma^\mu\psi)\gamma^5\gamma_\mu\psi = 0 . \quad (5)$$

We make use of the antisymmetric tensor

$$\varepsilon^{\mu\nu} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{6}$$

and note the relations

$$\gamma^5 \gamma^\mu = \varepsilon^{\mu\nu} \gamma_\nu \tag{7}$$

and

$$W^\mu = \overline{\psi} \gamma^5 \gamma^\mu \psi = \varepsilon^{\mu\nu} V_\nu . \tag{8}$$

The equation $\partial^\mu V_\mu = 0$ leads to

$$\partial^0 W^1 - \partial^1 W^0 = 0 . \tag{9}$$

This suggests

$$W^\mu = \partial^\mu \sigma . \tag{10}$$

Formal integration gives

$$\sigma(x) = 1/2 \int dx^{1\prime} \varepsilon(x^1 - x^{1\prime}) V^0(x^0, x^{1\prime}) . \tag{11}$$

Eqs. (3) and (10) combine to

$$A^\mu = \partial^\mu \sigma + 1/f \, \partial^\mu \varphi = 1/f \, \partial^\mu \alpha \tag{12}$$

where α is a free pseudoscalar field since $\partial^\mu A_\mu = 0$.

$$\partial^\mu \partial_\mu \alpha = 0 . \tag{13}$$

It will be shown that

$$\alpha = \varphi_{\text{in}} = \varphi_{\text{out}} . \tag{14}$$

Associated with α is a free scalar field β given by

$$\partial^\mu \beta = \varepsilon^{\mu\nu} \partial_\nu \alpha . \tag{15}$$

For the spinor field we obtain, using (12),

$$\begin{aligned} (i\gamma\partial - m)\psi + f\gamma^5\gamma\,\partial\alpha\psi + (c-1) f^2 (\overline{\psi}\gamma^5\gamma^\mu\psi)\gamma^5\gamma_\mu\psi = 0 , \\ (i\gamma\partial - m)\psi - f\gamma\,\partial\beta\psi - (c-1) f^2 (\overline{\psi}\gamma^\mu\psi)\gamma_\mu\psi = 0 . \end{aligned} \tag{16}$$

Defining a new spinor field χ by means of

$$\psi = e^{-if\beta}\chi \tag{17}$$

leads to

$$(i\gamma\partial - m)\chi = (c-1) f^2 (\overline{\chi}\gamma^\mu\chi)\gamma_\mu\chi . \tag{18}$$

χ is therefore a self-interacting Fermifield. The meson field φ can be defined in terms of χ (since $\psi^+\psi = \chi^+\chi$) and the free field α.

$$\begin{aligned} \varphi(x) = \alpha(x) - f/2 \int dx^{1\prime} \varepsilon(x^1 - x^{1\prime}) \chi^+ \chi(x^0, x^{1\prime}) , \\ \partial^\mu \partial_\mu \varphi = j = 2imf \overline{\chi}\gamma^5\chi . \end{aligned} \tag{19}$$

We show now that (19) implies $\varphi_{\text{in}} = \varphi_{\text{out}} = \alpha$. For this relation it is necessary that $\tilde{j}(k)$, the Fourier transform of $j(x)$ vanishes for $k^2 = 0$. This follows from (19):

$$\tilde{j}(k) = \text{const} \cdot \frac{(k^{02} - k^{12})}{k^1} \tilde{V}^0(k)$$

and, since

$$k^0 \tilde{V}^0 - k^1 \tilde{V}^1 = 0,$$

$$\tilde{j}(k) = \text{const} \cdot (k^0 - k^1)(\tilde{V}^0 + \tilde{V}^1) = \text{const} \cdot (k^0 + k^1)(\tilde{V}^0 - \tilde{V}^1).$$

Hence $\tilde{j}(k) = 0$ for $k^2 = 0$, unless $\tilde{V}^\mu$ in singular there.

