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Master Analytic Representations and Unified Representation Theory of Certain Orthogonal and Pseudo-Orthogonal Groups

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Abstract. The representation theory of the groups $SO(5)$, $SO(4, 1)$, $SO(6)$ and $SO(5, 1)$ is studied using the method of Master Analytic Representations (MAR). It is shown that a single analytic expression for the matrix elements of the generators of $SO(n + 1)$ and $SO(n, 1)$ in an $SO(n)$ basis yields all the unitary representations (for $n = 4,5$); and that the compact and non-compact groups have essentially the same analytic representation. Once the MAR of a group is worked out, the search for the unitary irreducible representations is reduced to a purely arithmetic operation. The utmost care has been exercised to conduct the discussions at an elementary level: knowledge of simple angular momentum theory is the only prerequisite.

Introduction

In the course of a study of the unitary irreducible representations of a variety of groups we have discovered that, in all cases where the representations have been analyzed in detail, in every linear representation of the (locally compact) Lie algebra, the matrix elements describing the representation was a specialization of certain analytic functions. We refer to this general representation as the Master Analytic Representation (MAR). It provides us with a method of studying the representations of various noncompact groups. In this paper we apply this method to study the representations of the groups $SO(5)$, $SO(4, 1)$, $SO(6)$ and $SO(5, 1)$.

The plan of the paper is as follows: in Sec. 1 we study the MAR for $SO(5)$ in an $SO(4)$ basis and deduce all the unitary representations of the $SO(5)$ group [1]. An elementary diagramatic analysis for this purpose is developed and exploited. We then use the Weyl trick [2] to write down the MAR for the de Sitter group SO(4, 1) and specialize it to find all the unitary irreducible representations of $SO(4, 1)$ and of its covering group [3]. Sec. 2 deals with the unitary representations of the $SO(5, 1)$ group

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and its covering group. The MAR of the $SO(6)$ and the $SO(5, 1)$ groups is labelled by three parameters. The values of these parameters which furnish the unitary irreducible representations of the covering group of $SO(5, 1)$ are obtained, thus furnishing a complete catalogue of these representations. This is a complete solution to the problem since the matrix elements of all the generators are known explicitly. We believe that these results are obtained here for the first time.

The earliest use of the idea of analytic continuation is due to DIRAC [4]. Various authors have recently carried out studies in representation theory which make use of some of the ideas of MAR. Particular mention must be made of the work of BARUT and FRONSDAL [5], HERMANN [6], and HOLMAN and BIEDENHARN [7]. The method of MAR have been used by one or other of the present authors in other investigations [8] and the theory was announced at the Third Coral Gables Conference [9]. An account with application is contained in a thesis by one of us (J.G. K.) [10]; and a more detailed presentation of the method of MAR is given in another paper [11].

I. Master Analytic Representations of the Orthogonal and Pseudo-Orthogonal Groups in Five Dimensions

The unimodular, real orthogonal group in n dimensions, $SO(n)$ [12], is generated by $\frac{1}{2} n(n-1)$ generators

$$A_{\alpha\beta} = -A_{\beta\alpha}, \alpha, \beta = 1, \ldots n \tag{I.1}$$

which obey the commutation rules (C.R.'s)

$$-i[A_{\alpha\beta}, A_{\mu\nu}] = g_{\alpha\nu}A_{\beta\mu} + g_{\beta\mu}A_{\alpha\nu} - g_{\alpha\mu}A_{\beta\nu} - g_{\beta\nu}A_{\alpha\mu} \tag{I.2}$$

with

$$g_{\alpha\beta} = \delta_{\alpha\beta}. \tag{I.3}$$

The $SO(n)$ groups have the feature that within any unitary irreducible representation (UIR) of $SO(n)$, the $SO(n-1)$ subgroup labels uniquely specify a state — that is to say, if we reduce a UIR of $SO(n)$ with respect to $SO(n-1)$, each UIR of $SO(n-1)$ will occur at most once.

We shall obtain the UIR's of the covering group of $SO(4, 1)$ [3] by the method of Master Analytic Representation (MAR) from the UIR's of the covering group of $SO(5)$ [1]. The group $SO(5)$ has ten generators, six of which are generators of the $SO(4)$ subgroup, and the remaining four transform as a four-vector with respect to this $SO(4)$ subgroup. We can consider the (covering group of the) $SO(4)$ subgroup as a direct product of (the covering groups of) two $SO(3)$ subgroups and use these two sets of $SO(3)$ labels to classify the states within an UIR of $SO(4)$. In this basis the four-vector is a combined spherical tensor of rank 1/2 with respect to each of the two $SO(3)$ subgroups. The advantage in using this basis is that the "magnetic quantum number" dependence of the matrix elements (M.E.) of the extra generators of $SO(5)$ will be given by the usual Clebsch-Gordan (C.G.) coefficients of $SO(3)$. If we had used the $SO(4)$ labels directly, then this magnetic quantum number dependence would be given by a C.G. coefficient of $SO(4)$, which is a little more difficult to handle [11]. In this section we shall use this $SO(3) \otimes SO(3)$ basis for the analysis of the UIR's of $SO(5)$ and $SO(4, 1)$.

We rewrite the C.R.'s (I.1) for the Lie algebra of $SO(5)$ as follows:

$$\begin{aligned} [X_{m_1}, X_{m_2}] &= -\sqrt{2}\, C(111; m_1 m_2 m_1 + m_2)\, X_{m_1 + m_2} \\ [Y_{m_1}, Y_{m_2}] &= -\sqrt{2}\, C(111; m_1 m_2 m_1 + m_2)\, Y_{m_1 + m_2} \\ [X_{m_1}, Y_{m_2}] &= 0 \end{aligned} \tag{I.4}$$

$$\begin{aligned} [X_m, Q_{m_1 m_2}] &= -\frac{\sqrt{3}}{2}\, C\left(1 \tfrac{1}{2} \tfrac{1}{2}\,; m m_1 m + m_1\right) Q_{m + m_1, m_2} \\ [Y_m, Q_{m_1 m_2}] &= -\frac{\sqrt{3}}{2}\, C\left(1 \tfrac{1}{2} \tfrac{1}{2}\,; m m_2 m + m_2\right) Q_{m_1, m + m_2} \end{aligned} \tag{I.5}$$

$$\begin{aligned} [Q_{m_1 m_2}, Q_{m_1' m_2'}] = 4 \Big\{ & C\left(\tfrac{1}{2} \tfrac{1}{2} 0; m_1 m_1'\, 0\right) C\left(\tfrac{1}{2} \tfrac{1}{2} 1; m_2 m_2'\, m_2 + m_2'\right) Y_{m_2 + m_2'} \\ & + C\left(\tfrac{1}{2} \tfrac{1}{2} 1; m_1 m_1'\, m_1 + m_1'\right) C\left(\tfrac{1}{2} \tfrac{1}{2} 0; m_2 m_2'\, 0\right) X_{m_1 + m_1'} \Big\}\,. \end{aligned} \tag{I.6}$$

Eqs. (I.4) imply that X_m and Y_m generate two commuting $SO(3)$ groups. Eqs. (I.5) imply that $Q_{m m'}$ is a combined spherical tensor rank 1/2 with respect to each of the $SO(3)$ groups generated by X_m and Y_m. Eq. (I.6) is characteristic of the semisimple structure of $SO(5)$.

