Commun. math. Phys. 10, 274—279 (1968)

An Existence Proof for the Gap Equation in the Superconductivity Theory

P. BILLARD and G. FANO*

Centre de Physique Théorique et Département de Physique Mathématique d'Université d'Aix Marseille

Received May 28, 1968

Abstract. An existence theorem for the "gap equation" in the superconductivity theory is given, as a consequence of the Schauder-Tychonoff theorem. Sufficient conditions on the kernel are given, which insure the existence of a solution amongst a particular class of continuous functions. The case of a positive kernel is studied in detail.

1. Introduction

For a non relativistic many-fermion system the existence of a "superfluid" or "superconducting" state is related to the appearence of non trivial solutions in a non linear integral equation, called the "gap equation".

Various approximation methods for finding the solution of the gap equation have been devised [1, 2, 3], which give rise to a "linearization" of the equation. All these methods produce solutions with the same non-analytic behaviour for small values of the interaction strength. A necessary condition for the appearence of non trivial solutions has been given a long time ago by COOPER, MILLS and SESSLER [4] (see also ref. 1). The convergence of an iterative procedure has been proved, under certain conditions, by KITAMURA [5]. Fixed point theorems were first used by ODEH [6]. We prove here an existence theorem under entirely different assumptions, which cover many cases of physical interest. We make use of the Schauder-Tychonoff theorem, which allows us to find a solution amongst a particular class of continuous functions.

2. The Existence Theorem

Let us consider the gap equation in its simplest form (i.e. the equation for the spherically symmetrical solutions at zero temperature):

$$\varphi(k) = \int_0^\infty K(k, k') \frac{\varphi(k')}{\sqrt{(k'^2 - 1)^2 + \varphi(k')^2}} \, dk' \tag{1}$$

* Supported by contract D. G. R. S. T. No. 6700 976.

and make the following hypotheses on the kernel $K(k, k')$:

(I) $\begin{cases} K \text{ is a measurable real valued bounded function on } \boldsymbol{R}^+ \times \boldsymbol{R}^+; \\ \text{let } M > 0 \text{ be a bound such that } |K(k, k')| \leqq M \text{ for every} \\ (k, k') \in \boldsymbol{R}^+ \times \boldsymbol{R}^+. \end{cases}$

(II) $\begin{cases} \text{There exists a compact interval } I = \{k\colon \xi_1 \leqq k \leqq \xi_2\},\ I \subset \boldsymbol{R}^+, \\ \xi_1 < 1 < \xi_2 \text{ such that } K(k, k') \geqq 0 \text{ for } (k, k') \in I \times I. \end{cases}$

(III) $\begin{cases} \text{There exist three positive numbers } a, A, \varepsilon\,(0 < a < A,\ \varepsilon > 0) \\ \text{such that the following inequalities hold.} \end{cases}$

(III$_1$) $$\int_I K(k, k') \frac{1}{\sqrt{(k'^2 - 1)^2 + a^2}}\, dk' \geqq 1 + \varepsilon \quad \text{for} \quad k \in I\,.$$

(III$_2$) $$\int_{\boldsymbol{R}^+ - I} |K(k, k')| \frac{1}{\sqrt{(k'^2 - 1)^2 + A^2}}\, dk' \leqq \frac{a}{A}\,\varepsilon \quad \text{for} \quad k \in I\,.$$

(III$_3$) $$\int_{\boldsymbol{R}^+} |K(k, k')| \frac{1}{\sqrt{(k'^2 - 1)^2 + A^2}}\, dk' \leqq 1 \quad \text{for all} \quad k \in \boldsymbol{R}^+\,.$$

(IV) There exists an $L > 0$ such that

$$\int_{\boldsymbol{R}^+} |K(k_1, k') - K(k_2, k')| \frac{A}{\sqrt{(k'^2 - 1)^2 + A^2}}\, dk' \leqq L|k_1 - k_2|$$

for every $(k_1, k_2) \in \boldsymbol{R}^+ \times \boldsymbol{R}^+$.

For the remainder of this section we will consider only kernels verifying conditions (I) . . . (IV).

Definition 1. Let $\mathscr{F}(\boldsymbol{R}^+)$ be the space of all continuous numerical functions on $\boldsymbol{R}^+$, with the topology of uniform convergence on compacts. $\mathscr{F}(\boldsymbol{R}^+)$ is a Fréchet space.

