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The Power Counting Theorem for Minkowski Metric*

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Abstract. Dyson's power counting theorem is proved for the case of Minkowski metric.

1. Introduction

A major complication in conventional renormalization theory has been the fact that even in renormalized form Feynman integrals are only conditionally convergent [1]. In this paper a simple method of circumventing this difficulty is discussed. We propose to write the Feynman propagator in the form

$$\frac{1}{l_0^2 - \boldsymbol{l}^2 - \mu^2 + i\varepsilon(\boldsymbol{l}^2 + \mu^2)} . \tag{1.1}$$

With this convention it is easy to see that the Feynman integrals are absolutely convergent for $\varepsilon > 0$ provided the hypothesis of the power counting theorem applies (Section 2). Of course, the integrals are not relativistically covariant as long as $\varepsilon > 0$. But it will be shown in section 3 that the limit exists and defines a covariant distribution.

2. Power Counting Theorem for Minkowski Metric

We consider Feynman integrals of the form

$$I(q\,\mu\,\varepsilon) = \int dk \frac{P(kq)}{\prod_{j=1}^{n} f_j(k\,q, \mu_j\,\varepsilon)} \tag{2.1}$$

where

$$\begin{aligned} q &= (q_1 \ldots q_r) , \quad k = (k_1 \ldots k_m) , \\ dk &= dk_1 \ldots dk_m , \\ \mu &= (\mu_1 \ldots \mu_n) , \quad \mu_j \geqq 0 , \end{aligned} \tag{2.2}$$

with q_j, k_j denoting Minkowski four vectors. The functions f_j denote Feynman denominators in the modified form

$$f_j(k\,q_j\,\mu_j\,\varepsilon) = l_{j0}^2 - \boldsymbol{l}_j^2 - \mu_j^2 + i\varepsilon(\boldsymbol{l}_j^2 + \mu_j^2) . \tag{2.3}$$

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The four vectors $\boldsymbol{l}_j$ are of the form

$$l_j = \sum_{j'=1}^{m} c_{j j'} k_{j'} + q_j \,. \tag{2.4}$$

We further introduce the Euclidean integral corresponding to (2.1)

$$I_E(q\,\mu) = \int dk \frac{P(kq)}{\prod_{j=1}^{n} e_j(k\,q_j\,\mu_j)} \tag{2.5}$$

with

$$e_j(k\,q_j\,\mu_j) = l_{j0}^2 + \boldsymbol{l}_j^2 + \mu_j^2 \,. \tag{2.6}$$

The following theorem shows that both integrals (2.1) and (2.5) have the same convergence properties.

Theorem 1. *The Minkowskian integral* (2.1) *is absolutely convergent* (*a.c.*) *if and only if the corresponding Euclidean integral is a.c.*

Proof. The inequalities

$$\frac{\boldsymbol{l}^2 + \mu^2}{|l_0^2 - \boldsymbol{l}^2 - \mu^2 + i\varepsilon(\boldsymbol{l}^2 + \mu^2)|} \leqq \frac{1}{\varepsilon}$$

$$\frac{l_0^2}{|l_0^2 - \boldsymbol{l}^2 - \mu^2 + i\varepsilon(\boldsymbol{l}^2 + \mu^2)|} \leqq \sqrt{1 + \frac{1}{\varepsilon^2}}$$

imply

$$\frac{l_0^2 + \boldsymbol{l}^2 + \mu^2}{l_0^2 - \boldsymbol{l}^2 - \mu^2 + i\varepsilon(\boldsymbol{l}^2 + \mu^2)} \leqq \frac{1}{\varepsilon} + \sqrt{1 + \frac{1}{\varepsilon^2}} \,. \tag{2.7}$$

The Minkowskian integral (2.1) is therefore majorized by the Euclidean integral (2.5). Also (2.5) is majorized by (2.1) according to the inequality

$$\frac{|l_0^2 - \boldsymbol{l}^2 - \mu^2 + i\varepsilon(\boldsymbol{l}^2 + \mu^2)|}{l_0^2 + \boldsymbol{l}^2 + \mu^2} \leqq \sqrt{1 + \varepsilon^2} \,. \tag{2.8}$$

Combining Theorem 1 with Weinberg's version of the power counting theorem for Euclidean metric [1–3] one obtains

