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Non-Existence of Spontaneous Magnetization in a One-Dimensional Ising Ferromagnet

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Abstract. It is proved that an infinite linear chain of spins $\mu_j = \pm 1$, with an interaction energy

$$H = - \sum J(i-j)\, \mu_i \mu_j\,,$$

has zero spontaneous magnetization at all finite temperatures, provided that $J(n)$ is non-negative and that

$$(\log \log N)^{-1} \sum_{1}^{N} n\, J(n) \to 0 \quad \text{as} \quad N \to \infty\,.$$

This shows that a theorem of RUELLE, establishing the absence of long-range order when the sum $\sum n\, J(n)$ converges, is not the best possible.

1. Result

This paper is a sequel to an earlier one [1] dealing with the existence of phase-transitions in the infinite Ising ferromagnet with energy

$$H = - \sum_{i>j} J(i-j)\, \mu_i\, \mu_j\,. \tag{1.1}$$

In [1] it was proved that a transition at a finite temperature from zero to nonzero spontaneous magnetization does occur if $J(n)$ is positive and monotonically decreasing and if

$$M_0 = \sum_{n=1}^{\infty} J(n) < \infty\,, \tag{1.2}$$

$$K_3' = \sum_{n=1}^{\infty} (\log \log (n+4))\, [n^3 J(n)]^{-1} < \infty\,. \tag{1.3}$$

On the other hand, RUELLE [2] has proved that if $J(n)$ is positive and

$$M_1 = \sum_{n=1}^{\infty} n\, J(n) < \infty\,, \tag{1.4}$$

then there is zero spontaneous magnetization at all temperatures. A gap remains between the conditions (1.3) and (1.4), including the particularly interesting case

$$J(n) = n^{-2}\,. \tag{1.5}$$

Within the gap the existence of spontaneous magnetization is still in doubt. KAC and THOMPSON [3] conjectured that (1.4) would be necessary as well as sufficient for the non-existence of spontaneous magnetization. In this paper we narrow the gap very slightly, not enough to deal with the case (1.5), but enough to exclude the Kac-Thompson conjecture.

Theorem. *In the infinite Ising ferromagnet with energy* (1.1), *there is zero spontaneous magnetization at all finite temperatures provided that* $J(n)$ *is non-negative and*

$$(\log\log N)^{-1} \sum_{n=1}^{N} n\, J(n) \to 0 \quad \text{as} \quad N \to \infty\,. \tag{1.6}$$

2. Proof

The proof of the theorem is similar to the proof of Theorem 6 in [1], and is entirely based on the work of GRIFFITHS [4]. The same idea which was applied to the "Hierarchical Model" in the proof of Theorem 6 is now applied directly to the linear model (1.1).

We denote by L_0 the Ising ferromagnet with the energy (1.1). For any positive integer p we define an Ising ferromagnet L_p which is obtained by locking together blocks of 2^p consecutive spins in L_0. Equivalently, L_p is obtained from L_{p-1} by locking together pairs of neighbouring spins. A single spin μ_j in L_p replaces a block of spins μ_k in L_0 with

$$(j-1)\,2^p + 1 \leqq k \leqq j\,2^p\,. \tag{2.1}$$

Therefore the model L_p has the energy

$$H_p = -\sum_{i>j} J_p(i-j)\,\mu_i\mu_j\,, \tag{2.2}$$

with

$$J_p(n) = \sum_{k=-2^p}^{2^p} [2^p - |k|]\, J(n.\, 2^p + k)\,. \tag{2.3}$$

The sum (1.2) calculated for the model L_p is

$$M_{p,0} = \sum_{n=1}^{\infty} J_p(n) = \sum_{k=1}^{\infty} J(k)\,\mathrm{Min}\,[k, 2^p]\,. \tag{2.4}$$

The condition (1.6) implies that M_0 given by (1.2) and all the $M_{p,0}$ given by (2.4) converge. Therefore the theorem of GALLAVOTTI and MIRACLE-SOLE [5] ensures that the models L_p are well-defined thermodynamic systems.

