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On the Cluster Property Above the Critical Temperature in Lattice Gases

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Abstract. We prove cluster properties of the correlation functions at high temperature and arbitrary activity. We obtain also results on clustering at complex temperatures and activities.

§ 1. Introduction

In a recent paper [1] it was shown that the correlation functions of a lattice gas with negative two-body interactions have some cluster property not only at low activity, as known before [2–4] but also for all values of the activity z inside the Lee-Yang circle defined by

$$|z| < \exp\beta A \qquad A = \sum_{0 \neq y \in Z^\nu} \varphi(y)\,. \tag{1}$$

Similar results [1] have been obtained for purely repulsive potentials but with the Lee-Yang circle replaced by the circle of convergence of the Mayer series.

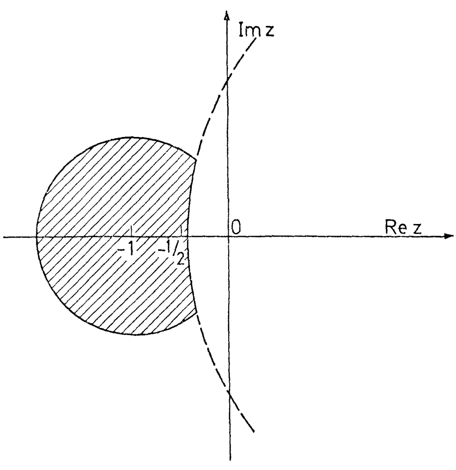

Fig. 1

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In Ref. [1] analyticity properties in β and z of the pressure and the correlation functions are proved for z in any compact not intersecting the Lee-Yang circumference $|z| = \exp\beta A$, in the attractive case, or for z in the interior of the circle of convergence of the Mayer series, in the repulsive case.

In this paper we deduce results, complementary to the above cited ones, concerning clustering and analyticity at high temperature for lattice gases with very general interactions (not necessarily attractive or repulsive and possibly involving many-body interactions).

We show that there exists $\beta_0 > 0$ such that the Ursell functions $u(X)$ are analytic in z and β for z in an open region $\tilde{C}$ of the complex plane including an open strip around the real positive axis (see Fig. 1: $\tilde{C}$ has the form of the complementary of the dashed set) and β in a neighborhood I_c of the set $|\mathrm{Re}\beta| < \beta_0$, $\mathrm{Im}\beta = 0$.

Furthermore if X is a configuration $X = \{x_1, x_2, \ldots, x_{N(X)}\}$ and $X = X_1 \cup X_2$ is any decomposition of X into two configurations at a distance $d(X_1, X_2)$, then there exist $\theta > 0$, $\alpha > 0$ such that:

$$|u(X_1 \cup X_2)| \leqq \theta^{(N(X_1)+N(X_2))^2} \exp - \alpha \frac{d(X_1, X_2)}{\lambda}\cdot, \quad (z, \beta) \in C \times I_c\,, \tag{2}$$

where $\lambda \leqq +\infty$ is the range of the interaction. If $\lambda = +\infty$ the following weaker result still holds for $(z, \beta) \in \tilde{C} \times I_c$:

$$\lim_{\substack{d(X_1,X_2)\to\infty \\ N(X_1)+N(X_2)\ \text{fixed}}} u(X_1 \cup X_2) = 0\,. \tag{3}$$

The techniques used to obtain (2) and (3) are similar to the ones used in Ref. [1] except that we obtain bounds on $u(X)$ and analyticity regions by using integral equations instead of the Lee-Yang theorem on attractive potentials or the Groeneveld alternating sign property for positive potentials.

We remark that (2) implies that

$$\sum_{\substack{0 \in X \\ N(X)\ \text{fixed}}} |u(X)| < +\infty\,. \tag{4}$$

§ 2. The Interaction Potentials

Suppose the particles are on a ν-dimensional lattice Z^ν and interact through symmetric translationally invariant many particle potentials $\Phi^{(k)}(x_1 \ldots x_k)$ and consider these as a function Φ on the finite configurations X defined by

$$\Phi(X) = \Phi^{(N(X))}(x_1 \cdots x_{N(X)}) \quad \text{if} \quad X = \{x_1, x_2, \cdots, x_{N(X)}\}$$

We consider only interactions involving a finite number of particles such that $\Phi^{(2)}(x, x) = +\infty$ and we call $\tilde{X}$ the set of the sites occupied by particles in X (so if $X \neq \tilde{X}$ the configuration X has zero probability).

Furthermore we suppose finiteness of the energy of the origin, i.e.

$$\|\Phi\| = \sum_{\substack{0 \in X \\ X = \tilde{X}}} |\Phi(X)| < +\infty \,. \tag{5}$$

We denote by $\mathfrak{B}$ the class of potentials described above. Since we shall be interested in the analyticity properties in the activity $z = \exp - \beta \Phi^{(1)}$ it is useful to write $\Phi = (\Phi^{(1)}, \Phi')$ where Φ' is obtained from Φ by setting the one particle potential $\Phi^{(1)}$ (which may be interpreted as minus the chemical potential) equal to zero.