Theorem 2. *Let all masses* $\mu_j \neq 0$. *The integral* (2.1) *is a.c. if* (2.1) *and any subintegral*

$$I(q\,\mu\,\varepsilon\,H) = \int_H dV \frac{P(kq)}{\prod_{j=1}^{n} f_j(k\,q_j\,\mu_j\,\varepsilon)} \tag{2.9}$$

have negative dimension. H *denotes a hyperplane in* R_{4m} *described by a set of linear equations*

$$\sum_{j=1}^{m} d_{ij} k_j = r_i\,, \qquad i = 1, \ldots, t \,. \tag{2.10}$$

The dimension of a rational integral is defined by $d = d' + d''$ *where* d' *is the number of integration variables and* d'' *the degree of the integrand with respect to the integration variables.*

3. Parametrized Integrals

Introducing Feynman parameters $\alpha = (\alpha_1 \ldots \alpha_n)$ in the usual manner we obtain from (2.1)

$$I(q\,\mu\,\varepsilon) = (n-1)! \int dk \int_{\mathcal{D}} d\alpha \frac{P(kq)}{\left(\sum_{j=1}^{n} \alpha_j f_j(k\,q_j\,\mu_j\,\varepsilon)\right)^n} \tag{3.1}$$

with

$$d\alpha = d\alpha_1 \ldots d\alpha_{n-1}, \; \alpha_n = 1 - \sum_{j=1}^{n-1} \alpha_i$$

$\mathcal{D}$ is the set of all points $(\alpha_1 \ldots \alpha_{n-1})$ satisfying

$$\alpha_1 \geqq 0, \ldots, \alpha_n \geqq 0 \quad \text{and} \quad \sum_{i=1}^{n} \alpha_i = 1\,. \tag{3.2}$$

For deriving the parametrized integral it is necessary to interchange the k- and α-integration in (3.1). To justify this we prove the following Lemma.

Lemma 1. *If the k-integral* (2.1) *is a.c. the k-α-integral* (3.1) *is also a.c.*

Proof. According to Theorem 1 the a.c. of (2.1) implies the convergence of

$$\bar{I}_E = \int dk \frac{|P(kq)|}{\prod_{j=1}^{n} e_j(k\,q_j\,\mu_j)}\,. \tag{3.3}$$

Hence also

$$\bar{I}_E = (n-1)! \int dk \int_{\mathcal{D}} d\alpha \frac{|P(kq)|}{\left(\sum_{j=1}^{n} \alpha_j e_j(k\,q_j\,\mu_j)\right)^n} \tag{3.4}$$

is convergent. Since

$$\frac{\sum_{j=1}^{n} \alpha_j e_j(k\,q_j\,\mu_j)}{\left|\sum_{j=1}^{n} \alpha_j f_j(k\,q\,\mu_j\,\varepsilon)\right|} \leqq \frac{n}{\varepsilon} + n\sqrt{1 + \frac{1}{\varepsilon^2}} \tag{3.5}$$

the integral

$$\int dk \int_{\mathcal{D}} d\alpha \frac{|P(kq)|}{\left|\sum_{j=1}^{n} \alpha_j f_j(k\,q_j\,\mu_j\,\varepsilon)\right|^n} \tag{3.6}$$

is majorized by (3.4). Hence (3.1) is a.c.

By FUBINI's theorem it is therefore permitted to interchange the k- and α-integration in (3.1). We thus obtain

$$I(q\,\mu\,\varepsilon) = \int_{\mathcal{D}} d\alpha\, B(q\,\mu\,\alpha\,\varepsilon) = \int_{\mathcal{D}-\mathcal{E}} d\alpha\, B(q\,\mu\,\alpha\,\varepsilon)\,, \tag{3.7}$$

$$B(q\,\mu\,\alpha\,\varepsilon) = (n-1)! \int dk \frac{P(kq)}{\left(\sum_{j=1}^{n} \alpha_j f_j(k\,q_j\,\mu_j\,\varepsilon)\right)^n}\,. \tag{3.8}$$

Here $\mathcal{E}$ denotes the set of all $\alpha \in \mathcal{D}$ for which at least one $\alpha_i = 0$ $(i = 1, \ldots, n)$. The integral B can be evaluated by standard methods.

