Commun. math. Phys. 13, 175—179 (1969)

A Converse to a Theorem by Friedrichs

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Received March 26, 1969

Abstract. It is proved that the requirement of implementability of a group of canonical transformations defines a class of irreducible representations of the CAR. As a corollary a converse to Friedrichs' theorem about canonical transformations implementable in the Fock representation is obtained.

A well known theorem due to K. O. FRIEDRICHS [1] states that a (linear) canonical transformation

$$b^+(f) = a^+(Af) + a^-(Bf) \qquad (*)$$

is unitarily implementable in the Fock representation of the canonical anticommutation relations if, and only if, B is of the Hilbert-Schmidt type (i.e.: $B^* B$ is of the trace class).

In this note we prove the following converse theorem: if in an irreducible representation of the canonical anticommutation relations all canonical transformations (*) with $B = 0$ are implementable then it is the Fock or the anti-Fock representation.

Before going further, let us recall the definitions.

Let H be separable Hilbert space (the space of the test functions). By a representation of the canonical anticommutation relations (CAR) over H in a Hilbert space $\mathscr{H}$ we mean a linear mapping $a^+\colon H \to \mathscr{L}(\mathscr{H})$ such that if $a^-(f) := a^+(f)^*$, $f \in H$, then:

$$a^-(f)\, a^+(g) + a^+(g)\, a^-(f) = (f|g) \quad \text{and} \quad a^+(f)\, a^+(g) + a^+(g)\, a^+(f) = 0\,.$$

A Fock (resp.: an anti-Fock) representation of the CAR is an irreducible representation for which there exists $\Omega \in \mathscr{H}$ such, that $a^-(f)\,\Omega = 0$ (resp.: $a^+(f)\,\Omega = 0$) for all $f \in H$.

One says that a pair (A, B), A linear and B an antilinear operators in H, defines a canonical transformation if b^+ defined by (*) is a representation of the CAR.

If, in addition, there exists such unitary $U \in \mathscr{L}(\mathscr{H})$ that

$$b^+(f) = U a^+(f)\, U^{-1}, \qquad f \in H\,.$$

then it is said that canonical transformation is (unitarily) implementable in the given representation.

In Sections 1—4, we give an analysis of the following situation: Let S_1 denote the multiplicative group of the complex numbers of modulus one, $G = S_1 \times S_1 \times \ldots$ and let $\{f_i\}_{i=1}^{\infty}$ be some orthonormal basis of H.

Let $g = (z_1, z_2, \ldots) \in G$ and let A_g be such an unitary operator that $A_g f_i = z_i f_i$, $i = 1, 2, \ldots$, and $B_g = 0$; the pair $(A_g B_g)$ defines a canonical transformation.

We will investigate the restrictions imposed on an irreducible representation of the CAR by the following condition: all the canonical transformations defined by (A_g, B_g), $g \in G$, are implementable.

By factorization, the analysis is applicable to the factor type I representations and, with taking needed care, to the representations of the canonical commutation relations too.

1. Let us choose for each $g \in G$ a unitary operator U_g such that:

$$a^+(A_g f) = U_g a^+(f)\, U_g^{-1}, \quad f \in H .$$

U_g is defined by this condition up to a factor of modulus one.

In what follows we will consider G as a topological, compact and separable group: a denumerable product of the compact and separable, topological groups S_1.

In G there exists a natural Borel structure generated by that topology and in this section we want to prove that for every $x, y \in \mathscr{H}$ the function $g \to |(x \,|\, U_g y)|$ is a Borel function.

Let $a_i^{\pm} := a^{\pm}(f_i)$ and let a_i denote a_i^+ or a_i^-. Because of the estimation $\|a(f)\| \leqq \|f\|$, $f \in H$, [2], separability of H and irreducibility of our representation the Hilbert space $\mathscr{H}$ is separable and the *-algebra $\mathscr{A}$ generated by $\{a_i\}$ acts irreducibly in $\mathscr{H}$.

Because of $U_g a_i^+ U_g^{-1} = z_i a_i^+$, $g \to U_g a_i U_g^{-1}$ is a norm continuous function for every i and therefore for each $A \in \mathscr{A}$ $g \to U_g A\, U_g^{-1}$ is norm continuous too.

Let us consider some $A \in \mathscr{L}(\mathscr{H})$. Because of the irreducibility of $\mathscr{A}$ and separability of $\mathscr{H}$ there exists a sequence $A_n \in \mathscr{A}$ such that $A_n \to A$ weakly ([3], § 3).