To summarize these formal considerations:

We expect that the model given by the Lagrangian (1) implies, as shown by Eq. (18), non-trivial scattering and production amplitudes for the spinor particles. The mesons, while described by the non-free field φ(x), are effectively non-interacting ($\varphi_{\text{in}} = \varphi_{\text{out}}$)[1].

III. Local Quantum Fields

In the following we construct local quantum fields $\psi(x)$ and $\varphi(x)$ which correspond to the solution of our formal Lagrangian (1).

Of course, unless $c = 1$, we have to assume that an operator solution $\chi(x)$ exists for the renormalizable interaction formally given by Eq. (18). From now on, all operators and constants denote renormalized quantities.

It appears natural to treat the case $c = 1$ first, since it involves free fields only.

The Case $c = 1$. Let a free spinor field χ of mass m, a free pseudoscalar field α of mass zero and its associated scalar field β be given. For the definition of mass zero scalar or pseudoscalar fields we follow Wightman's procedure [3].

The connection between α and β in momentum space reads

$$\begin{aligned} \alpha(x) &= \frac{1}{(2\pi)^{1/2}} \int \frac{dk^1}{2|k^1|} \left\{ e^{-ikx} a(k^1) + e^{ikx} a^+(k^1) \right\}, \\ \beta(x) &= \frac{1}{(2\pi)^{1/2}} \int \frac{dk^1}{2|k^1|} \varepsilon(-k^1) \left\{ e^{-ikx} a(k^1) + e^{ikx} a^+(k^1) \right\}. \end{aligned} \tag{20}$$

This gives for the equal-time commutator

$$[\alpha(x), \beta(y)]_{x^0 = y^0} = -i/2\,\varepsilon(x^1 - y^1). \tag{21}$$

[1] However, their presence leads to an infraparticle situation as discussed by Schroer [2]. The quantized theory will not contain single particle states for the spinor particles. Therefore the scattering matrix cannot be defined as usual.

It is known [3] that the axial-vector associated with a free spinor field is indeed the derivative of a local pseudoscalar field:

$$W^\mu(x) = :\bar\chi(x)\gamma^5\gamma^\mu\chi(x): = \partial^\mu\sigma(x), \tag{22}$$

$$\sigma(x) = 1/2 \int dx^{1\prime}\varepsilon(x^1 - x^{1\prime}) : \chi^+\chi : (x^0, x^{1\prime}). \tag{23}$$

This is equivalent to Wightman's definition. The locality of $\sigma(x)$ follows from (23). However, as noted by Wightman [3], $\sigma(x)$ is not relatively local to χ. In fact, we get for the commutator

$$[\sigma(x), \chi(y)]_{x^0=y^0} = -1/2\,\varepsilon(x^1 - y^1)\chi(y). \tag{24}$$

Next, we define

$$\psi(x) = :e^{-if\beta(x)}:\chi(x) \tag{25}$$

and

$$\varphi(x) = \alpha(x) - f\sigma(x). \tag{26}$$

It is important that ψ and φ are not only local fields but also relatively local to each other. Namely,

$$\begin{aligned}[\varphi(x), \psi(y)]_{x^0=y^0} &= [\alpha(x), :e^{-if\beta(y)}:]\,\chi(y) - f:e^{-if\beta(y)}:[\sigma(x), \chi(y)] \\ &= -f/2\,\varepsilon(x^1 - y^1)\psi(y) + f/2\,\varepsilon(x^1 - y^1)\psi(y) = 0.\end{aligned}$$

We note further that $\psi(x)$ satisfies the positive definiteness condition [3]. Therefore, through their Wightman functions, ψ and φ generate a local theory.

It remains to show that ψ and φ satisfy field equations which correspond to the classical relations (4) and (5). Clearly a limiting procedure is necessary to define the operator products which will occur in these equations.