We shortly indicate the derivation. Since

$$\alpha_1 > 0, \ldots, \alpha_n > 0 \quad \text{in} \quad \mathcal{D} - \mathcal{E}$$

we need compute B for positive α only. We consider the quadratic form

$$F(k\, q\, \alpha) = \sum_{i=1}^{n} \alpha_i\, l_i^2 = \sum a_{ij}\, k_i\, k_j + 2 \sum Q_j\, k_j + Q\,. \tag{3.9}$$

The matrix a_{ij} depends on α only and has the properties

$$\begin{aligned} &a_{ij} = a_{ji} \\ &d(\alpha) = \det(a_{ij}) > 0 \quad \text{for} \quad \alpha_1, \ldots, \alpha_n > 0\,. \end{aligned} \tag{3.10}$$

The stationary value of F is

$$F_0(q\, \alpha) = F(k^0\, q\, \alpha) \tag{3.11}$$

where k^0 is the solution of

$$\sum_{j=1}^{m} a_{ij}\, k_j^0 + Q_i = 0\,. \tag{3.12}$$

One obtains

$$F = \sum a_{ij}\, k_i'\, k_j' + F_0(q\, \alpha) = \sum_{j=1}^{m} \lambda_j\, \tilde{k}_j^2 + F_0(q\, \alpha) \tag{3.13}$$

with the transformations

$$k_i' = k_i - k_i^0\,, \qquad \tilde{k}_i = \sum_{j=1}^{m} O_{ij}\, k_j'\,, \qquad O = (O_{ij}) \text{ orthogonal}\,. \tag{3.14}$$

With this result we obtain for the denominator of (3.8)

$$\begin{aligned} &\sum_{j=1}^{n} \alpha_j\, f_j(k\, q_j\, \mu_j\, \varepsilon) = \sum_{j=1}^{m} \lambda_j\, \tilde{k}_j^2 + i\, \varepsilon \sum_{j=1}^{m} \lambda_j\, \tilde{\boldsymbol{k}}_j^2 + C\,, \\ &\qquad C = F_0(q\, \alpha) + i\, \varepsilon\, F_0(\boldsymbol{q}\, \alpha) - (1 - i\, \varepsilon) \sum_{j=1}^{n} \alpha_j\, \mu_j^2\,, \\ &F_0(\boldsymbol{q}\, \alpha) = F_0(q\, \alpha) \quad \text{with} \quad q_j^0 = 0\,. \end{aligned} \tag{3.15}$$

Introducing the polynomial $\overline{P}$ by

$$\begin{aligned} &\overline{P}(\bar{k}\, q\, \alpha) = P(k\, q) \\ &\qquad k = O^{-1}\, \lambda^{-1/2}\, \bar{k} + k^0 \end{aligned} \qquad \lambda = (\lambda_i \delta_{ij}) \tag{3.16}$$

we obtain

$$B = \frac{(n-1)!}{d(\alpha)^2} \int dk\, \frac{\overline{P}(k\, q\, \alpha)}{(\sum k_j^2 + i\, \varepsilon \sum \boldsymbol{k}_j^2 + C)^n}\,. \tag{3.17}$$

We decompose $\overline{P}$ into parts which are homogeneous in each $k_{i\mu}$ separately

$$\overline{P}(k\, q\, \alpha) = \sum_{\nu_{10} \ldots \nu_{m3}} \prod_{i=0}^{m} \prod_{\mu=0}^{3} (k_{i\mu})^{\nu_{i\mu}}\, \Pi_{\nu_{10} \ldots \nu_{m3}}(q\, \alpha)\,. \tag{3.18}$$

So B becomes

$$B = \sum_{\gamma_{10} \ldots \gamma_{m3}} B_{\gamma_{10} \ldots \gamma_{m3}}\, \Pi_{2\gamma_{10}, \ldots, 2\gamma_{m3}} \tag{3.19}$$

with the coefficients

$$B_{\gamma_{10}\ldots\gamma_{m3}} = \frac{(n-1)!}{d(\alpha)^2} \int dk \, \frac{(k_{10})^{2\gamma_{10}} \ldots (k m_3)^{2\gamma_{m3}}}{(\Sigma k_j^2 + i\,\varepsilon\, \Sigma \boldsymbol{k}_j^2 + C)^n} \,. \tag{3.20}$$

The remaining integrations are elementary and can easily be carried out. We state the final result in the following lemma.