Representations of SO(5) and its Covering Group

We are interested in hermitian representations of the $SO(5)$ Lie algebra; this implies

$$Q^+_{mn} = (-1)^{m-n} Q_{-m, -n}\,. \tag{I.7}$$

We introduce a basis labelled by the quantum numbers $(j_1 m_1)$ pertaining to a UIR of the X_m, and the quantum numbers $(j_2 m_2)$ pertaining to a UIR of the Y_m:

$$|j_1 j_2 m_1 m_2\rangle\,. \tag{I.8}$$

The hermiticity requirement (I.7) implies then:

$$\begin{aligned} \langle j_1' j_2' m_1' m_2' |\, Q_{mn} \,| j_1 j_2 m_1 m_2 \rangle \\ = (-1)^{m-n} \langle j_1 j_2 m_1 m_2 |\, Q_{-m, -n} \,| j_1' j_2' m_1' m_2' \rangle^*\,. \end{aligned} \tag{I.9}$$

The Wigner-Eckart theorem as applied to the two $SO(3)$ subgroups enables us to factor out the $(m_1 m_2)$ dependence of the M.E.'s of Q as follows:

$$\langle j_1' j_2' m_1' m_2' | Q_{mn} | j_1 j_2 m_1 m_2 \rangle = C\left(j_1 \tfrac{1}{2} j_1'; m_1 m m_1'\right) C\left(j_2 \tfrac{1}{2} j_2'; m_2 n m_2'\right) \langle j_1' j_2' \| Q \| j_1 j_2 \rangle . \tag{I.10}$$

Let us define:

$$\begin{aligned} \left\langle j_1 + \tfrac{1}{2}\, j_2 + \tfrac{1}{2} \right\| Q \left\| j_1 j_2 \right\rangle &\equiv a(j_1 j_2) \\ \left\langle j_1 + \tfrac{1}{2}\, j_2 - \tfrac{1}{2} \right\| Q \left\| j_1 j_2 \right\rangle &\equiv b(j_1 j_2) \\ \left\langle j_1 - \tfrac{1}{2}\, j_2 + \tfrac{1}{2} \right\| Q \left\| j_1 j_2 \right\rangle &\equiv c(j_1 j_2) \\ \left\langle j_1 - \tfrac{1}{2}\, j_2 - \tfrac{1}{2} \right\| Q \left\| j_1 j_2 \right\rangle &\equiv d(j_1 j_2) . \end{aligned} \tag{I.11}$$

On account of (I.7) we then have:

$$\begin{aligned} [(2j_1 + 2)(2j_2 + 2)/(2j_1 + 1)(2j_2 + 1)]^{1/2} a(j_1 j_2) &= d^*\left(j_1 + \tfrac{1}{2}\, j_2 + \tfrac{1}{2}\right) \\ [(2j_1 + 2)(2j_2)/(2j_1 + 1)(2j_2 + 1)]^{1/2} b(j_1 j_2) &= -c^*\left(j_1 + \tfrac{1}{2}\, j_2 - \tfrac{1}{2}\right) \end{aligned} \tag{I.12}$$

So we are left with two unknown reduced M.E., $a(j_1 j_2)$ and $b(j_1 j_2)$. We sketch below the method for obtaining them.

We consider the C.R.

$$[Q_{-1/2\,-1/2}, Q_{1/2\,1/2}] = -2(X_0 + Y_0) \tag{I.13}$$

and sandwich it between the states $|j_1 j_2 m_1 m_2\rangle$ and $\langle j_1' j_2' m_1 m_2|$. For $j_1' = j_1 + 1, j_2' = j_2$, we get, using (I.10) and (I.11):

$$\frac{a(j_1 j_2)}{a\left(j_1 + \frac{1}{2}, j_2 - \frac{1}{2}\right)} = \left[\frac{2j_2}{2j_2 + 2}\right]^{1/2} \frac{b(j_1 j_2)}{b\left(j_1 + \frac{1}{2}, j_2 + \frac{1}{2}\right)} . \tag{I.14}$$

For $j_1' = j_1, j_2' = j_2 - 1$, we get:

$$\frac{b(j_1 j_2)}{b\left(j_1 - \frac{1}{2}, j_2 - \frac{1}{2}\right)} = \left[\frac{2j_1}{2j_1 + 2}\right]^{1/2} \frac{d(j_1 j_2)}{d\left(j_1 + \frac{1}{2}, j_2 - \frac{1}{2}\right)} . \tag{I.15}$$

With $j_1' = j_1 - 1, j_2' = j_2$, we get:

$$\frac{d(j_1 j_2)}{d\left(j_1 - \frac{1}{2}, j_2 + \frac{1}{2}\right)} = \left[\frac{2j_2 + 2}{2j_2}\right]^{1/2} \frac{c(j_1 j_2)}{c\left(j_1 - \frac{1}{2}, j_2 - \frac{1}{2}\right)} . \tag{I.16}$$

Combining (I.12), (I.14) and (I.15), we find:

$$|a(j_1 j_2)|^2 = \frac{(2j_1 + 3)}{(2j_1 + 2)} \cdot \frac{(2j_2 + 1)}{(2j_1 + 2)} \left| a\left(j_1 + \frac{1}{2}, j_2 - \frac{1}{2}\right)\right|^2 . \tag{I.17}$$

Similarly, (I.12), (I.15) and (I.16) lead to

$$|b(j_1 j_2)|^2 = \frac{(2j_1 + 1)}{(2j_1 + 2)} \frac{(2j_2 - 1)}{(2j_2)} \cdot \left| b\left(j_1 - \frac{1}{2}, j_2 - \frac{1}{2}\right)\right|^2 . \tag{I.18}$$

It is advantageous at this point to change variables and define

$$x = j_1 + j_2 + 1$$
$$y = j_1 - j_2$$
$$A(x, y) \equiv |a(j_1 j_2)|^2$$
$$B(x, y) \equiv |b(j_1 j_2)|^2 \tag{I.19}$$

Eqs. (I.17) and (I.18) then read as follows:

$$[(x+1)^2 - y^2]\, A(x, y) = [(x+1)^2 - (y+1)^2]\, A(x, y+1) \equiv V(x) \tag{I.20}$$

$$\begin{aligned}[x^2 - (y+1)^2]\, B(x, y) = [(x-1)^2 \\ - (y+1)^2]\, B(x-1, y) \equiv - W(y)\end{aligned} \tag{I.21}$$