As a consequence of Eqs. (22) and (26)

$$\partial^\mu\partial_\mu\varphi = j = 2imf :\bar\chi\gamma^5\chi:$$

The definition of $j(x)$ in terms of ψ is simplified by the fact that

$$:\bar\chi\gamma^5\chi:(x) = \bar\chi(x)\gamma^5\chi(x) = \lim_{\substack{\varepsilon\to 0\\ \varepsilon^2<0}} \{\bar\chi(x+\varepsilon)\gamma^5\chi(x) + \bar\chi(x)\gamma^5\chi(x+\varepsilon)\}. \tag{27}$$

According to (25)

$$\begin{aligned}\bar\psi(x+\varepsilon)\gamma^5\psi(x) &= :e^{if\beta(x+\varepsilon)}: :e^{-if\beta(x)}: \bar\chi(x+\varepsilon)\gamma^5\chi(x) \\ &= e^{if^2 D^+(\varepsilon)} :e^{if[\beta(x+\varepsilon)-\beta(x)]}: \bar\chi(x+\varepsilon)\gamma^5\chi(x) \\ &= |\varepsilon^2|^{-f^2/4\pi} :e^{if[\beta(x+\varepsilon)-\beta(x)]}: \bar\chi(x+\varepsilon)\gamma^5\chi(x).\end{aligned}$$

This leads to

$$j(x) = imf \lim_{\varepsilon\to 0} |\varepsilon^2|^{f^2/4\pi}\{\bar\psi(x+\varepsilon)\gamma^5\psi(x) + \bar\psi(x)\gamma^5\psi(x+\varepsilon)\}. \tag{28}$$

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* We use the notation of Gelfand-Schilow [4].

To obtain an equation of motion for $\psi(x)$, we note that (25) implies

$$(i\gamma\partial - m)\,\psi(x) = g(x) = f\,{:}\gamma\partial\beta(x)\,e^{-if\beta(x)}{:}\,\chi(x) \tag{29}$$

and we use the identity

$${:}\partial_\mu\beta(y)\,e^{-if\beta(x)}{:}\,\chi(x) = \partial_\mu\beta(y)\,\psi(x) - f\,\partial_\mu D^+(y-x)\,\psi(x)\,. \tag{30}$$

Adding the two expressions with $y = x+\varepsilon$ and $y = x-\varepsilon$ we obtain in the limit

$$g(x) = f/2 \lim_{\varepsilon\to 0}\{\gamma\partial\beta(x+\varepsilon)\,\psi(x) + \gamma\partial\beta(x-\varepsilon)\,\psi(x)\}$$

and with

$$\gamma\partial\beta = -\gamma^5\gamma\partial\alpha = -\gamma^5\gamma\partial\varphi + f\gamma V,$$

$$g(x) = -f/2 \lim_{\varepsilon\to 0}\{\gamma^5\gamma\partial\varphi(x+\varepsilon) - f\gamma V(x+\varepsilon) + (\varepsilon\to-\varepsilon)\}\,\psi(x)\,. \tag{31}$$

The vector current $V^\mu = {:}\bar{\chi}\gamma^\mu\chi{:}$ which occurs in (31) can again be defined in terms of the interacting fields by an analogous procedure. The result is

$$\begin{aligned} V^\mu = \lim_{\substack{\delta\to 0\\ \delta^2<0}} \frac{1}{2(1+f^2/\pi)} &\left\{\left[g^{\mu\nu}(1+f^2/\pi) - f^2/\pi\,\frac{\delta^\mu\delta^\nu}{\delta^2}\right]\right. \\ &\left.\times\,|\delta^2|^{f^2/4\pi}[\bar{\psi}(x+\delta)\,\gamma_\nu\psi(x) + \bar{\psi}(x)\,\gamma_\nu\psi(x+\delta)] - \frac{2f}{\pi}\frac{\delta^\mu\delta_\nu}{\delta^2}\,\varepsilon^{\nu\varrho}\partial_\varrho\varphi(x)\right\}. \end{aligned} \tag{32}$$

The General Case. If $c \neq 1$, we make the following assumptions.