Lemma 2. *If the integral* (2.1) *is a.c. it can be brought into the parametrized form*

$$I(q\,\mu\,\varepsilon) = \int_{\mathscr{D}} d\,\alpha\, B(q\,\mu\,\alpha\,\varepsilon) \tag{3.21}$$

where (except for the set $\mathscr{E}$ of measure zero) the integrand B is given by

$$B(q\,\mu\,\alpha\,\varepsilon) = \sum_{\gamma_{10}\ldots\gamma_{m3}} B_{\gamma_{10}\ldots\gamma_{m3}}(q\,\alpha\,\varepsilon)\, \Pi_{2\gamma_{10},\ldots,2\gamma_{m3}}(q\,\alpha)\,. \tag{3.22}$$

The Π are polynomials in q defined by (3.18). *The functions B are explicitly*

$$B_{\gamma_{10}\ldots\gamma_{m3}} = \frac{(\gamma_{10})\ldots(\gamma_{m3})\;(n-\gamma-2m-1)!\;\pi^2\, i^m}{2^\gamma (i\,\varepsilon - 1)^\sigma\, d(\alpha)^2\, C^{n-\gamma-2m}} \tag{3.23}$$

with

$$\begin{aligned}
C &= F_0(q\,\alpha) + i\,\varepsilon\, F_0(\boldsymbol{q}\,\alpha) - (1 - i\,\varepsilon) \sum_{j=1}^{n} \alpha_j\,\mu_j^2\,,\\
\gamma &= \sum_{j=1}^{n} \sum_{\mu=0}^{3} \gamma_{j\mu}\,,\\
\sigma &= \sum_{j=1}^{n} \sum_{\mu=1}^{3} \gamma_{j\mu} + \frac{3m}{2}\,,\\
(N) &= 1\cdot 3\cdot 5 \ldots (2N-1) \quad \text{for} \quad N = 1, 2, \ldots,\\
(0) &= 1\,.
\end{aligned} \tag{3.24}$$

Formula (3.23) is still unsatisfactory insofar as the denominator contains terms which are not covariant. The following theorem, however, shows that these terms may be dropped in the limit $\varepsilon \to +0$. Since this limit will be considered in the sense of distributions we have to specify the domain of definition for the variables $q_1 \ldots q_n$. We assume that q varies over a $4r$-dimensional vector space described by

$$\begin{aligned}
&q = \beta\, p\\
&\beta = (\beta_{ij})\,, \quad p = (p_1 \ldots p_r)\,, \quad r \leqq n
\end{aligned} \tag{3.25}$$

where the p_j are arbitrary four vectors and the $n \times r$ matrix β has rank r. This assumption covers all situations where Feynman integrals are considered as distributions in the external variables.

Theorem 3. *For $\varepsilon \to +0$ the functions $I(q(p), \mu, \varepsilon)$ converge strongly in $\mathscr{S}'(R_{4r})$ to a tempered distribution*

$$I(q(p), \mu) = \lim_{\varepsilon \to +0} I(q(p), \mu, \varepsilon) \quad \text{in} \quad \mathscr{S}'(R_{4r})\,. \tag{3.26}$$

$I(q(p), \mu)$ can as well be expressed as the strong limit of covariant parametrized integrals

$$I(q(p), \mu) = \lim_{\varepsilon \to +0} \Phi(q(p), \mu, \varepsilon) \quad \text{in} \quad \mathcal{S}'(R_{4r}) \tag{3.27}$$

where

$$\Phi(q\,\mu\,\varepsilon) = \int_{\mathcal{D}} d\alpha\, \Psi(q\,\mu\,\alpha\,\varepsilon)$$

$$\Psi(q\,\mu\,\alpha\,\varepsilon) = \sum_{\gamma_{10}\ldots\gamma_{m3}} \Xi_{\gamma_{10}\ldots\gamma_{m3}}(q\,\alpha)\, \Pi_{2\gamma_{10},\ldots,2\gamma_{m3}}(q\,\alpha)$$

$$\Xi_{\gamma_{10}\ldots\gamma_{m3}} = \frac{(\gamma_{10})\cdots(\gamma_{m3})\;(n-\gamma-2m-1)!\;\pi^2\, i^m}{2^\gamma\, d(\alpha)^2\, \Delta^{n-\gamma-2m}} \tag{3.28}$$

$$\Delta = F_0(q\,\alpha) - \sum_{j=1}^{m} \alpha_j\,\mu_j^2 + i\,\varepsilon \sum_{j=1}^{m} \alpha_j\,\mu_j^2\,.$$