Next we need an inhomogeneous equation for the reduced M.E.'s $A(x, y)$ and $B(x, y)$. For this we consider the diagonal matrix element of the C.R. (I.13) for the state $|j_1 j_2 m_1 m_2\rangle$ to obtain:

$$\begin{aligned}&\frac{|a(j_1 j_2)|^2}{(2j_1+1)(2j_2+1)}[2m_1(j_2+1) + 2m_2(j_1+1)] + \frac{|b(j_1 j_2)|^2}{(2j_1+1)(2j_2+1)} \\ &\cdot [2m_1 j_2 - 2m_2(j_1+1)] + \frac{\left|b\left(j_1 - \frac{1}{2}, j_2 + \frac{1}{2}\right)\right|^2}{2j_1(2j_2+2)}[-2m_1(j_2+1) + 2m_2 j_1] \\ &+ \frac{\left|a\left(j_1 - \frac{1}{2}, j_2 - \frac{1}{2}\right)\right|^2}{2j_1 2j_2}[-2m_1 j_2 - 2m_2 j_1] = -2(m_1 + m_2)\end{aligned} \tag{I.22}$$

Since m_1 and m_2 vary independently of one another, we may equate the coefficients of m_1 on the two sides of (I.22), and similarly for m_2. The resulting equations, expressed in terms of $V(x)$ and $W(y)$, read:

$$\begin{aligned}\frac{V(x) - W(y)}{x+y+1} - \frac{V(x-1) - W(y-1)}{x+y-1} &= -2(x^2 - y^2) \\ \frac{V(x) - W(y-1)}{x-y+1} - \frac{V(x-1) - W(y)}{x-y-1} &= -2(x^2 - y^2)\end{aligned} \tag{I.23}$$

We have to solve these two equations to determine $V(x)$ and $W(y)$.

Let $x = e$ and $y = f - 1$ be zeros of $V(x)$ and $W(y)$ respectively. That is to say,

$$V(e) = W(f-1) = 0 \tag{I.24}$$

Then setting $x = e + 1$, $y = f$ in (I.23) enables us to solve for $V(e+1)$ and $W(f)$:

$$\begin{aligned}V(e+1) &= -2(e+1)(e+f+1)(e-f+2) \\ W(f) &= 2f(e-f)(e+f+1)\end{aligned} \tag{I.25}$$

We now make explicit use of the that $W(y)$ is independent of x, so that $W(y)$ can be evaluated at any value of x, and in particular at $x = e + 1$. We set $x = e + 1$ in Eqs. (I.23) and use (I.24) and (I.25) to get two equations in $W(y)$ and $W(y-1)$. Solving them, we find:

$$W(y) = -(e-y)(e+y+1)(f+y)(f-y-1) \tag{I.26}$$

We use the same trick to evaluate $V(x)$; we set $y = f$ in (I.23), use (I.25), and solve for $V(x)$. The solution is:

$$V(x) = (x+f)(x-f+1)(e-x)(e+x+1) \tag{I.27}$$

Thus we have solved for the independent reduced M.E. $a(j_1 j_2)$ and $b(j_1 j_2)$. To analyze the representations, it is convenient to write the final expressions in the following form:

$$\begin{aligned} a(j_1 j_2) &= \left[\frac{\{x(x+1)-(f-1)f\}\{e(e+1)-x(x+1)\}}{(x+1)^2-y^2}\right]^{1/2} \\ b(j_1 j_2) &= \left[\frac{\{e(e+1)-y(y+1)\}\{(f-1)f-y(y+1)\}}{x^2-(y+1)^2}\right]^{1/2} \end{aligned} \tag{I.28}$$

These analytic functions with the definitions (I.10) and (I.11) furnish the Master Analytic Representation with parameters e and f.

To proceed with the specialization to unitary representations (of the covering group) of $SO(5)$, we note that the state labels must vary in such a domain that the master analytic functions remain real. Since these functions are symmetric in $e(e+1)$ and $(f-1)f$, we may assume without loss of generality that

$$e(e+1) \geqq (f-1)f;\ e \geqq -\frac{1}{2},\ f \geqq \frac{1}{2};\ e+1 \geqq f \tag{I.29}$$

The quantities $e(e+1)$ and $(f-1)f$ cannot both be negative because $a(j_1 j_2)$ would become imaginary. (Remember that $x = j_1 + j_2 + 1 > 0$.) We now wish to study the behaviour of the expressions for the matrix elements when the state labels x and y vary. The denominators within the square root do not change sign; we are therefore interested in the behaviour of the polynomials in the numerators of the two expressions in (I.28). This is facilitated by considering the following diagram in which we plot the sign of the polynomial numerators as x and y from $-\infty$ to

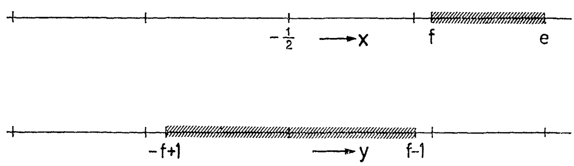

Fig. 1. MAR diagram for $SO(5)$. The shaded regions represent the domain of variation of x and y respectively

$+\infty$. Nonnegativity of the master functions together with the information $x = j_1 + j_2 + 1 \geqq 1$, $|y| \leqq x - 1$, show that the only region of interest is

$$e \geqq x \geqq f, \quad f-1 \geqq y \geqq 1-f$$

so that

$$e-1 \geqq x-1 \geqq f-1 \geqq y \geqq 1-f \tag{I.30}$$

which are the Gel'fand-Tseitlin branching rules for $SO(5)$. Each UIR of $SO(5)$ is labelled by two parameters e, f which are both integral or both half integral which satisfy $e \geqq f \geqq 1$.

The UIR's of $SO(4)$ included in a UIR of $SO(5)$ are obtained by requiring the transition matrix elements for transitions leading out of the allowed set to vanish. Thus, since $a(j_1, j_2)$ vanishes at $j_1 + j_2 + 1 = e$, the transition from the state labelled by $(j_1 j_2)$ (with $j_1 + j_2 + 1 = e$) to $(j_1 + 1/2, j_2 + 1/2)$ is not possible. On the other hand, for $j_1 + j_2 + 1 = f - 1$, the same transition is forbidden, but this means that $j_1 + j_2 + 1 = f$ is the lower limit, since that state cannot be connected with the state with $j_1' = j_1 - 1/2$, and $j_2' = j_2 - 1/2$. It is then clear that the parameters of the MAR, which originally entered as the arbitrary constants of the solution to the recurrence relation for (or, better, as the zeros of the polynomials associated with,) the matrix elements, serve to specify the range of variation of the state labels. Due reflection will reveal that this is no mere accident and this characterizes the MAR for any compact group.