Therefore, for $y \in \mathscr{H}$, $g \in G$:

$$\begin{aligned}(y \,|U_g A\, U_g^{-1} y) = (U_g^{-1} y|\, A\, U_g^{-1} y) &= \lim_{n\to\infty} (U_g^{-1} y \,|\, A_n U_g^{-1} y) \\ &= \lim_{n\to\infty} (y \,|\, U_g A_n U_g^{-1} y) .\end{aligned}$$

Thus, the function $g \to (y \,|\, U_g A\, U_g^{-1} y)$ being a point limit of a sequence $g \to (y \,|\, U_g A_n U_g^{-1} y)$ of continuous functions, is a Borel function on G. Taking for A the orthogonal projection on subspace generated by x we have:

$$(y \,|\, U_g A\, U_g^{-1} y) = |(y \,|\, U_g x)|^2$$

and therefore for every $x, y \in \mathscr{H}$, $g \to |(y \,|\, U_g x)|$ is a Borel function on G.

2. In this section using the very strong results of G. W. Mackey [4], (see however, the remark in the end of this section) we will prove the following.

Lemma. *There exists a finite dimensional x subspace of $\mathscr{H}$ invariant for all U_g.*

For the proof let us proceed as follows: from Section 2, we know that $g \to |(x|U_g y)|$ is a Borel function on G for all $x, y \in \mathscr{H}$. Using the method of proof of the Theorem 2.2 in [4] we may define such function f on G, $|f(g)| = 1$, that for every $x, y \in \mathscr{H}$ $g \to (x|f(g)\,U_g y)$ is a Borel function. Now, if $V_g := f(g)\,U_g$ then $g \to V_g$ is a projective representation of G in the sence of [4] with multiplier σ defined by: $V_{g_1 g_2} = \sigma(g_1, g_2)\,V_{g_1} V_{g_2}$. After introducing in $S_1 \times G$ a multiplication; $(\lambda_1, g_1)(\lambda_2 g_2) = \left(\frac{\lambda_1 \lambda_2}{\sigma(g_1, g_2)}, g_1 g_2\right)$ one obtains a group G^σ and $(\lambda, g) \to \lambda V_g$ is an ordinary representation of this group.

Now Theorem 2.1 of [4] asserts that in G there exists a topology that makes G^σ a locally compact group, $(\lambda, g) \to \lambda V_g$ is a continuous representation of this group and, moreover, the Haar measure in G^σ is the product of the Haar measures of G and S_1. The Haar measures of G and S_1, because of their compactness, are finite and therefore the Haar measure of G^σ is finite too. But a locally compact group with finite Haar measure must be compact ([5], § 8) and therefore we arrived at a continuous representation $(\lambda, g) \to \lambda V_g$ of the compact group G^σ.

For such representations it is known that the Hilbert space of representation splits into a sum of finite dimensional invariant subspaces and thus the Lemma follows from proportionality of U_g and λV_g.

Remark. In fact the use of the results of Ref. [4] about connection of Borel structure and topology in separable groups is not essential. After introducing in G^σ the product measure and product Borel structure we see that G^σ becomes a Borel group with a left and right invariant finite measure and $(\lambda, g) \to \lambda V_g$ is a measurable representation of G^σ in a separable Hilbert space. But for such representations the existence of a finite dimensional invariant subspace may be proved directly. I have not found the proof published but it proceeds, with minor changes, as for compact groups. As it is rather long, it is not given here.

3. Let us define $N_k = a_k^+ a_k$; $\{N_k\}$ is a commuting set of projectors Now we show that there exists in $\mathscr{H}$ a common eigenvector for all N_k, $k = 1, 2, \ldots$

It is easy to verify that for $z = e^{it} \in S_1$: $e^{itN_k} a_k^+ e^{-itN_k} = z a_k^+$ and $e^{itN_k} a_y^+ e^{-itN_k} = \bar{z} a_j^+$ for $j \neq k$. If therefore, p_k denote the injection $S_1 \to G$ for which $p_k(z)$ has the $k-th$ component equal to z and the remaining equal to one then e^{itN_k} must be proportional to $U_{pk(z)}$ (they implement the same canonical transformation).

From the proceding section we have that there exists $\mathscr{H}'$ — a finite dimensional subspace of $\mathscr{H}$ such, that $e^{itN_k}\mathscr{H}' \subset \mathscr{H}'$. $\{e^{itN_k}|\mathscr{H}'\}_{t,k}$ is a commuting family of unitary operators in a finite dimensional space and therefore there exists in $\mathscr{H}'$ common eigenvector for this family. The same vector is of course a common eigenvector for $\{N_k\}$.

4. Let us consider in more detail such irreducible representations of canonical anticommutation relations for which there exists common eigenvector x for $\{N_k\}$. N_k are projectors and therefore $N_k x = 0$ or $N_k x = x$.