Let long-range order in the model L_p be measured by the coefficient

$$g_p(k) = k^{-2}\left\langle\left(\sum_{j=1}^{k}\mu_j\right)^2\right\rangle_p\,, \tag{2.5}$$

the average being taken in L_p at some fixed temperature T. The spontaneous magnetization m_p of L_p is then given by

$$m_p^2 = \lim_{k \to \infty} g_p(k) , \qquad 0 \leqq m_p \leqq 1 . \tag{2.6}$$

The limit exists according to an argument of GRIFFITHS [4]. Let now P_p be the probability that two neighbouring spins are parallel in the model L_p. From (2.5) we deduce

$$g_p(2k) = (2k)^{-2} \sum_{j=1}^{k} P_p \left\langle 2\mu_{2j} \left(\sum_{i=1}^{2k} \mu_i \right) \right\rangle_{p,L} , \tag{2.7}$$

where the suffix L means that the spins μ_{2j-1} and μ_{2j} are to be locked together in L_p while taking the average. By GRIFFITHS [4], the average in (2.7) can only increase if all neighbouring spin-pairs are locked together, thus converting the model L_p into L_{p+1}. Thus (2.7) implies

$$g_p(2k) \leqq P_p \, g_{p+1}(k) . \tag{2.8}$$

Letting $k \to \infty$ according to (2.6),

$$m_p^2 \leqq P_p \, m_{p+1}^2 , \tag{2.9}$$

and therefore

$$m_0^2 \leqq \prod_{p=0}^{\infty} P_p . \tag{2.10}$$

Since m_0 is the spontaneous magnetization of the model (1.1), the theorem is proved if we can show that the product on the right of (2.10) diverges to zero.

An upper bound to P_p is obtained from the theorem of GRIFFITHS [4] which states that the probability for the spins (μ, μ') to be parallel is increased if all the remaining spins are locked in an orientation parallel to μ'. We thus find

$$P_p \leqq [1 + \exp(-4\beta M_{p,0})]^{-1} , \tag{2.11}$$

with $M_{p,0}$ given by (2.4). Hence $m_0 = 0$ provided that the series

$$S = \sum_{p=0}^{\infty} \exp(-4\beta M_{p,0}) \tag{2.12}$$

diverges. Now (1.6) implies that for every $\varepsilon > 0$ and all sufficiently large p

$$\sum_{n=1}^{2^p} n \, J(n) < \varepsilon \log p . \tag{2.13}$$

Therefore (2.4) gives for all large p

$$\begin{aligned} M_{p,0} &= \sum_{n=1}^{2^p} n\,J(n) + \sum_{q=p}^{\infty} \sum_{n=2^q+1}^{2^{q+1}} 2^p\,J(n) \\ &\leq \sum_{n=1}^{2^p} n\,J(n) + \sum_{q=p}^{\infty} 2^{p-q} \sum_{n=1}^{2^{q+1}} n\,J(n) \\ &< \varepsilon \log p + \sum_{q=p}^{\infty} 2^{p-q}\,\varepsilon \log(1+q) \\ &< 4\,\varepsilon \log p\,. \end{aligned} \tag{2.14}$$

Hence the terms of the series (2.12) satisfy

$$\exp(-4\beta M_{p,0}) > p^{-16\varepsilon\beta}\,, \qquad p > p_0(\varepsilon)\,. \tag{2.15}$$

Choosing $\varepsilon = (16\beta)^{-1}$, the series diverges and the theorem is thereby proved.

Addendum

References to two earlier papers, FISHER [6] and GRIFFITHS [7], ought to have been included in my paper [1]. I am grateful to the authors for bringing these papers to my attention. FISHER [6] is relevant to my work in two respects. Firstly, FISHER studies a one-dimensional spin-system with long-range interactions, solves it exactly, and proves that under suitable conditions a phase-transition exists. He carries through this beautiful and complete analysis for a model which is at least as "realistic" as my hierarchical model. Secondly, FISHER states explicitly the conjecture which appears as Corollary 1 to Theorem 1 in my paper [1], and attributes this conjecture to KAC [8]. GRIFFITHS [7] has greatly clarified the interrelations between the various alternative definitions of "spontaneous magnetization" in an Ising ferromagnet. I regret that in writing my paper [1] I did not make use of GRIFFITHS' nomenclature, and I urge anybody writing on this subject in future to do so.

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