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Absence of Interaction in Lie Field Theories*

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Abstract. It is shown that theories of relativistic Lie fields cannot lead to scattering or reactions, even if an infinite number of Lie fields is present.

1. Introduction

Generalized free fields can be considered to be the lowest order case of models of fields whose commutator can be expanded in products of Heisenberg fields containing up to a given number of factors. The next more complicated case, in which the commutator is a linear functional of the field, shares with the case of generalized free fields the property of being a soluble model, but has a more complicated structure, which might lead to scattering and reactions. For this reason, we suggested study of this case, now called Lie fields, and pointed out that the vacuum expectation values are determined by recursion in terms of the distributions introduced in the ansatz for the commutator and thus determine the field. At the same time, we pointed out that translation invariance, Lorentz invariance, locality, and positive energy spectrum can easily be satisfied by themselves, but that the Jacobi identity for the commutator and the positive definiteness conditions for the vacuum expectation values, both of which are essential for a consistent model of field theory, seemed to be difficult to fulfill [1]. Recently, Lowenstein [2] has shown that the case of Lie fields is not empty by giving non-trivial examples of scalar Lie fields. These examples make it worthwhile to study further the possibility that Lie field theories give rise to scattering and reactions. Glaser [3] has shown that a Lie field theory with a finite number of fields gives an elastic scattering amplitude whose absorptive part has only a finite number of partial waves in the s-channel. Such an absorptive part is a polynomial in $z = \cos\theta$ and fails to have the singularities in z which are required by unitarity and crossing [4]. In this article we will show that no scattering or reactions can occur in a Lie field theory, even if an infinite number of Lie fields is present. Our argument is based on unitarity, as expressed by the commutation relations of the in and out

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fields, together with a study of the contradiction between the necessity of principal value mass-shell singularities to obtain scattering and reactions[1] and the restrictions on such singularities imposed by the stability of the one-particle states. We assume Poincaré invariance, but we do not make explicit use of local commutativity[2].

We write the commutation relations in momentum space as[3]

$$[\Phi_\alpha(p), \Phi_\beta(q)] = \varepsilon(p)\, \varrho_{\alpha\beta}(p^2)\, \delta(p+q) + g_{\alpha\beta\gamma}(p, q)\, \Phi_\gamma(p+q)\,, \quad (1)$$

where α, β, γ stand for both internal and space-time indices and can take infinitely many values, and γ is summed. For simplicity, we consider only systems of hermitian Bose-like fields with the linear ansatz for the commutators. (Non-hermitian fields can be decomposed into pairs of hermitian fields.) Arguments similar to those below show that our results also hold for the cases in which Fermi-like fields, with the linear ansatz for the anti-commutator, are included. We assume that $\langle\Phi_\alpha(p)\rangle_0 = 0$.

2. Proof that Lie Field Theories Cannot Lead to Scattering or Reactions

The essential idea of the proof that Lie field theories cannot lead to scattering or reactions is that scattering or reactions require the in and out fields to differ, which requires principal value mass-shell singularities in the matrix elements of the Lie fields. The positive-definiteness conditions imply that $\varrho_{\alpha\beta}(p^2)$ is a positive matrix. Hermiticity implies that ϱ is an hermitian matrix, and can be diagonalized. If we choose fields which diagonalize ϱ, then the non-zero elements of ϱ must be positive measures and cannot have principal value singularities. Thus such singularities can occur only in the function $g_{\alpha\beta\gamma}(p, q)$ which is the coefficient of the term in the commutator which is linear in the field[4]. Since g does not know the position, in a vacuum expectation value, of the fields whose momenta occur in its arguments, principal value singularities will occur in the momenta of fields standing next to the vacuum, in violation of the stability of the one-particle states, if such singularities are present in g. Therefore g cannot have such singularities and no scattering or reactions can occur.

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[1] Such singularities are necessary for scattering and reactions in both the Lehmann-Symanzik-Zimmermann and Haag-Ruelle formulations of scattering theory.

[2] Local commutativity is used implicitly to the extent that the existence of asymptotic fields depends on it.

[3] Replacing the field $\Phi_\alpha(x)$ by sources $j_\alpha(x) = (\Box + m_\alpha^2)\, \Phi_\alpha(x)$ does not lead to a more general form, since in momentum space $j_\alpha(k) = (m_\alpha^2 - k^2)\, \Phi_\alpha(k)$, and the $\Phi_\alpha(k)$ obey commutation relations of the form of Eq. (1) with a different $\varrho_{\alpha\beta}$ and $g_{\alpha\beta\gamma}$.

[4] This holds only for particles created by a linear functional of the Lie field. We treat the general case, where polynomials in the fields create the particles, below.