Let $\mathscr{N}_0(x) = \{k: N_k x = 0\}$ and $\mathscr{N}_1(x) = \{k: N_k x = x\}$. Let a'^+ and a''^+ be irreducible representations in $\mathscr{H}'$ and $\mathscr{H}''$ respectively. Suppose that $x' \in \mathscr{H}'$ and $x'' \in \mathscr{H}''$ are common eigenvectors for $\{N'_k\}$ and $\{N''_k\}$. Then:

The representations a'^+ and a''^+ are equivalent if, and only if, the set $\mathscr{N}_0(x') \cap \mathscr{N}_1(x'')$ is finite.

For proof let us suppose that the set $\mathscr{N}_0(x') \cap \mathscr{N}_1(x'')$ is not finite and let $\{i_1, i_2, \ldots\}$ be some enumeration of its elements. Let us define:

$$\overline{N}'_k = \frac{1}{k}(N'_{i_1} + \cdots + N'_{i_k}) \quad \text{and} \quad \overline{N}''_k = \frac{1}{k}(N''_{i_1} + \cdots + N''_{i_k}) .$$

Calculations give: $\|\overline{N}'_k\| \leqq 1$ and $\|[a'_i, \overline{N}'_k]\| \leqq 1/k$. Therefore, if for some vector $x \in \mathscr{H}'$ there exists the strong limit $s - \lim\limits_{k\to\infty} \overline{N}'_k x$ then the sequence $\overline{N}'_k$ is strongly convergent and the limit is a scalar operator. But $\overline{N}'_k x' = 0$ hence $s - \lim\limits_{k\to\infty} \overline{N}'_k = 0$. Applying the same to $\overline{N}''_k$ we see that $s - \lim\limits_{k\to\infty} \overline{N}''_k = I$.

Now if there exists such unitary $U: \mathscr{H}'' \to \mathscr{H}'$ that $U a'^+(f)\, U^{-1} = a''^+(f)$, $f \in H$, then: $U N'_k U^{-1} = N''_k$, $U \overline{N}'_k U^{-1} = \overline{N}''_k$ and therefore $U s - \lim\limits_{k\to\infty} \overline{N}'_k U^{-1} = s - \lim\limits_{k\to\infty} \overline{N}''_k$, thus, we arrived to a contradiction.

The proof of the "if" part is straightforward (see also the next section).

The representations described in this section are called in [6]: the translated canonical representations.

5. In this section we prove a converse to the Friedrichs' theorem formulated in the introduction.

From Section 3, we know that there exists a common eigenvector x for $\{N_k\}$. Let $\mathscr{N}_0(x)$ and $\mathscr{N}_1(x)$ be as in the preceding section. Let us first observe that if $i \in \mathscr{N}_1(x)$ then $a_i^+ x = 0$, because of $a_i^+ N_i = 0$, and $a_i^- x \neq 0$, because of $(a_i^+ a_i^- + a_i^- a_i^+)\, x = x$.

Therefore if $\mathscr{N}_1(x)$ is finite: $\mathscr{N}_1(x) = \{i_1, \ldots, i_k\}$ and $x' = a_{i_1}^+ \ldots a_{i_k}^+ x$ then $x' \neq 0$, $\mathscr{N}_1(x')$ is an empty set and $a_i^- x = 0$ for all i. Thus our representation is a Fock one. Similarly, if $\mathscr{N}_0(x)$ is finite we have an anti-Fock representation.

But another possibility leads to a contradiction. For, let $\mathcal{N}_0(x) = \{i_1, i_2, \ldots\}$ and $\mathcal{N}_1(x) = \{j_1, j_2, \ldots\}$ both be infinite. Let A be such a unitary operator in H that $Af_{i_k} = f_{j_k}$, $Af_{j_k} = f_{i_k}$ and let $B := 0$. Then the pair A, B defines a canonical transformation and from assertion of the theorem to be proved there exists unitary operator $U \in \mathcal{L}(\mathcal{H})$ such, that:

$$a(Af) = U a(f) U^{-1}, \quad f \in H .$$

Let $\overline{N}'_k = 1/k\,(N_{i_1} + \cdots + N_{i_k})$ and $\overline{N}''_k = 1/k(N_{j_1} + \cdots + N_{j_k})$. As in Section 4, we arrive at a contradiction by: $s - \lim_{k\to\infty} \overline{N}'_k = 0$, $s - \lim_{k\to\infty} \overline{N}''_k = I$ and $U\overline{N}'_k U^{-1} = \overline{N}''_k$.

Theorem is therefore proved.

The author wishes to thank Professor AMNON KATZ for remarks concerning the manuscript.

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