It is also useful to introduce the potential $\mathscr{L}\Phi \in \mathfrak{B}$ defined, for $X = \tilde{X}$, by

$$(\mathscr{L}\Phi)(X) = (-1)^{N(X)} \sum_{\substack{S \supset X \\ S = \tilde{S}}} \Phi(S) \,, \qquad \Phi \in \mathfrak{B} \tag{6}$$

(this potential is related to the symmetry properties of the lattice gas under the exchange of particles and holes [5]) and the quantities:

$$A = \sum_{\substack{0 \in S \\ S = \tilde{S}}} \Phi(S) \qquad \text{(energy of the origin)} \,, \tag{7}$$

$$\left.\begin{aligned} \gamma &= [1 + 2(\exp(e^{\beta_0 \|\Phi'\|} - 1) - 1)] \,, \\ \gamma' &= [1 + 2(\exp(e^{\beta_0 \|(\mathscr{L}\Phi)'\|} - 1) - 1)] \,, \end{aligned}\right. \tag{8}$$

$$\alpha = \exp \beta_0 \|\Phi'\| \,, \qquad \alpha' = \exp \beta_0 \|(\mathscr{L}\Phi)'\| \,. \tag{9}$$

§ 3. Review of Useful Results

The following properties are proven in Refs. [2–6]:

i) Consider the two circles centered on the real axis respectively at $z = \frac{1}{\alpha(\gamma^2 - 1)}$ and $z = -\alpha' \exp \beta_0 A$ [where α, γ, γ', A are defined in (7), (8), (9)] and with respective radii: $\frac{\gamma}{\alpha(\gamma^2 - 1)}$ and $\gamma' \alpha' \exp \beta_0 A$. Suppose β_0 is so small that the maximum real z on the left circumference is, as in Fig. 1, inside the right circle.

If C is any closed set in the complementary $\tilde{C}$ of the dashed region drawn in Fig. 1 it is possible to choose β_0 small enough that there exists a neighborhood I_c of the set $|\mathrm{Re}\beta| < \beta_0$, $\mathrm{Im}\beta = 0$ such that the correlation functions $\varrho_{\beta\Phi}(X)$ are analytic functions in $(\beta, z) \in I_c \times C$. In this case there also exist [5, 6] a constant $\theta_0 > 1$ such that

$$|\varrho_{\beta\Phi}(X)| \leqq \theta_0^{N(X)} \qquad (\beta, z) \in I_c \times C \,. \tag{10}$$

Remark. What is really proved in Refs. [5, 6] is that if β is in a suitable neighborhood of the set $|\mathrm{Re}\beta| < \beta_0$, $\mathrm{Im}\beta > 0$, with β_0 sufficiently small, and z is both in C and inside the right circle we have $|\varrho_{\beta\Phi}(X)| \leqq \theta_0'$. However under the same conditions but with z outside the left circle we have, using the symmetry between holes and particles, that $\varrho_{\beta\Phi}(X) = \sum_{S \subset X} (-1)^{N(S)} \varrho_{\beta \mathscr{L}\Phi}(X)$ (for $X = \tilde{X}$) and $|\varrho_{\beta\mathscr{L}\Phi}(X)| \leqq \theta_0''$. Inequality (10) is a consequence of these two facts.

ii) The Ursell functions are given, for sufficiently small $|z|$, by the convergent [2–4] series:

$$u(X) = z^{N(X)} \sum_{n \geqq 0} z^n c_n(X) , \tag{11}$$

where the coefficients $c_n(X)$ are of the form

$$c_n(X) = \sum_{Y, N(Y) = n} U(X \cup Y)$$

and the functions $U(S)$ are defined by (apart from combinatorial factors):

$$U(S) = \sum_{\Gamma} \prod_{T \in \Gamma} (e^{-\beta \Phi(T)} - 1) , \tag{13}$$

where $\sum_{\Gamma}$ means sum over all the "connected diagrams" contained in S.

The coefficients $c_n(X)$ have the following properties [1–4]

$$\lim_{\substack{d(X_1, X_2) \to \infty \\ N(X_1) + N(X_2) \text{ fixed}}} c_n(X_1 \cup X_2) = 0 , \tag{14}$$

$$c_n(X_1 \cup X_2) = 0 \quad \text{if} \quad \frac{d(X_1, X_2)}{\lambda} > N(X_1) + N(X_2) + n , \tag{15}$$

where λ denotes the range of the interactions; (14) can be proved in the same manner used for two-body interactions in Ref. [1], Eq. (15) is obvious.