The group $SO(5)$ has two Casimir operators Q, R of the second and fourth degree respectively. They take the values

$$\begin{aligned} Q &= \frac{1}{2} A_{\alpha\beta} A_{\alpha\beta} = e(e+1) + f(f-1) - 2 \\ R &= W_\alpha W_\alpha = e(e+1)\, f(f-1) \end{aligned} \tag{I.31}$$

and $$W_\alpha = \frac{1}{8} \varepsilon_{\alpha\beta\gamma\mu\nu} A_{\beta\gamma} A_{\mu\nu}$$

for these representations.

Representations of the de Sitter Group $SO(4,1)$ and its Covering Group

The method of MAR demands that we carry out Weyl's trick, [i.e., $Q \to P = iQ$, and then $\{\boldsymbol{X}, \boldsymbol{Y}, P\}$ generate $SO(4,1)$], and then analytically continue the M.E. into regions in which the corresponding operators are hermitian. It is trivial to verify that the Weyl trick gives quantities satisfying the C.R.'s for generators of $SO(4,1)$. Defining the reduced M.E. of P by the equation

$$\langle j_1' j_2' m_1' m_2' |\, P_{mn} \,| j_1 j_2 m_1 m_2 \rangle = C\left(j_1 \tfrac{1}{2} j_1'; m_1 m m_1'\right) C\left(j_2 \tfrac{1}{2} j_2'; m_2 n m_2'\right) \langle j_1' j_2' \| P \| j_1 j_2 \rangle \ldots \tag{I.32}$$

the first step in the method of MAR is the following identification:

$$\langle j_1' j_2' \| P \| j_1 j_2 \rangle = i \langle j_1' j_2' \| Q \| j_1 j_2 \rangle \ldots \tag{I.33}$$

The two independent M.E. of P are

$$\begin{aligned}\left\langle j_1+\frac{1}{2}, j_2+\frac{1}{2}\right\| P \left\| j_1 j_2\right\rangle &\equiv [A'(x,y)]^{1/2}\\ &= \left[\frac{\{x(x+1)-e(e+1)\}\{x(x+1)-(f-1)f\}}{(x+1)^2-y^2}\right]^{1/2}\\ \left\langle j_1+\frac{1}{2}, j_2-\frac{1}{2}\right\| P \left\| j_1 j_2\right\rangle &\equiv [B'(x,y)]^{1/2}\\ &= \left[\frac{\{e(e+1)-y(y+1)\}\{y(y+1)-(f-1)f\}}{x^2-(y+1)^2}\right]^{1/2}\end{aligned} \tag{I.34}$$

The next step in the method of MAR is to analytically continue the range of variation of x, y in $A'(x, y)$ and $B'(x, y)$ and choose e, f so that P is hermitian, i.e., $A'(x, y)$ and $B'(x, y)$ are real and nonnegative.

We note that previously e and f turned out to be limiting points in the ranges of x and y, while here *e and f must be considered as parameters which can a priori take any arbitrary complex values*. However, though e and f may be complex parameters, both $e(e+1)$ and $(f-1)f$ must be real. For, if $e(e+1)$ were complex, then the reality of $B'(x, y)$ implies that $(f-1)f$ is the complex conjugate of $e(e+1)$; but then $B'(x, y)$ is negative.

We consider the functions

$$\begin{aligned}\alpha(x) &= [(x+1)^2-y^2]\, A'(x,y)\\ &= [x(x+1)-e(e+1)]\,[x(x+1)-(f-1)f]\\ \beta(y) &= [x^2-(y+1)^2]\, B'(x,y)\\ &= [e(e+1)-y(y+1)]\,[y(y+1)-(f-1)f]\end{aligned} \tag{I.35}$$

For hermitian representations of $SO(4, 1)$, the following conditions are satisfied:

$$\begin{aligned}x = j_1+j_2+1 \geqq |j_1-j_2|+1 = |y|+1 \geqq 1\,,\\ \alpha(x) \geqq 0, \qquad \beta(y) \geqq 0\end{aligned} \tag{I.36}$$

Since $\alpha(x)$ and $\beta(y)$ are both symmetric in $e(e+1)$ and $(f-1)f$, and since $e(e+1)$ and $(f-1)f$ must both be real, we may assume without loss of generality that

$$e(e+1) \geqq (f-1)f \tag{I.37}$$

Just as in the case of $SO(5)$, we can first prove that $e(e+1)$ and $(f-1)f$ cannot both be negative. For if they were, then the function $\beta(y)$ is negative for all integral and half integral values of y except possibly for $y = -1/2$. If $y = -1/2$ is the only of y to be considered, we can also restrict x to be half odd integral. Then we see that $\alpha(x)$ is nonzero for all positive half odd integral values of x; in particular, $\alpha(x)$ is nonzero for $x = 1/2$. This means that we have a nonvanishing transition matrix element from $x = 3/2$ down to $x = 1/2$. Since $x = j_1 + j_2 + 1 \geqq 1$, the former is "allowed" while the latter is not. We conclude that $e(e+1)$ and $(f-1)f$ cannot both be negative. In view of the inequality (I.37), $e(e+1)$ is positive or zero.

The reality of $e(e+1)$ and $(f-1)f$ implies that either e is itself real or a complex number of the form $e = -1/2 + i\sigma$; similarly, f is either real or of the form $1/2 + i\varrho$. However, we have just seen that $e(e+1)$ is nonnegative. We conlude that e is a real number in the range $0 \leqq e < \infty$. [Note that e appears only in the combination $e(e+1)$.] As for f, it is either real and in the range $1/2 \leqq f < \infty$, or complex of the form $f = 1/2 + i\varrho$.

We now divide the analysis into the cases where f is real, and f is complex.

i) *Representations of the D Series*: *f real, f* $\geqq 1$. We begin with the following situation:

$$e(e+1) > (f-1)f > 0 \tag{I.38}$$

and consider later the possibility of replacing the inequalities by equalities. (I.38) is equivalent to

$$e+1 > f > 1 \tag{I.39}$$

In the integral case, the least possible values of e and f would be 2, while in the half-odd-integral case, this least value would be 3/2. The zeros of $\alpha(x)$ and $\beta(y)$ occur at the points $e, f-1, -f, -e-1$ which obey:

$$e > f-1 > -f > -e-1 \tag{I.40}$$

In particular, note that no two of these zeros coincide. The signs of $\alpha(x)$ and $\beta(y)$ are plotted in Fig. 2. [The shaded regions are the regions of positive $\alpha(x)$ and $\beta(y)$ which qualify for the domain of variation of x and y respectively in UIR.]