1. A local spinor field χ and a local conserved vector current $V^\mu\{\chi\}$, which correspond to the renormalized solution of the formal field equation

$$(i\gamma\partial - m_0)\,\chi = \lambda_0(\bar{\chi}\gamma^\mu\chi)\,\gamma_\mu\chi \tag{18'}$$

exist as operator-valued tempered distributions acting in a Hilbert space H equipped with a unique, cyclic vacuum state Φ_0. The charge Q, which is associated with V^μ, satisfies

$$[Q,\chi(x)] = -\chi(x), \qquad [Q, V^0(x)] = 0\,.$$

2. In the charge zero sector, the energy-momentum spectrum has a lowest mass $\mu > 0$.

We may then define an operator $\sigma(f)$, $f \in S^\star$ as follows:

Let B be a quasilocal operator and

$$\Phi = B\Phi_0\,,$$

$$\begin{aligned}\sigma(f)\,\Phi = 1/2 \textstyle\int dx^{1\prime}\varepsilon(x^{1\prime})\,[V^0 * f(0,x^{1\prime}), B]\,\Phi_0 \\ - B\textstyle\int d^2x'\bar{D} * f(x')\,\partial'_\mu\varepsilon^{\mu\nu}V_\nu(x')\,\Phi_0\,.\end{aligned} \tag{33}$$

To show that $\|\sigma(f)\Phi\| < \infty$, we note first that

$$\int d^2x' \bar{D} * f(x') \partial'_\mu \varepsilon^{\mu\nu} V_\nu(x') \Phi_0 = \int d^2x' \mathring{f}(x') \partial'_\mu \varepsilon^{\mu\nu} V_\nu(x') \Phi_0 = C_f \Phi_0$$

with $\mathring{f}(x) \in S$, and C_f therefore a quasilocal operator. In fact, due to the form of the spectrum,

$$F[\bar{D} * f](k) = 1/k^2 \tilde{f}(k)$$

can be replaced by

$$\mathring{f}(k) = g(k^2)\, 1/k^2 \tilde{f}(k)$$

where $g(h^2) \in C^\infty$ is real and

$$g(k^2) = 1, \quad k^2 > 2/3\mu^2,$$
$$g(k^2) = 0, \quad k^2 < 1/3\mu^2.$$

Hence

$$\mathring{f}(x) = F^{-1}[g(k^2) F[\bar{D} * f]] \in S.$$

Secondly, we note that the integral

$$\int dx^{1\prime} \varepsilon(x^{1\prime}) [V^0 * f(0, x^{1\prime}), B] \Phi_0$$

is strongly convergent. This follows by observing that

$$(\Phi_0, B'[V^0 * f(0, x^{1\prime}), B] \Phi_0) \in C^\infty$$

and the norm

$$\|[V^0 * f(0, x^{1\prime}), B] \Phi_0\|$$

decreases fast for $|x^{1\prime}| \to \infty$ due to the quasilocality of B.

It can be shown in a straightforward manner that σ has the following properties:

a) For each test function $f \in S$, $\sigma(f)$ is defined on a domain $D(\sigma)$ of vectors, dense in H. $D(\sigma)$ is a linear set containing the domain D formed by all finite linear combinations of vectors obtained by applying polynomials in the smeared $\chi, \bar{\chi}$ and $V^\mu\{\chi, \bar{\chi}\}$ fields to the vacuum state.

Furthermore, $D(\sigma)$ satisfies

$$U(a, \Lambda) D(\sigma) \subset D(\sigma), \quad O_j(f) D(\sigma) \subset D(\sigma), \quad O_j^+(f) D(\sigma) \subset D(\sigma)$$

where O_j denotes any one of the fields χ, V^μ, σ. The Wightman functions of $\chi, \bar{\chi}$, V^μ and σ are tempered distributions.

b) $\sigma x)/_{D(\sigma)}$ is a hermitian field.

c) $\sigma(x)/_{D(\sigma)}$ is a pseudoscalar field.

d) $\partial^\mu \sigma(x) = W^\mu(x) = \varepsilon^{\mu\nu} V_\nu(x)$ on $D(\sigma)$.

e) $[\sigma(x), \chi(y)] = -1/2\, \varepsilon(x^1 - y^1) \chi(y)$ on $D(\sigma)$ for $(x-y)^2 < 0$.

f) $[\sigma(x), V^\mu(y)] = 0$ on $D(\sigma)$ for $(x-y)^2 < 0$.

g) $[\sigma(x), \sigma(y)] = 0$ on $D(\sigma)$ for $(x-y)^2 < 0$.