We consider now the following subset of $\mathscr{F}(\boldsymbol{R}^+)$:

$$\mathscr{K} = \mathscr{K}(\xi_1, \xi_2, a, A, L) = \Big\{ f \in \mathscr{F}(\boldsymbol{R}^+)\colon f \text{ real valued},$$

$$\|f\|_\infty = \sup_{k \in \boldsymbol{R}^+} |f(k)| \leqq A, \inf_{k \in I} f(k) \geqq a, \lambda(f) = \sup_{\substack{(k_1, k_2) \in \boldsymbol{R}^+ \times \boldsymbol{R}^+ \\ k_1 \neq k_2}} \left| \frac{f(k_1) - f(k_2)}{k_1 - k_2} \right| \leqq L \Big\}.$$

It is straightforward to prove the following proposition:

Proposition 1. *$\mathscr{K}$ is a convex compact subset of $\mathscr{F}(\boldsymbol{R}^+)$, and $0 \notin \mathscr{K}$.*

Furthermore we have:

Proposition 2. *The application $T\colon \mathscr{K} \to \mathscr{F}(\boldsymbol{R}^+)$ defined, for every $f \in \mathscr{K}$ by*

$$(T(f))(k) = \int_{\boldsymbol{R}^+} K(k, k') \frac{f(k')}{\sqrt{(k'^2 - 1)^2 + f(k')^2}}\, dk' \quad (k \in \boldsymbol{R}^+)$$

is a continuous mapping of $\mathscr{K}$ into $\mathscr{K}$.

Proof. By (I), $(T(f))(k)$ is defined for every $k \in \boldsymbol{R}^+$ because

$$\left|\frac{f(k')}{\sqrt{(k'^2-1)^2+f(k')^2}}\right| \leqq \frac{A}{\sqrt{(k'^2-1)^2+A^2}} \text{ for every } k' \in \boldsymbol{R}^+ \tag{2}$$

and $T(f)$ is real valued because of (I).

By (IV) and the condition $\|f\|_\infty \leqq A$ we have:

$$|T(f)(k_1) - T(f)(k_2)| \leqq \int_0^\infty |K(k_1,k') - K(k_2,k')| \frac{A}{\sqrt{(k'^2-1)^2+A^2}}$$

$$\leqq L|k_1 - k_2| \quad \text{for} \quad (k_1,k_2) \in \boldsymbol{R}^+ \times \boldsymbol{R}^+, k_1 \neq k_2\,.$$

Therefore $\lambda(T(f)) \leqq L$ which implies in particular $T(f) \in \mathscr{F}(\boldsymbol{R}^+)$. Furthermore, for every $k \in \boldsymbol{R}^+$ we have by (III_3)

$$|T(f)(k)| \leqq \int_0^\infty |K(k,k')| \frac{A}{\sqrt{(k'^2-1)^2+A^2}}\, dk' \leqq A\,.$$

Consequently $\|T(f)\|_\infty \leqq A$.

For $k \in I$, we have by (II), (III_1), (III_2) and the inequality $\inf_{k' \in I} f(k') \geqq a$

$$T(f)(k) \geqq \int_I K(k,k') \frac{a}{\sqrt{(k'^2-1)^2+a^2}}\, dk' - \int_{\boldsymbol{R}^+ - I} |K(k,k')|$$

$$\cdot \frac{A}{\sqrt{(k'^2-1)^2+A^2}}\, dk' \geqq a(1+\varepsilon) - a\varepsilon = a; \quad \text{so} \quad \inf_{k \in I} T(f)(k) \geqq a\,.$$

Therefore $T(\mathscr{K}) \subset \mathscr{K}$. It still remains to prove the continuity of T.

As $\mathscr{K} \subset \mathscr{F}(\boldsymbol{R}^+)$ is a metrizable space, in order to prove the continuity of T on $\mathscr{K}$ it is sufficient to show that from

$$f_n \in \mathscr{K} \quad (n = 1, 2 \ldots), f \in \mathscr{K}, f_n \xrightarrow[n\to\infty]{} f \quad \text{in} \quad \mathscr{K}$$

it follows that $T(f_n) \xrightarrow[n\to\infty]{} T(f)$ in $\mathscr{K}$.