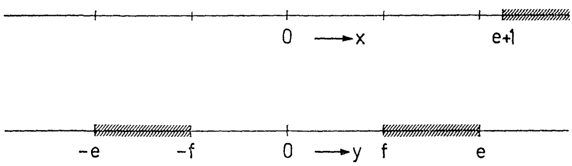

Fig. 2. MAR diagram for general discrete series

$\alpha(x)$ is strictly positive for $e+1 \leqq x < \infty$, and vanishes for $x = e$. Hence the states with $x \geqq e+1$ cannot be connected to those with $x < e+1$. The range $-f \leqq x \leqq f-1$ is one for which $\alpha(x)$ is nonnegative, and $\alpha(x)$ vanishes at $x = f-1$ and $x = -f$. Hence the states with $1-f \leqq x \leqq f-1$ cannot be connected to those with $x > f-1$ or $x < 1-f$. Similarly, $\beta(y)$ is nonnegative for $f-1 \leqq y \leqq e$, and for $-e-1 \leqq y \leqq -f$, and vanishes for $y = e, f-1, -f, -e-1$. Hence the set of states with $f \leqq y \leqq e$ cannot be connected to states with y outside this interval; and similarly, states with $-e \leqq y \leqq -f$ cannot be connected to states with y outside this range. Remembering that $x \geqq |y| + 1$, we see that the only representations in this case are the following:

$$D^{+}_{e,f} : x \geqq e + 1; \quad e \geqq y \geqq f;$$
$$D^{-}_{e,f} : x \geqq e + 1; \quad -f \geqq y \geqq -e; \quad e + 1 > f > 1 \tag{I.41}$$

In terms of j_1 and j_2, the ranges are

$$j_1 + j_2 = e, e + 1, e + 2, \ldots \infty$$
$$\pm (j_1 - j_2) = f, f + 1, \ldots e \tag{I.42}$$

In the representations $D^{\pm}_{e,f}$ given by (I.41) or (I.42), we must remember that e and f are both integral or both half-odd-integral, obeying the inequalities (I.39).

Now we consider the possibilities that arise when the inequalities in (I.38) are replaced by equalities. Suppose we take

$$e(e + 1) = (f - 1) f > 0 \tag{I.43}$$

This is equivalent to

$$e + 1 = f > 1 \tag{I.44}$$

$\alpha(x)$ and $\beta(y)$ become

$$\alpha(x) = [x(x + 1) - e(e + 1)]^2$$
$$\beta(y) = - [y(y + 1) - e(e + 1)]^2 \tag{I.45}$$

$\beta(y)$ is strictly negative for all y except $y = e$ or $y = - e - 1$, where it vanishes. So in principle the states in the range $e \geqq y \geqq - e$ are connected only to one another, and not to states with y outside this range. However, since $e \neq 0$, there are at least two distinct values of y in the range $e \geqq y \geqq - e$; in particular, $e - 1 \geqq - e$. But then the transition matrix element $\beta(e - 1)$ from $y = e$ down to $y = e - 1$ becomes nonzero and negative. We conclude that we have *no* representations for

$$e(e + 1) = (f - 1) f > 0 \tag{I.43}$$

Consider next the possibility where the second inequality alone in (I.38) is replaced by an equality:

$$e(e + 1) > (f - 1) f = 0 ,$$
$$\text{i.e.,} \quad e(e + 1) > 0, \quad f = 1 \tag{I.46}$$

This is the same as the requirement $f = 1$, $e > 0$, so that the values of e and f under consideration are

$$f = 1; e = 1, 2, 3, \tag{I.47}$$

We have, first of all, a set of representations very similar to the $D^{\pm}_{e,f}$ of (I.41); namely:

$$D^{+}_{e,1} : x \geqq e + 1; \quad e \geqq y \geqq 1 ;$$
$$D^{-}_{e,1} : x \geqq e + 1; \quad -1 \geqq y \geqq - e ; \quad e \geqq 1 . \tag{I.48}$$

These representations were not included in (I.41) because there $f = 1$ was not allowed. We can now consider the representations $D^{\pm}_{e,f}$ of (I.41) with $f > 1$, and the representations $D^{\pm}_{e,1}$ of (I.48), as constituting one class of representations. However, the situation described by (I.46) leads to further representations because of the following reason. The function $\beta(y)$ vanishes for $y = f - 1 = 0$, and for $y = -f = -1$. This vanishing of $\beta(y)$ for two consecutive integral values of y implies that the states with $y = 0$ form a set which cannot be connected to any other states with $y \neq 0$. We find therefore the following class of representations.

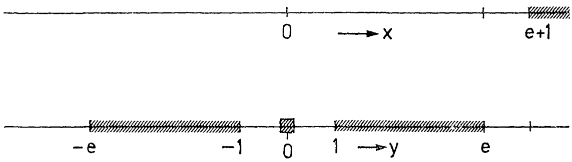

Fig. 3. MAR diagram for the case $f = 1$ showing $D^{\pm}_{e,1}$ and $D^{0}_{e,1}$

$$D^{0}_{e,1} : x \geqq e + 1, \quad y = 0; \quad e = 1, 2, \ldots \tag{I.49}$$

The ranges of j_1 and j_2 in $D^{\pm}_{e,1}$ and $D^{0}_{e,1}$ are given by (see Fig. 1.3)

$$\begin{aligned} D^{\pm}_{e,1} &: j_1 + j_2 = e, e + 1, \ldots \infty, \\ &\pm (j_1 - j_2) = 1, 2, \ldots e; \quad e \geqq 1 \text{ (a)} \\ D^{0}_{e,1} &: j_1 + j_2 = e, e + 1, \ldots \infty, \\ &j_1 - j_2 = 0; \quad e \geqq 1 \ldots \text{(b)} \end{aligned} \tag{I.50}$$

We have a still further exceptional situation which occurs when $e = f = 1$ so that the zeros $e, f - 1, -f, -e - 1$ are the consecutive

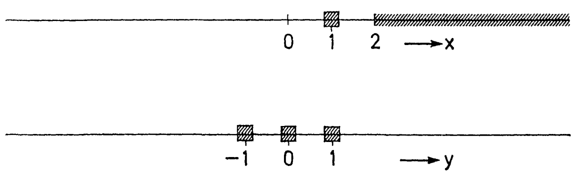

Fig. 4. MAR diagram for $e = f = 1$ showing $D^{\pm}_{11}$, D^{0}_{11} and I

integers $1, 0, -1, -2$. In this case in addition to the representations D^{+}_{11}, D^{-}_{11}, D^{0}_{11}, we have yet another possibility because the state with $x = 1, y = 0$ is disconnected from all other states. This gives rise to the identity representation (see Fig. I.4):

$$I : x = 1, y = 0 \tag{I.51}$$

The "range" of values of j_1, j_2 are

$$I : j_1 = j_2 = 0 \tag{I.52}$$

Last of all we consider the possiblity that both inequalities in (I.38) are replaced by equalities:

$$e(e+1) = (f-1)f = 0 : e = 0, f = 1 \tag{I.53}$$

The functions $\alpha(x)$ and $\beta(y)$ are explicitly:

$$\alpha(x) = x^2(x+1)^2; \quad \beta(y) = -y^2(y+1)^2 \tag{I.54}$$

We find a single representation which should be added to the class $D^0_{e,1}$ of (I.49). We have the representation

$$D^0_{0,1} : x \geqq 1, y = 0 \tag{I.55}$$

The range of values of j_1 and j_2 is:

$$\begin{aligned} D^0_{0,1} : j_1 + j_2 &= 0, 1, 2, \ldots \infty \\ j_1 - j_2 &= 0 \end{aligned} \tag{I.56}$$