The distribution $I(q(p), \mu)$ is explicitly given by the mapping

$$\varphi(p) \to \lim_{\varepsilon \to +0} \int dp\, \varphi(p)\, \Phi(q(p), \mu, \varepsilon) \quad \text{for} \quad \varphi \in \mathcal{S}(R_{4r})\,. \tag{3.29}$$

Proof. The limit $\varepsilon \to +0$ of parametrized Feynman integrals has been discussed by HEPP [4]. Since the integral (3.21–23) is of a slightly different type we first reformulate the problem such that Hepp's method can be applied.

The function $I(q\,\mu\,\varepsilon)$ is of the form

$$I(q(p), \mu, \varepsilon) = \int_{\mathcal{D}} B(q(p), \mu, \alpha, \varepsilon)$$

$$B(q(p), \mu, \alpha, \varepsilon) = \frac{N(p\,\mu\,\varepsilon\,\alpha)}{(i\,\varepsilon - 1)^s\, C^{n-2m}} \tag{3.30}$$

where N is a polynomial in p and ε. The function $I(q(p), \mu, \varepsilon)$ converges in $\mathcal{S}'(R_{4r})$ to a distribution if the limit

$$\lim_{\varepsilon \to +0} \int dp\, \varphi(p)\, I(q(p), \mu, \varepsilon) \tag{3.31}$$

exists for every $\varphi \in \mathcal{S}(R_{4r})$. Hence the existence of (3.31) must be shown.

We have

$$\int dp\, \varphi(p)\, I(q(p), \mu, \varepsilon)$$

$$= \sum_{\nu_{10}\ldots\nu_{r3}\nu} \int dp \int_{\mathcal{D}} d\alpha\, \varphi(p)\, N^{(\nu)}_{\nu_{10}\ldots\nu_{r3}}(\mu\,\alpha)\, \frac{\varepsilon^\nu\, P_{10}^{\nu_{10}} \cdots P_{r3}^{\nu_{r3}}}{(i\,\varepsilon - 1)^s\, C^{n-2m}} \tag{3.32}$$

where $N^{(\nu)}_{\nu_{10}\ldots\nu_{r3}}$ denote the coefficients of the polynomial N with respect to p and ε

$$N(q(p), \mu, \alpha, \varepsilon) = \sum_{\nu_{10}\ldots\nu_{r3}\nu} N^{(\nu)}_{\nu_{10}\ldots\nu_{r3}}(\mu\,\alpha)\, \varepsilon^\nu\, p_{10}^{\nu_{10}} \cdots p_{r3}^{\nu_{r3}}\,. \tag{3.33}$$

Each term on the right hand side of (3.32) is a.c. with respect to the p- and α-integration as can be seen in the following way. First we note that (3.30) is a.c. since

$$\int d\alpha\, |B(q\,\mu\,\alpha\,\varepsilon)| \leqq (n-1)! \int_{\mathcal{D}} d\alpha \int dk \left| \frac{P(k\,q)}{(\sum \alpha_j\, f_j(k\,q_j\,\mu_j))^n} \right|$$

(Lemma 1, p. 3). The a.c. of (3.30) implies the a.c. of

$$\int_{\mathscr{D}} d\,\alpha\, N(q\,\mu\,\alpha\,\varepsilon) \tag{3.34}$$

because

$$|C| \leqq M \quad \text{for} \quad \alpha \in \mathscr{D}\,.$$

Applying Lemma 3a of ref. [3] we obtain that

$$\int_{\mathscr{D}} d\,\alpha\, N^{(\nu)}_{\nu_{10}\ldots\nu_{r3}}(\mu\,\alpha) \quad \text{is a.c.} \tag{3.35}$$