In order to see that this is the case, let's fix an arbitrary number $\eta > 0$ and write

$$|T(f_n)(k) - T(f)(k)|$$

$$\leqq \int_0^\infty |K(k,k')| \left| \frac{f_n(k')}{\sqrt{(k'^2-1)^2+f_n(k')^2}} - \frac{f(k')}{\sqrt{(k'^2-1)^2+f(k')^2}} \right| dk'$$

$$= \int_0^{k_1} + \int_{k_1}^\infty = J_1 + J_2\,.$$

If k_1 is chosen large enough so that

$$\int_{k_1}^{\infty} M \frac{2A}{\sqrt{(k'^2 - 1)^2 + A^2}}\, dk' \leqq \frac{\eta}{2}, \quad \xi_2 \leqq k_1 < \infty$$

we have $J_2 \leqq \frac{\eta}{2}$ independently of $k \in \boldsymbol{R}^+$ and n. k_1 being fixed by this condition, there is uniform convergence of

$$r_n(k') = \frac{f_n(k')}{\sqrt{(k'^2 - 1)^2 + f_n(k')^2}} - \frac{f(k')}{\sqrt{(k'^2 - 1)^2 + f(k')^2}}$$

to zero on the interval $[0, k_1]$, when $n \to \infty$. This follows from the inequality:

$$|r_n(k')| \leqq \frac{1}{\inf[a^2, (\xi_2^2 - 1)^2, (\xi_2^1 - 1)^2]} \cdot \left| f_n(k') \sqrt{(k'^2 - 1)^2 + f(k')^2} - f(k') \sqrt{(k'^2 - 1)^2 + f_n(k')^2} \right|.$$

As $J_1 \leqq M \int_0^{k_1} |r_n(k')|\, dk'$, there exists an entire n_0 such that for $n \geqq n_0$, $J_1 \leqq \frac{\eta}{2}$ independently of k. Therefore $n \geqq n_0 \Rightarrow \|T(f_n) - T(f)\|_\infty \leqq \eta$ which proves Proposition 2.

Theorem. *Eq.* (1) *admits at least one solution* $\varphi \in \mathscr{K}$. *(Therefore in particular* $\varphi \neq 0$.*)*

Proof. The theorem follows immediately from Propositions 1 and 2 by applying the Schauder-Tychonoff theorem [7].

Remark. Condition IV holds if the following condition is verified:

$$\text{(V)} \quad \begin{cases} \text{There exists } N > 0 \text{ such that, for every fixed } k' \in \boldsymbol{R}^+, \text{ the function} \\ K_{k'} : K_{k'}(k) = K(k, k')\ (k \in \boldsymbol{R}^+) \text{ verifies } \lambda(K_{k'}) \leqq N. \end{cases}$$

This happens in particular if for every fixed $k' \in \boldsymbol{R}^+$, the function $K_{k'}$ is continuous on $\boldsymbol{R}^+$, differentiable on $\boldsymbol{R}^+$ except at most for a denumerable set of points of $\boldsymbol{R}^+$, and the absolute value of this derivative is majorized by N.

In general condition III_1 can be satisfied with a sufficiently small $a > 0$, and condition III_3 can be satisfied with a sufficiently large $A > 0$. In order to produce a large clase of kernels fulfilling all the conditions, it is then sufficient to consider kernels which vanish sufficiently fast outside of $I \times I$ (in order to verify condition III_2) and which are sufficiently regular (in order to verify condition IV).

3. The Case of a Positive Kernel

If $K(k, k') > 0$ for every $(k, k') \in \boldsymbol{R}^+ \times \boldsymbol{R}^+$, one is tempted to put $I = \boldsymbol{R}^+$, because the inequality III_2 is then automatically satisfied.

However, since in all reasonable physical cases $\lim_{k\to\infty} K(k, k') = 0$, it is not possible in general to find in a such that the inequality III_1, written with $I = \boldsymbol{R}^+$, is satisfied.

In order to avoid this difficulty we consider, in the place of $\mathscr{K}$, the following subset of $\mathscr{F}(\boldsymbol{R}^+)$:

$$\mathscr{K}' = \mathscr{K}'(a, A, L) = \Big\{ f \in \mathscr{F}(\boldsymbol{R}^+) : f \text{ real valued} > 0\,,$$

$$\|f\|_\infty = \sup_{k\in\boldsymbol{R}^+} f(k) \leqq A,\, f(k) \geqq aK(k, 1),\, \lambda(f) \leqq L \Big\}$$

and we make the following hypotheses on the kernel $K(k, k')$:

(I′) K is a measurable bounded function > 0 on $\boldsymbol{R}^+ \times \boldsymbol{R}^+$; let M be a bound such that $K(k, k') \leqq M$ for every

$$(k, k') \in \boldsymbol{R}^+ \times \boldsymbol{R}^+ .$$

(II′) There exists an $a > 0$ such that the following inequality holds:

$$\int_{\boldsymbol{R}^+} \frac{K(k, k')\, K(k', 1)}{K(k, 1)\sqrt{(k'^2 - 1)^2 + a^2 K(k', 1)^2}}\, dk' \geqq 1 \quad \text{for all} \quad k \in \boldsymbol{R}^+ .$$