It follows then that ψ and φ as defined by Eqs. (25) and (26) are local and relatively local fields.

If the assumption 2 is strengthened to:

2'. The energy-momentum spectrum of the operator field χ has the structure

$$\{p/p=0; \quad p^2=m^2, p^0>0; \quad p^2 \geqq m_1^2 > m^2, p^0>0\},$$

it can be proved that $D(\sigma)$ contains the set D_0^{ex} of "non-overlapping" asymptotic states formed from χ,

$$D(\sigma) \supset D_0^{\mathrm{ex}} \qquad (\mathrm{ex} = \mathrm{in}, \mathrm{out}),$$

and that for any two Ψ, $\Psi' \in D_0^{\mathrm{ex}}$

$$(\Psi', \sigma(f)\,\Psi)$$

is a tempered distribution, regarded as a functional of f.

Further, it will be shown in an appendix that for vectors

$$\Psi^{\mathrm{ex}}, \Psi'^{\mathrm{ex}} \in D_0^{\mathrm{ex}} \cap D'(\{k \mid k^2 < m^2\}),$$

$$t^{2L}\{(\Psi'^{\mathrm{out}}, \tilde{\sigma}(k)\,\Psi^{\mathrm{out}})e^{-it(k_0 \pm k_1)} \mp \pi\delta(k)\,(\Psi'^{\mathrm{out}}, Q\,\Psi^{\mathrm{out}})\} \xrightarrow[t\to+\infty]{} 0,$$

$$t^{2M}\{(\Psi'^{\mathrm{in}}, \tilde{\sigma}(k)\,\Psi^{\mathrm{out}})\,e^{-it(k_0 \pm k_1)} \mp \pi\varepsilon(t)\,\delta(k)\,(\Psi'^{\mathrm{in}}, Q\,\Psi^{\mathrm{out}})\} \xrightarrow[|t|\to\infty]{} 0,$$

$$t^{2N}\{(\Psi'^{\mathrm{in}}, \tilde{\sigma}(k)\,\Psi^{\mathrm{in}})\,e^{-it(k_0 \pm k_1)} \pm \pi\delta(k)\,(\Psi'^{\mathrm{in}}, Q\,\Psi^{\mathrm{in}})\} \xrightarrow[t\to-\infty]{} 0$$

for arbitrary non-negative integers L, M and N.

Finally we add to the assumptions 1 and 2' yet another assumption concerning asymptotic completeness:

3. The set of all asymptotic states formed from χ is dense in H.

It follows then that the pseudoscalar particles are effectively free, i.e.

$$\varphi_{\mathrm{in}} = \varphi_{\mathrm{out}} = \alpha$$

in the sense of Schroer [2].

We do not attempt to give a precise form of field equations for the ψ- and φ-fields. By formal manipulations we can of course obtain equations which correspond to (4) and (5). Also, we do not discuss the correct mathematical description of the substitution $\varphi \to \varphi + \mathrm{const.}$

We note finally that the definition of $\sigma(f)$ given in Eq. (33) and the subsequent introduction of ψ and φ by means of (25), (20) and (26) can be carried through in cases more general than the self-interacting spinor field (e.g. $\chi(x)$ could be a charged scalar field) which we have studied.

IV. Conclusions

Our results are summarized in the following statement, valid in two-dimensional space-time:

If a local field χ with a conserved vector current $V^\mu\{\chi\}$ is given whose energy-momentum spectrum does not contain massless particles, a local pseudoscalar field σ can be defined by Eq. (33) such that

$$\partial^\mu \sigma = \varepsilon^{\mu\nu} V_\nu .$$

Then, taking the tensor product [3] of χ with a free pseudoscalar mass-zero field α, we may define local and relatively local fields ψ and φ by means of Eqs. (20). (25) and (26). This leads to a local theory which contains a conserved axial vector. The $ps(pv)$ theory is a simple example of this situation.