To carry out this demonstration in detail, we note that stability of the vacuum implies $\langle 0\,0\,i\,i\rangle_0 = \langle i\,0\,i\,i\rangle_0$, where 0 and i stand for out and in fields, respectively. We will show that $\langle i\,0\,i\,i\rangle_0 = \langle i\,i\,i\,i\rangle_0$, so that the elastic scattering amplitude vanishes for all energy and momentum transfer, and, by the optical theorem, the processes $2 \to n$ do not occur.

We can see that $\langle i\,0\,i\,i\rangle_0 = \langle i\,i\,i\,i\rangle_0$ in a Lie field theory by inspection of the four-point function. Of course these vacuum expectation values of in and out fields will have support on the respective mass shells.[5] In addition, the difference $D \equiv \langle i\,0\,i\,i\rangle_0 - \langle i\,i\,i\,i\rangle_0$ will receive contributions only from terms in g with mass-shell principal value in at least one argument, and, since [in, in] and [out, out] must be c-numbers, only from those terms in g in which one argument is the momentum of an out field and the other argument is the momentum of an in field. The terms which can contribute are all quadratic in g. In order to have momenta corresponding to both in and out fields in each g, two out fields would be required. Since only one is present, $D = 0$, and there is no scattering or reactions.

In detail, the recursive evaluation [1] gives

$$
\begin{aligned}
\langle \Phi_\alpha(p)\, \Phi_\beta(q)\, \Phi_\gamma(r)\, \Phi_\delta(s)\rangle_0 = {} & \theta(p)\, \{\theta(q)\, \varrho_{\alpha\delta}(p^2)\, \varrho_{\beta\gamma}(q^2)\, \delta\,(p+s)\, \delta(q+r) \\
& + \theta(q)\, \varrho_{\alpha\gamma}(p^2)\, \varrho_{\beta\delta}(q^2)\, \delta(p+r)\, \delta(q+s) \\
& + \theta(r)\, \varrho_{\alpha\beta}(p^2)\, \varrho_{\gamma\delta}(r^2)\, \delta(p+q)\, \delta(r+s)\} \\
& + \theta(p)\, \{\theta(q)\, \theta(r)\, g_{\alpha\delta\mu}(p, s)\, g_{\beta\mu\nu}(q, p+s)\, \varrho_{\gamma\nu}(r^2) \\
& + \theta(q)\, \theta\,(-s)\, g_{\alpha\gamma\mu}(p, r)\, g_{\beta\mu\nu}(q, p+r)\, \varrho_{\nu\delta}(s^2) \\
& + \theta(p+q)\, \theta(r)\, g_{\alpha\beta\mu}(p, q)\, g_{\mu\delta\nu}(p+q, s)\, \varrho_{\gamma\nu}(r^2) \\
& + \theta(p+q)\, \theta(-s)\, g_{\alpha\beta\mu}(p, q)\, g_{\mu\gamma\nu}(p+q, r)\, \varrho_{\nu\delta}(s^2) \\
& + \theta(q)\, \theta(q+r)\, g_{\alpha\delta\mu}(p, s)\, g_{\beta\gamma\nu}(q, r)\, \varrho_{\nu\mu}((q+r)^2) \\
& + \theta(q)\, \theta(p+r)\, g_{\alpha\gamma\mu}(p, r)\, g_{\beta\delta\nu}(q, s)\, \varrho_{\mu\nu}((p+r)^2)\}\, \delta(p+q+r+s)\,,
\end{aligned}
\tag{2}
$$

where repeated indices are summed. The terms quadratic in ϱ in the first curly bracket do not contribute to D because they do not have principal values. The first four terms quadratic in g in the second curly

[5] We use the recipes $\delta_\alpha(p) \to \delta_\alpha(p), t \to \pm\infty$, and $P \dfrac{1}{p^2 - \alpha^2} \to \mp i\pi\,\varepsilon(p)\,\delta_\alpha(p)$, $t \to \pm\infty$, where α is short for m_α, and $\delta_\alpha(p) \equiv \delta(p^2 - \alpha^2)$, to read off the vacuum expectation values of the asymptotic fields from the vacuum expectation values of the Heisenberg fields. These recipes follow, among other ways, from the LSZ asymptotic limit in momentum space,

$$\Phi_\alpha^{\substack{\text{out}\\ \text{in}}}(p) = \lim_{t\to\pm\infty} \varepsilon(p)\,\delta_\alpha(p) \int dq^0\ \Phi(q^0, \boldsymbol{p})\,(q^0 + p^0)\,\chi(p^0 + q^0)\,\exp[-i(q^0 - p^0)\,t]\,,$$

where $\chi \in \mathscr{D}$ and $\chi(0) = 1$, and the limit is weak. The LSZ asymptotic limit has been shown to hold in Wightman's framework between states in at least one of which no pair of asymptotic particles moves with the same velocity and the momentum-space single-particle wave functions are in $\mathscr{D}$[5,6].