Let us summarize the results of this subsection. Assuming f to be real, we considered here the possibility $f \geqq 1$. We then find the following representation of $SO(4, 1)$, in all of which e and f are quantized:

$$\begin{aligned} &\underline{D^{\pm}_{e,f}} : f = 1, 3/2, 2, 5/2, \ldots; \quad e = 1, 3/2, 2, \ldots; \quad \underline{\underline{e+1 > f}} \\ &\underline{D^0_{e,1}} : f = 1; \quad e = 0, 1, 2, \ldots \\ &\underline{I} : j_1 = j_2 = 0 \end{aligned} \tag{I.57}$$

ii) *Representations of the C Series.* $f = 1/2 + i\varrho$, $0 < \varrho < \infty$.

We have in this case:

$$e(e+1) \geqq 0, e \geqq 0; f(f-1) = -\frac{1}{4} - \varrho^2; -\frac{1}{4} > f(f-1) > -\infty \ldots \tag{I.58}$$

The zeros and signs of $\alpha(x)$, $\beta(y)$ are given in Fig. (I.5):

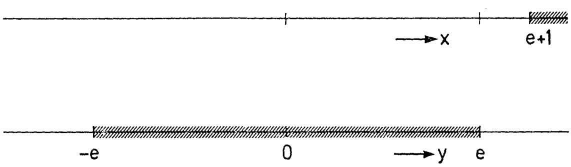

Fig. 5. MAR diagram for continuous series

The functions $\alpha(x)$ and $\beta(y)$ are

$$\begin{aligned} \alpha(x) &= \{x(x+1) - e(e+1)\}\left\{\left(x + \frac{1}{2}\right)^2 + \varrho^2\right\} \\ \beta(y) &= \{e(e+1) - y(y+1)\}\left\{\left(y + \frac{1}{2}\right)^2 + \varrho^2\right\} \end{aligned} \tag{I.59}$$

A straightforward analysis using the diagram in Fig. I.5 yields the following general class of representations:

$$C_{e,\varrho}: x \geqq e+1,\ e \geqq y \geqq -e;\quad e = 0, \frac{1}{2}, 1, \ldots \infty \tag{I.60}$$

so that the ranges of j_1, j_2 are given by

$$\begin{aligned} C_{e,\varrho}: j_1 + j_2 &= e, e+1, e+2, \ldots \infty; \\ j_1 - j_2 &= e, e-1, \ldots -e; \\ e &= 0, \frac{1}{2}, 1, \ldots; \quad \varrho > 0 \end{aligned} \tag{I.61}$$

There is no distinction like $D^{\pm}_{e,f}$ for the C Series, and the range of x and y is independent of the value of ϱ. No exceptional cases arise for the C Series.

In the above we assumed $\varrho > 0$, thus excluding the possibility $\varrho = 0$, when $f = 1/2$ becomes real. Let us now look at this possibility. For integral x and y, we have the same kinds of representations as before, because $\beta(y)$ does not acquire any new zeros:

$$\begin{aligned} C^{(0)}_{e,0}: j_1 + j_2 &= e, e+1, e+2, \ldots \infty; \\ j_1 - j_2 &= e, e-1, \ldots -e; \\ e &= 0, 1, 2, \ldots \end{aligned} \tag{I.62}$$

However, for half integral values of x and y, $\beta(y)$ acquires an extra zero, at $y = -1/2$. Then the states with $e \geqq y \geqq 1/2$ are connected only to one another, and not to any other values of y; similarly the states with $-1/2 \geqq y \geqq -e$ are connected only to themselves. We find two new families of representations which could be classed along with $D^{\pm}_{e,f}$ of (I.41) and (I.48). They are:

$$\begin{aligned} D^{+}_{e,1/2}: x \geqq e+1,\ e \geqq y \geqq \frac{1}{2};\ e = \frac{1}{2}, 3/2, \ldots \\ D^{-}_{e,1/2}: x \geqq e+1,\ -\frac{1}{2} \geqq y \geqq -e;\ e = \frac{1}{2}, 3/2, \ldots \end{aligned} \tag{I.63}$$

The corresponding ranges of j_1 and j_2 are:

$$\begin{aligned} D^{\pm}_{e,1/2}: j_1 + j_2 &= e, e+1, e+2, \ldots \infty; \\ \pm(j_1 - j_2) &= \frac{1}{2}, 3/2, \ldots e; \\ e &= \frac{1}{2}, 3/2, 5/2, \ldots \infty. \end{aligned} \tag{I.64}$$

iii) *Representations of the E Series.* $f = 1/2 + r,\ 0 < r < 1/2$.

From the expressions (I.59) for $\alpha(x)$ and $\beta(y)$, we can see that when x and y are both *integral*, we have a family of representations of the following kind:

$$\begin{aligned} E_{e,r}: x \geqq e+1;\quad e \geqq y \geqq -e;\quad 0 < r < \frac{1}{2} \\ e = 0, 1, 2, \ldots \end{aligned} \tag{I.65}$$

The range of j_1 and j_2 is provided by:

$$E_{e,r}: j_1 + j_2 = e, e+1, \ldots \infty;$$
$$j_1 - j_2 = -e, -e+1, \ldots +e; \qquad e = 0, 1, 2, \ldots \tag{I.66}$$

If one so chooses, these $E_{e,r}$ representations could be put into the same class as the $C_{e,\varrho}$ representations of (I.60).

In all these cases, discrete as well as continouous, the Casimir operators have the values

$$\begin{aligned} Q &= e(e+1) + f(f-1) - 2 \\ R &= e(e+1)\, f(f-1) \end{aligned} \tag{I.67}$$

We note that the values of the Casimir operators do not label the representations uniquely; $D^+_{e,f}$ and $D^-_{e,f}$ have the same Casimir operators; $D^+_{e,1}$, $D^-_{e,1}$ and $D^0_{e,1}$ all have the same Casimir operators; and finally D^+_{11}, D^-_{11}, D^0_{11} and I have zero for both Casimir operators. To distinguish between different representations with the same Casimir operators, we should use "nonanalytic labels" like the domain of variation of the state labels x and y. It is also relevant to notice that these representations which have the same value for the two Casimir operators have matrix elements which are the same analytic functions; and they are all associated with the same unitary represntation of the compact group, if such a representation with the same values of the Casimir operators exists.