Another local extension of the χ-theory is of course given by a free scalar field β and ψ as defined by Eq. (25). If χ is a free spinor field, this is the well-known derivative coupling of a scalar field [2]. In contrast to the pseudoscalar case, the scalar extension is not restricted to mass-zero particles or to two-dimensional space time.

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Appendix

We want to prove that for any non-negative integer L and for Ψ^{out}, $\Psi'^{\text{out}} \in D_0^{\text{out}}$

$$t^{2L}\{(\Psi'^{\text{out}}, \tilde{\sigma}(k)\Psi^{\text{out}})\, e^{-it(k_0+k_1)} - \pi\delta(k)\,(\Psi'^{\text{out}}, Q\,\Psi^{\text{out}})\} \xrightarrow[t\to+\infty]{} 0 \quad \text{in} \quad D'(\{k \,|\, k^2 < m^2\}) . \tag{A.1}$$

Without loss of generality we may assume that Ψ^{out} describes i outgoing particles and $\bar{i}$ antiparticles while Ψ'^{out} describes i' outgoing particles and $\bar{i}'$ antiparticles.

We take a test function $\tilde{f} \in S$ whose support is contained in $\{k/h^2 < m^2\}$ and form

$$\begin{aligned} &t^{2L}\{\textstyle\int dk\, \tilde{f}(k)\, [(\Psi'^{\text{out}}, \tilde{\sigma}(k)\, \Psi^{\text{out}})\, e^{-it(k_0+k_1)} - \pi\delta(k)\,(\Psi'^{\text{out}}, Q\,\Psi^{\text{out}})]\} \\ &\quad = t^{2L}\{\textstyle\int dx\, f(x;t,t)\,(\Psi'^{\text{out}}, \sigma(x)\,\Psi^{\text{out}}) - (i-\bar{i})\,\pi \tilde{f}(0)\,(\Psi'^{\text{out}}, \Psi^{\text{out}})\} \end{aligned} \tag{A.2}$$

where

$$f(x;t,s) = \frac{1}{2\pi} \int dk\, e^{ikx} e^{-itk_0 - isk_1} \tilde{f}(k)$$

is for any fixed, finite t and s a test function $\in S$ whose support is concentrated around the point $x^0 = t$, $x^1 = s$. For $t > 0$

$$t^{2L}\{\textstyle\int dx\, f(x;t,t)\,(\Psi'^{\text{out}}, \sigma(x)\,\Psi^{\text{out}}) - (i-\bar{i})\,\pi \tilde{f}(0)\,(\Psi'^{\text{out}}, \Psi^{\text{out}})\}$$

can be replaced by

$$t^{2L}\left\{\int d^{2i'}y'\, d^{2\bar{i}'}\bar{y}'\, d^2x\, d^{2i}y\, d^{2\bar{i}}\bar{y}\, \psi'_{(\alpha');(\beta')}(y',\bar{y}';t,t)\, f(x;t,t)\right.$$

$$\times \psi^*_{(\alpha);(\beta)}(y,\bar{y};t,t)\Bigg(\Phi_0, \prod_{\pi=1}^{i'} \chi_{\alpha'_\pi}(y'_{(\pi)}) \prod_{\varrho=1}^{\bar{i}'} \tilde{\chi}_{\beta'_\varrho}(\bar{y}'_{(\varrho)})\,\sigma(x) \qquad \text{(A.3)}$$

$$\times \prod_{\sigma=1}^{\bar{i}} \tilde{\chi}^+_{\beta_{\bar{i}+1-\sigma}}(\bar{y}_{(\bar{i}+1-\sigma)}) \prod_{\tau=1}^{i} \chi^+_{\alpha_{i+1-\tau}}(y_{(i+1-\tau)})\,\Phi_0\Bigg)$$