On the other hand we have $|C| \geqq \gamma > 0$ for $\alpha \in \mathscr{D}$ since $\mu_i > 0$. Hence each term on the right hand side of (3.32) is a.c. according to

$$\int d\,p \int_{\mathscr{D}} d\,\alpha \left|\varphi(p)\, N^{(\nu)}_{\nu_{10}\ldots\nu_{r3}} \frac{p_{10}^{\nu_{10}}\cdots p_{r3}^{\nu_{r3}}}{C^{n-2m}}\right|$$
$$\leqq A \int d\,p\,|\varphi(p)\, p_{10}^{\nu_{10}}\cdots p_{r3}^{\nu_{r3}}| \int_{\mathscr{D}} d\,\alpha\, |N^{(\nu)}_{\nu_{10}\ldots\nu_{r3}}|\,.$$

It is therefore permitted to interchange the p- and α-integration in (3.32) and we obtain

$$\int d\,p\,\varphi(p)\, I(p\,\mu\,\varepsilon)$$
$$= \sum_{\nu_{10}\ldots\nu_{r3}\,\nu} \int_{\mathscr{D}} d\,\alpha\, N^{(\nu)}_{\nu_{10}\ldots\nu_{r3}}(\mu\,\alpha) \int d\,p\, \frac{\varepsilon^{\nu}\, p_{10}^{\nu_{10}}\cdots p_{r3}^{\nu_{r3}}\,\varphi(p)}{(i\,\varepsilon - 1)^{s}\, C^{n-2m}}\,. \tag{3.36}$$

For proving the existence of (3.31) it is therefore sufficient to show that the limit $\varepsilon \to +0$ of

$$\int_{\mathscr{D}} d\,\alpha \int d\,p\;\psi(p)\, \frac{N^{(\nu)}_{\nu_{10}\ldots\nu_{r3}}(\mu\,\alpha)}{C^{n-2m}} \tag{3.37}$$

exists for every $\psi \in \mathscr{S}(R_{4r})$. C is given by (3.24) with $q(p)$ substituted for q. $F_0(q(p), \alpha)$ is a quadratic form in the p_j

$$F_0(q(p), \alpha) = \sum_{i,j=1}^{r} A_{ij}(\alpha)\, p_i\, p_j \tag{3.38}$$

(3.37) has the form of the parametrized integrals for which the limit $\varepsilon \to +0$ was studied by Hepp (Eq. (4.9–10) of ref. [4]). According to (3.35) the coefficients $N^{(\nu)}_{\nu_{10}\ldots\nu_{r3}}$ are absolutely integrable in $\mathscr{D}$. We will further show that the coefficients $A_{ij}(\alpha)$ (originally defined for $\alpha_j > 0$) can be extended to continuous functions of α in $\alpha_j \geqq 0$. To this end we introduce the quadratic form

$$G(x\,y\,\alpha) = \sum_{i=1}^{n} \alpha_i\, z_i^2$$
$$z_i = \sum_{j=1}^{n} c_{ij}\, x_j + \sum_{j=1}^{r} \beta_{ij}\, u_j \tag{3.39}$$

which is obtained from $F(k, q(p), \alpha)$ by replacing the four vectors k_j, p_j by real variables x_j or y_j resp. Then we define

$$G_0(y\,\alpha) = \operatorname*{Inf}_x G(x\,y\,\alpha) = \sum_{i,j=1} A_{ij}(\alpha)\, y_i\, y_j\,. \tag{3.40}$$

Here the $A_{ij}(\alpha)$ are continuous functions of α in $\alpha_i \geqq 0$ and coincide with the coefficients of (3.38) for $\alpha_j > 0$.

Following HEPP it can now be shown that the limit (3.31) exists. Hence $I(q(p), \mu, \varepsilon)$ approaches a distribution in $\mathscr{S}'(R_{4r})$ for $\varepsilon \to +0$.

By a similar argument it follows that also $\Phi(q(p), \mu, \varepsilon)$ converges in $\mathscr{S}'(R_{4r})$ for $\varepsilon \to +0$.

We further have

$$\lim_{\varepsilon \to +0} \big(I(q(p), \mu, \varepsilon) - \Phi(q(p), \mu, \varepsilon)\big) = 0$$

since

$$\lim_{\varepsilon \to +0} \int dq\, \varphi(q) \left\{\frac{1}{(1 - i\,\varepsilon)^\sigma C^{n-2m-\gamma}} - \frac{1}{\Delta^{n-2m-\gamma}}\right\} = 0$$

for every $\varphi \in \mathscr{S}(R_{4r})$.

This completes the proof of the theorem.

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