We conclude this section by listing together all the classes of UIR's of $SO(4, 1)$, including in each case the spectrum of UIR's of $SO(4)$ that appear. (The trivial identity representation is omitted.)

$$D^+_{e,f}: e \geqq f \geqq \frac{1}{2}; \quad e - f = \text{integer};$$
$$f \leqq y \leqq e; \quad e+1 \leqq x < \infty.$$
$$D^-_{e,f}: e \geqq f \geqq \frac{1}{2}; \quad e - f = \text{integer};$$
$$-e \leqq y \leqq -f; \quad e+1 \leqq x < \infty.$$
$$D^0_{e,1}: e \geqq 0; \quad f = 1; \quad e = \text{integer};$$
$$y = 0; \quad e+1 \leqq x < \infty.$$
$$C^{(0)}_{e,\varrho}\ e \geqq 0; \quad f = \frac{1}{2} + i\varrho; \quad e = \text{integer}; \quad 0 \leqq \varrho < \infty;$$
$$-e \leqq y \leqq e; \quad e+1 \leqq x < \infty.$$
$$C^{(1/2)}_{e,\varrho}: e \geqq \frac{1}{2}; \quad f = \frac{1}{2} + i\varrho; \quad 2e = \text{odd integer}; \quad 0 < \varrho < \infty;$$
$$-e \leqq y \leqq e; \quad e+1 \leqq x < \infty.$$
$$E_{e,r}: e \geqq 0; \quad f = \frac{1}{2} + r; \quad e = \text{integer}; \quad 0 < r < \frac{1}{2};$$
$$-e \leqq y \leqq e; \quad e+1 \leqq x < \infty.$$

(In $D^{\pm}_{e,f}$, e and f are simultaneously integral or simultaneously half-odd integral.) In any given representation, the reduced matrix elements of P_{mn} linking one UIR of $O(4)$ to another are given by (I.34).

II. Representations of Orthogonal and Pseudo-Orthogonal Groups in Six Dimensions

Representations of $SO(6)$ and its Covering Group

In this section we would like to make use of similar methods to find all UIR of $SO(6)$ and $SO(5, 1)$. The computation of the master analytic functions proceed along lines very similar to the one sketched above for $SO(5)$, except that the algebraic computation is twice as long and more than twice as laborious. In our study of the UIR of $SO(5)$ and $SO(4, 1)$ in an $O(4)$ basis we made use of the special circumstance that (the covering group of) the $SO(4)$ group is isomorphic with the direct product (of the covering groups) of two commuting $SO(3)$ groups. However the final expressions (Eqs. (I.28)) for the reduced matrix elements are in fact given in terms of the $O(4)$ labels defined by Eqs. (I.19). Since the C.G. coefficients of $SO(5)$ are not familiar quantities we shall not bother to write down the explicit reduced matrix elements of those generators $A_{6\mu}$, $1 \leqq \mu \leqq 5$ of $SO(6)$ which are not in $SO(5)$. These quantities form the components of a five-vector with respect to $SO(5)$ and it is therefore sufficient to know the complete set of matrix elements of any one of them. We choose the component A_{65}. Since A_{65} transforms as a scalar with respect to a $SO(4)$ subgroup of $SO(5)$, its matrix elements are independent of the labels j, m which occur as the magnetic quantum numbers of the $O(4)$ representations. Since a state in a UIR of $SO(5)$ labelled by the two nonnegative parameters e, f requires as magnetic quantum numbers x, y the labels of the $SO(4)$ representation and the quantities j, m for its complete specification, we are interested in matrix elements of the form

$$\langle e' f' x' y' j' m' | A_{65} | e f x y j m \rangle \,. \tag{II.1}$$

But in view of the fact that A_{65} is scalar with respect to $SO(4)$, these matrix elements vanish unless

$$x' = x \qquad y' = y \qquad j' = j \qquad m' = m$$

and for this nonvanishing case they are independent of j and m. The relevant matrix elements may be denoted as:

$$\langle e' f' x y | A_{65} | e f x y \rangle \,. \tag{II.2}$$

It is seen that the only nonvanishing matrix elements are obtained for the cases when

$$(e', f') = (e, f)\,, \quad (e \pm 1, f)\,, \quad (e, f \pm 1)\,.$$

By a direct computation we obtain:

$$\langle efxy| A_{65} |efxy\rangle = \frac{xy}{e(e+1)\,f(f-1)} \cdot (\alpha+1)\,\beta(\gamma-1)$$

$$\langle e+1fxy| A_{65} |efxy\rangle = \sqrt{\frac{\{(e+1)^2 - x^2\}\{(e+1)^2 - y^2\}}{(e+1)^2 \cdot \{4(e+1)^2 - 1\}\{e+1)^2 - f^2\}\{(e+1)^2 - (f-1)^2\}}}$$

$$\cdot \sqrt{\{(\alpha+1)^2 - (e+1)^2\}\{(e+1)^2 - \beta^2\}\{(e+1)^2 - (\gamma-1)^2\}}$$

$$\equiv \langle efxy| A_{65} |e+1fxy\rangle^*$$

$$\langle ef+1xy| A_{65} |efxy\rangle = \sqrt{\frac{(x^2 - f^2)(f^2 - y^2)}{f^2(4f^2 - 1)\{(e+1)^2 - f^2\}\{e^2 - f^2\}}}$$

$$\cdot \sqrt{\{(\alpha+1)^2 - f^2\}\{\beta^2 - f^2\}\{f^2 - (\gamma-1)^2\}} \qquad \text{(II.3)}$$

$$= \langle efxy| A_{65} |ef+1xy\rangle^*.$$

All $SO(6)$ representations are to be obtained from an analysis of these matrix elements.

The Gel'fand-Tseitlin "branching rules" for the $SO(4)$ "content" of a UIR (e, f) of $SO(5)$ are [1]:

$$e \geqq x \geqq f \qquad f-1 \geqq y \geqq 1-f \qquad e \geqq f \geqq 1\,. \qquad \text{(II.4)}$$

These assure us that the poles of the quantities within the square roots may be ignored altogether. We may therefore consider the behaviour of the polynomials.

$$\begin{aligned} E(e) &= \{(\alpha+1)^2 - (e+1)^2\}\{(e+1)^2 - \beta^2\}\{(e+1)^2 - (\gamma-1)^2\} \\ F(f) &= \{(\alpha+1)^2 - f^2\}\{\beta^2 - f^2\}\{f^2 - (\gamma-1)^2\} = -E(f-1) \end{aligned} \qquad \text{(II.5)}$$

and their variation as e and f assume both integral or both half integral values satisfying $e \geqq f \geqq 1$.