(III′) There exists an $L > 0$ such that

$$\int_{\boldsymbol{R}^+} |K(k_1, k') - K(k_2, k')| \frac{A}{\sqrt{(k'^2 - 1)^2 + A^2}}\, dk' \leqq L|k_1 - k_2|$$

for every $(k_1, k_2) \in \boldsymbol{R}^+ \times \boldsymbol{R}^+$,
where A is a positive number verifying the inequalities:

$$M \int_{\boldsymbol{R}^+} \frac{1}{\sqrt{(k'^2 - 1)^2 + A^2}}\, dk' \leqq 1; \quad A > aK(1, 1) .$$

It is straightforward to prove that propositions 1 and 2, as well as the existence theorem, hold equally well if we replace $\mathscr{K}$ by $\mathscr{K}'$, and we take into account the new hypotheses (I′), (II′), (III′) on the kernel. In particular if $f \in \mathscr{K}'$, we have, making use of the inequality (II′):

$$T(f)(k) = \int_{\boldsymbol{R}^+} K(k,k') \frac{f(k')}{\sqrt{(k'^2 - 1)^2 + f(k')^2}}\, dk'$$

$$\geqq \int_{\boldsymbol{R}^+} K(k,k') \frac{aK(k', 1)}{\sqrt{(k'^2 - 1)^2 + a^2 K(k', 1)^2}}\, dk' \geqq aK(k, 1) .$$

Example. Let us consider the kernel

$$K(k, k') = \frac{Vk'}{\pi k} \int_0^\infty dr\, e^{-\alpha r} \sin kr \sin k'r$$

$$= \frac{2V\alpha}{\pi} \frac{k'^2}{[\alpha^2 + (k + k')^2]\,[\alpha^2 + (k - k')^2]}\, (\alpha > 0,\, V > 0) .$$

This kernel arises naturally in physical situations (see ref. 2); it corresponds to an attractive two body potential of the form $V(r) = V e^{-\alpha r}$, r being the interparticle distance. It is easy to verify that

$$K(k, k') \leqq \frac{2V}{\pi\alpha}\,; \left|\frac{\partial K(k, k')}{\partial k}\right| \leqq \frac{8V}{\pi\alpha} \sup\left(1, \frac{1}{\alpha^2}\right)$$

for every $(k, k') \in \boldsymbol{R}^+ \times \boldsymbol{R}^+$
and therefore the kernel verifies the conditions (I′), (III′) (see the preceding Remark). It is also immediate to see that the function

$$D(k') = \inf_{k \in \boldsymbol{R}^+} \frac{K(k, k')}{K(k, 1)}$$

is continuous and strictly positive for $k' > 0$.

Therefore choosing a sufficiently small in order that

$$\int_{\boldsymbol{R}^+} D(k') \frac{K(k', 1)}{\sqrt{(k'^2 - 1)^2 + a^2 K(k', 1)^2}}\, dk' \geqq 1$$

also condition (II′) is verified, and the existence theorem applies.

Acknowledgements. One of us (G. F) would like to thank Professor D. Kastler for the hospitality received at the University of Aix-Marseille, and the D.G.R.S.T. for financial support.

References

1. Bogoliubov, N. N., V. V. Tolmachev, and D. V. Shirkov: A new method in the theory of superconductivity. Consultant Bureau, Inc. (1959).
2. Fano, G., and A. Tomasini: Nuovo Cimento **18**, 1247 (1960); Fano, G., M. Savoia, and A. Tomasini: Nuovo Cimento **21**, 854 (1961).
3. Emery, V. J., and A. M. Sessler: Phys. Rev. **119**, 43 (1960); 248 (1960).
4. Cooper, L. N., R. L. Mills, and A. M. Sessler: Phys. Rev. **114**, 1377 (1959).
5. Kitamura, M.: Progr. Theor. Phys. **30**, 435 (1963).
6. Odeh, F.: I.B.M. Journal Research Development (USA) 8, 187 (1964).
7. Dunford, N., and J. T. Schwartz: Linear operators, Part I, p. 456. New York: Interscience Publishers Inc.

P. Billard
Département de Physique
Mathématique, Faculté des Sciences
Université d'Aix Marseille
Place Victor Hugo
F 13 Marseille 3e

G. Fano
Istituto di Fisica
dell' Università
Bologna, Italia