bracket do not contribute to D because of the stability of one particle states (they correspond to $(s, \delta) \to (p, \alpha) + (q, \beta) + (r, \delta)$, for example). The final two terms in the curly bracket give vanishing contributions, because in both terms there is a g factor both of whose arguments are momenta belonging to in fields after the asymptotic limit is applied and D is formed. Since

$$\left[\Phi_\alpha^{\overset{\text{out}}{\text{in}}}(p), \Phi_\beta^{\overset{\text{out}}{\text{in}}}(q)\right] = \text{c-no} . \tag{3}$$

the contribution of the term with g in the commutator, Eq. (1), to the [in, in] asymptotic limit must vanish[6].

Now we consider the case of particles created by quasi-local polynomials in the Lie fields and, again, show that no scattering or reactions can occur. It suffices to consider a quasi-local homogeneous polynomial A_τ in the Lie fields:

$$A_\tau(p) = \int dp_1 \ldots dp_n \delta(p - \Sigma_1^n p_j) F_\tau^{(\alpha_1, \ldots, \alpha_n)}(p_1, \ldots, p_n) \Phi_{\alpha_1}(p_1) \ldots \Phi_{\alpha_n}(p_n), \tag{4}$$

where τ stands for all the internal indices of A, the α's are summed on, and we have used Poincaré invariance to write this form. For scattering or reactions to occur, it is necessary that $[A_\tau^{\text{out}}, \Phi_\alpha^{\text{in}}] \neq [A_\tau^{\text{in}}, \Phi_\alpha^{\text{in}}]$ or, more generally, $[A_\tau^{\text{out}}, A_\sigma'^{\text{in}}] \neq [A_\tau^{\text{in}}, A_\sigma'^{\text{in}}]$. The commutator, Eq. (1), leads to

$$\begin{aligned}
[A_\tau(p), \Phi_\varkappa(q)] &= \int dp_1 \ldots dp_n \, \delta(p - \Sigma_1^n p_j) \, F_\tau^{(\alpha_1, \ldots \alpha_n)}(p_1, \ldots, p_n) \\
&\cdot \sum_{i=1}^{n} [\varepsilon(p_i) \varrho_{\alpha_i \varkappa}(p_i^2) \delta(p_i + q) \Phi_{\alpha_1}(p_1) \cdots \Phi_{\alpha_{i-1}}(p_{i-1}) \Phi_{\alpha_{i+1}}(p_{i+1}) \cdots \Phi_{\alpha_n}(p_n) \\
&+ g_{\alpha_i \varkappa \beta}(p_i, q) \Phi_{\alpha_1}(p_1) \cdots \Phi_{\alpha_{i-1}}(p_{i-1}) \Phi_\beta(p_i + q) \Phi_{\alpha_{i+1}}(p_{i+1}) \cdots \Phi_{\alpha_n}(p_n)] .
\end{aligned} \tag{5}$$

Again, vacuum expectation values of A_τ must have mass-shell principal value singularities in order to have $A_\tau^{\text{out}} \neq A_\tau^{\text{in}}$. The only place where p^2 occurs as an independent variable on the right hand of Eq. (5), and therefore the only place where mass-shell principal values can occur, is in F, which can have such a singularity in the sum of its arguments. Again, however, such singularities cannot occur in the functions F because then these singularities would occur in momenta belonging to fields standing next to the vacuum, and violate the stability of one-particle states.

[6] This condition, together with the analogous condition for the [out, out] asymptotic limit requires the form

$$\begin{aligned}
g_{\alpha\beta\gamma}^{(\text{sing.})}(p, q) &= \left(\pi^2 \varepsilon(p) \varepsilon(q) \delta_\alpha(p) \delta_\beta(q) + P \frac{1}{p^2 - \alpha^2} P \frac{1}{q^2 - \beta^2}\right) g_{\alpha\beta\gamma}^{(\text{I})}(p, q) \\
&+ \left(\varepsilon(p) \delta_\alpha(p) P \frac{1}{q^2 - \beta^2} - P \frac{1}{p^2 - \alpha^2} \varepsilon(q) \delta_\beta(q)\right) g_{\alpha\beta\gamma}^{(\text{II})}(p, q)
\end{aligned}$$

for the part of g which has mass-shell singularities in both arguments.

Therefore $[A_\tau^{\text{out}} - A_\tau^{\text{in}}, \Phi_\alpha^{\text{in}}] = 0$. A similar argument shows that $[A_\tau^{\text{out}} - A_\tau^{\text{in}}, A_\sigma'^{\text{in}}] = 0$, and thus $A_\tau^{\text{out}} = A_\tau^{\text{in}}$, and no scattering or reactions occur for particles created by polynomials in the Lie fields.

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