$$\left. -(i-\bar{i})\,\pi\,\tilde{f}(0)\,(\Psi'^{\text{out}} Q\, \Psi^{\text{out}})\right\}$$

with an error of $0(|t|^{-\infty})$ [5]. Here $\psi_{(\alpha);(\beta)}(y,\bar{y};t,s)$ is defined by

$$\psi_{(\alpha);(\beta)}(y,y;t,s) = \frac{1}{(2\pi)^{i+\bar{i}}}\int d^{2i}p\, d^{2\bar{i}}\bar{p}\, e^{i\sum_{\tau=1}^{i} p^{(\tau)}y_{(\tau)} + i\sum_{\sigma=1}^{\bar{i}} \bar{p}^{(\sigma)}\bar{y}_{(\sigma)}}$$

$$\times \exp\left\{-it\left[\sum_{\tau=1}^{i} p_0^{(\tau)} + \sum_{\sigma=1}^{\bar{i}} \bar{p}_0^{(\sigma)}\right]\right.$$

$$\left. + is\left[\sum_{\tau=1}^{i}\sqrt{m^2+(p_1^{(\tau)})^2} + \sum_{\sigma=1}^{\bar{i}}\sqrt{m^2+(\bar{p}_1^{(\sigma)})^2}\right]\right\}\tilde{\psi}(p,\bar{p}),$$

$$\tilde{\psi}(p,p') \in S(G^{i+\bar{i}})$$

non-overlapping,

$$G = \{p/p_0 > 0, \quad |p^2 - m^2| < c < \text{Min}\{m^2, m_1^2 < m^2\}.$$

$$t^{2L}\{\textstyle\int dx\, f(x;t,t)\,(\Psi'^{\text{out}}, \sigma(x)\,\Psi^{\text{out}}) - (i-\bar{i})\pi\tilde{f}(0)\,(\Psi'^{\text{out}}, \Psi^{\text{out}})\}$$

$$= t^{2L}\{\textstyle\int d^{2i'}y'\, d^{2\bar{i}'}\bar{y}'\, d^2x\, d^{2i}y\, d^{2\bar{i}}\bar{y}\, \psi'_{(\alpha');(\beta')}(y',\bar{y}';0,t)\, f(x;0,t)$$

$$\times \psi^*_{(\alpha);(\beta)}(y,\bar{y};0,t)\left[\sum_\sigma\Bigg(\Phi_0, \prod_{\pi=1}^{i'} \chi_{\alpha'_\pi}(y'_{(\pi)}) \prod_{\varrho=1}^{\bar{i}'} \tilde{\chi}_{\beta'_\varrho}(\bar{y}'_{(\varrho)})\, \tilde{\chi}^+_{\beta_{\bar{i}}}(\bar{y}_{(\bar{i})})\cdots\right.$$

$$\times [\sigma(x), \tilde{\chi}^+_{\beta_\sigma}(\bar{y}_\sigma)]\cdots \tilde{\chi}^+_{\beta_1}(\bar{y}_{(1)}) \prod_{\tau=1}^{i} \chi^+_{\alpha_{i+1-\tau}}(y_{(i+1-\tau)})\,\Phi_0\Bigg) \qquad \text{(A.4)}$$

$$+\sum_\tau\Bigg(\Phi_0, \prod_{\pi=0}^{i'} \chi_{\alpha'_\pi}(y'_{(\pi)}) \prod_{\varrho=1}^{\bar{i}'} \tilde{\chi}_{\beta'_\varrho}(\bar{y}'_{(\varrho)}) \prod_{\sigma=1}^{\bar{i}} \tilde{\chi}^+_{\beta_{\bar{i}+1-\sigma}}(\bar{y}_{(i+1-\sigma)})\, \chi^+_{\alpha_i}(y_{(i)})\cdots$$

$$\times [\sigma(x), \chi^+_{\alpha_\tau}(y_{(\tau)})]\cdots \chi^+_{\alpha_1}(y_{(1)})\,\Phi_0\Bigg)$$

$$\left. -(i-\bar{i})\,\pi\,\tilde{f}(0)\,(\Psi'^{\text{out}}, \Psi^{\text{out}})\right\} + O(|t|^{-\infty}).$$

This is a simple consequence of translation invariance and the spectrum condition.