§ 4. The Cluster Properties

The Ursell functions are defined in terms of correlation functions:

$$u(X) = - \sum_{k \geqq 1} \frac{(-1)^k}{k} \sum_{\substack{\{X_1 \ldots X_k\} \\ U_i X_i = X, X_i \neq \emptyset}} \prod_{i=1}^{k} \varrho(X_i) , \tag{16}$$

where the second sum runs over the ordered families of k configurations such that $U_i X_i = X$. Then using (16) we find

$$|u(X)| \leqq (2^{N(X)} \theta_0)^{N(X)} \qquad N(X) \neq 0 \qquad (\beta, z) \in I_c \times C . \tag{17}$$

Let us now choose the analyticity region C to be the closure of a simply connected open set containing the origin[1]. Then by Rieman's theorem [7] there exists a conformal one-to-one mapping $z \to t(z)$ which maps the interior of C onto the interior of the unit circle of the complex t-plane and leaves the origin invariant. Thus the Ursell functions have a power-series expansion in t, convergent for $|t| < 1$:

$$u(X) = t^{N(X)} \sum_{r \geqq 0} t^r \gamma_r(X) , \tag{18}$$

where, for t sufficiently small and $z = \sum_{k=1}^{\infty} c_k t^k$,

$$\gamma_r(X) = \sum_{n \geqq 0}^{r} c_n(X) \sum_{\substack{k_1 \ldots k_{n+N(X)} \\ k_i \geqq 1 \sum k_i = r + N(X)}} c_{k_1} c_{k_2} \ldots c_{k_{N(X)+n}} . \tag{19}$$

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[1] At this point we could apply a very general theorem by ZERNER [8] to obtain the promised result (we owe this remark to RUELLE).

Hence, from (14) and (15) it follows that

$$\gamma_r(X_1 \cup X_2) = 0 \quad \text{if} \quad \frac{d(X_1, X_2)}{\lambda} > N(X_1) + N(X_2) + r \tag{20}$$

and, if $\lambda = +\infty$:

$$\lim_{\substack{d(X_1, X_2) \to \infty \\ N(X_1) + N(X_2) \text{ fixed}}} \gamma_r(X_1 \cup X_2) = 0 \,. \tag{21}$$

Now using the Cauchy formula (integrating over a circle of radius as near as we want to 1) one finds from (18) and (17) that

$$|\gamma_r(X)| = \frac{1}{2\pi} \left| \oint \frac{u(X)\, dt}{t^{N(X)+r+1}} \right| \leqq (2^{N(X)}\, \theta_0)^{N(X)} \,. \tag{22}$$

Hence the series (18) converges uniformly in X and t provided $N(X)$ is fixed and t is inside any compact contained in the interior of C. Thus from (21) we can deduce property (3) for $(\beta, z) \in I_c \times C$ and from (20) we deduce [for $(\beta, z) \in I_c \times C$]:

$$|u(X_1 \cup X_2)| \leqq \frac{|t(z)|^{d(X_1, X_2)/\lambda}}{1 - |t(z)|} (2^{N(X)}\, \theta_0)^{N(X)} \qquad |t(z)| < 1 \,. \tag{23}$$

§ 5. Possible Improvements

The results at hand are incomplete in two respects. First in the case of infinite range interactions one has only the quite weak cluster property (3) whereas one expects that a cluster property of the form (4) holds.

Secondly one has little information, even in the case of finite range interactions about the dependence of the cluster property on $N(X)$. It is known [2–4] that at sufficiently low $|z|$ one has a cluster property, both for finite or infinite range forces of the form

$$\sum_{\substack{0 \in X \\ N(X) \text{ arbitrary}}} |u(X)| < +\infty \,. \tag{24}$$

It seems that, at least for what concerns the second problem, one has to use some different techniques like, for instance, integral equation methods of Refs. [2, 4, 5].

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References

1. Lebowitz, J. L., and O. Penrose: Analytic and clustering properties of thermodynamic functions and distribution functions for classical lattice and continuous systems. Commun. math. Phys. **11**, 99 (1968).
2. Ruelle, D.: Correlation functions of classical gases. Ann. Phys. **25**, 109 (1963).
3. Penrose, O.: Convergence of fugacity expansions for fluids and lattice gases. J. Math. Phys. **4**, 1312 (1963).
4. Gallavotti, G., and S. Miracle-Sole: Correlation functions of a lattice system. Comm. Math. Phys. **7**, 274 (1968).
5. — —, and D. Robinson: Analyticity properties of a lattice gas. Phys. Lett. **25** A, 493 (1967).
6. Ruelle, D.: Statistical Mechanics. III. 2.7 New York: Benjamin (to be published).
7. Ahlfors, L. A.: Complex analysis. IV. 4.2. New York: McGraw Hill 1953.
8. Zerner, M.: Théorie de Hartogs et singularités des distributions. Bull. Soc. Math. France **90**, 165 (1962). — A simple account of this paper is in Math. Rev. **26**, abstract 1754 (1963).

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