Moreover, we commit only an error of $0(|t|^{-\infty})$ if we replace the commutators

$$[\sigma(x), \tilde{\chi}^{+}_{\beta_\sigma}(\bar{y}_{(\sigma)})] \quad \text{and} \quad [\sigma(x), \chi^{+}_{\alpha_\tau}(y_{(\tau)})]$$

by

$$-1/2\,\varepsilon(x^1-\bar{y}^1_{(\sigma)})\,\tilde{\chi}^{+}_{\beta_\sigma}(\bar{y}_{(\sigma)}) \quad \text{and} \quad 1/2\,\varepsilon(x^1-y^1_{(\tau)})\,\chi^{+}_{\alpha_\tau}(y_{(\tau)})$$

respectively. This is so because the main contributions in x and $\bar{y}_{(\sigma)}$ and in x and $y_{(\tau)}$ separate by a space-like distance that increases linearly in t and because of (e). We observe that the main contributions in x come from points which lie to the right of the main constributions of all other variables. Hence

$$\begin{aligned}
&\int dk\,\tilde{f}(k)\{t^{2L}[(\Psi'^{\text{out}}, \tilde{\sigma}(k)\,\Psi^{\text{out}})\,e^{-it(h_0+h_1)} - \pi\delta(k)\,(\Psi'^{\text{out}}, Q\,\Psi^{\text{out}})]\}\\
&= t^{2L}\{(i-\bar{i})\,1/2\int d^{2i'}y'\,d^{2\bar{i}'}\bar{y}'\,d^2x\,d^{2\bar{i}}\bar{y}\,d^{2i}y\,\psi'_{(\alpha');(\beta')}(y',\bar{y}';t,t)\\
&\quad\times f(x;0,t)\,\psi^*_{(\alpha);(\beta)}(y,\bar{y};t,t)\Big(\Phi_0, \prod_{\pi=1}^{i'}\chi_{\alpha'_\pi}(y'_{(\pi)})\prod_{\varrho=1}^{\bar{i}'}\tilde{\chi}_{\beta_\varrho}(\bar{y}'_{(\varrho)})\\
&\quad\times\prod_{\sigma=1}^{\bar{i}}\tilde{\chi}^{+\,+\,1-\sigma}_{\beta_{\bar{i}}}(\bar{y}_{(\bar{i}+1-\sigma)})\prod_{\tau=1}^{i}\chi^{+}_{\alpha_{i+1-\tau}}(y_{(i+1-\tau)})\,\Phi_0\Big)\\
&\quad-(i-\bar{i})\,\pi\,\tilde{f}(0)\,(\Psi'^{\text{out}}, \Psi^{\text{out}}) + O(|t|^{-\infty})\,.
\end{aligned} \tag{A.5}$$

According to Hepp [5], the curled bracket itself is of the type

$$\pi\cdot(i-\bar{i})\,\tilde{f}(o)\cdot O(|t|^{-\infty})$$

Hence we obtain

$$\int dk\,\tilde{f}(k)\{t^{2L}[(\Psi'^{\text{out}}, \tilde{\sigma}(k)\,\Psi^{\text{out}})\,e^{-it(k_0+k_1)} - \pi\delta(k)\,(\Psi'^{\text{out}}, Q\,\Psi^{\text{out}})]\} \xrightarrow[t\to+\infty]{} 0$$

i.e.

$$t^{2L}[(\Psi'^{\text{out}}, \tilde{\sigma}(k)\,\Psi^{\text{out}})\,e^{-it(k_0+k_1)} - \pi\delta(k)\,(\Psi'^{\text{out}}, Q\,\Psi^{\text{out}})] \xrightarrow[t\to+\infty]{} 0$$
$$\text{in} \quad D'(\{k/k^2<m^2\})\,.$$

The other assertions follow in an analogous manner.

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