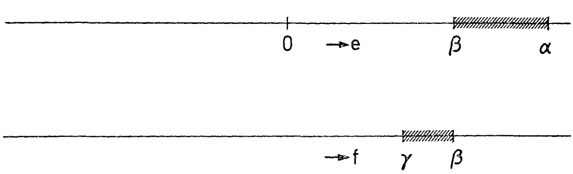

Fig. 6. MAR diagram for $SO(6)$ for $D_{\alpha\beta\gamma}$. Shaded portions indicate the relevant domains of variation of f and e respectively

The UIR of (the covering group of) $SO(6)$ are to be sought by finding suitable values of α, β, γ and corresponding domains for the state labels such that the master analytic functions represent a hermitian operator A_{65}. These considerations are faciliated by the use of diagrams.

Nonnegativity of the polynomials together with reality of the diagonal matrix element in (II.3) require that α, β, γ be all real. Recalling that $E(e_1) = 0$ implies that $e_1 \to e_1 + 1$ transition is forbidden, $F(f_1) = 0$ implies that $f_1 \to f_1 + 1$ transition is forbidden and that $e \geqq f \geqq 1$ we find that for positive α, β, γ only one class of representations $D_{\alpha\beta\gamma}$ labelled by the three parameters α, β, γ occur with the branching rule:

$$D_{\alpha\beta\gamma}: \gamma \leqq f \leqq \beta \leqq e \leqq \alpha \,. \tag{II.6}$$

The explicit matrix elements are already given by (II.3). The requirement on the parameters are that α, β, γ be all nonnegative integers or all nonnegative half integers.

If any one (or all three) of the three parameters α, β, γ change sign we get a representation which is inequivalent to the original one unless $(\alpha + 1)\,\beta(\gamma - 1) = 0$. Without loss of generality we could choose α, β to be nonnegative and γ to be of either sign if it is nonzero. The representations $D_{\alpha\beta\gamma}$ and $D_{\alpha\beta,2-\gamma}$ are conjugate representations and can be obtained by the outer automorphism:

$$\begin{aligned} &A_{\mu\nu} \to A_{\mu\nu} \\ &A_{6\mu} \to -A_{6\mu} \qquad 1 \leqq \mu, \nu \leqq 5 \end{aligned} \tag{II.7}$$

which is equivalent to the operation of space inversion in the carrier space of $SO(5)$.

Every UIR of $SO(6)$ is thus labelled by three numbers α, β, γ all of which are integral or all half integral and the first two nonnegative. For $\gamma > 1$ we find

$$D_{\alpha\beta\gamma}: \gamma \leqq f \leqq \beta \leqq e \leqq \alpha \,. \tag{II.8}$$

For $\gamma < 1$, we find:

$$D_{\alpha\beta\gamma}: 2 - \gamma \leqq f \leqq \beta \leqq e \leqq \alpha \,.$$

These two expressions may be amalgamated into the single expression

$$D_{\alpha\beta\gamma}: 1 + |\gamma - 1| \leqq f \leqq \beta \leqq e \leqq \alpha \,. \tag{II.9}$$

These representations are conjugates of each other. When $\gamma = 1$ we get self-conjugate representations.

All these representations were found by GEL'FAND and TSEITLIN [1].

The representations with $\beta = \gamma = 1$ have $f = 1$ and hence, by virtue of the $SO(5)$ branding rules (II.4), $y = 0$ so that these give the "symmetric tensor" representations with α labelling the rank.

Representations of $SO(5, 1)$ and its Covering Group

We can now carry out the method of MAR and use Weyl's trick to find the UIR of $SO(5, 1)$ (and of its covering group). Accordingly we let

$$A_{6\mu} \to i A'_{6\mu}\,; \quad A_{\mu\nu} \to A_{\mu\nu}\,; \quad 1 \leqq \mu, \nu \leqq 5 \tag{II.10}$$

and search for values of the parameters α, β, γ and the domain of e, f such that we get matrix elements which represent a hermitian operator. Since

$$\langle efxy | A'_{65} | efxy \rangle = \frac{xy}{e(e+1)f(f-1)} i(\alpha+1)\beta(\gamma-1) \qquad \text{(II.11)}$$

has to be real if all the quantities $\alpha+1$, β, $\gamma-1$ are nonzero, not all of them can be real; let us choose γ to be complex in such a case. Since the other matrix elements are, according to (II.10) also simply multiplied by i we are led to consider the polynomials

$$\begin{aligned} E'(e) &= \{(e+1)^2-(\alpha+1)^2\}\{(e+1)^2-\beta^2\}\{(e+1)^2-(\gamma-1)^2\} \\ F'(f) &= \{(\alpha+1)^2-f^2\}\{f^2-\beta^2\}\{f^2-(\gamma-1)^2\}\,. \end{aligned} \qquad \text{(II.12)}$$

The polynomials will change their sign only at isolated zeros. For large values of f, $F'(f)$ is negative and if all the quantities $\alpha+1$, β, $\gamma-1$ were complex there would be no domain where f could range. Hence at least one of these quantities should be real; let us choose it to be $\alpha+1$ if only one of the three quantities is real, otherwise let it be the largest of the real quantities among $\alpha+1$, β, $\gamma-1$. If β is complex (and γ is already complex) $F'(f)$ will have only two real zeros, symmetric about the origin. But the domain so given for the range of variation of f is inadmissible. (Recall the $SO(5)$ branching rule (II.4) which says $f \geqq 1$.) Hence β must be real, or else $(\gamma-1)$ must be zero; but this simply is equivalent to $\beta=0$ and γ being complex. Thus we deduce that α, β must be real and γ complex; reconsideration of (II.11) now tells us that $(\gamma-1)$ is pure imaginary or zero. We now study the possible MAR diagrams systematically.

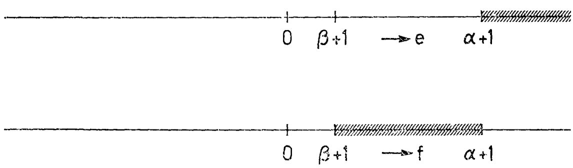

Fig. 7. MAR diagram for the principal series of $SO(5,1)$

i) *The Principal Series* $\alpha \geqq \beta > 0$. We begin with the case:

$$\alpha > \beta \geqq 0\,; \qquad \gamma = 1 + i\tau\,, \qquad \tau^2 > 0 \qquad \text{(II.13)}$$

and consider later the possibility of replacing one or more of the inequalities by equalities. Again remembering that $e \geqq f \geqq 1$ we find the allowed domain of variation

$$\begin{aligned} P_{\alpha\beta\tau} : \beta + 1 \leqq f \leqq \alpha + 1 \leqq e < \infty \\ \alpha > \beta \geqq 0\,; \qquad \tau^2 > 0 \end{aligned} \qquad \text{(II.14)}$$

independent of the (positive or negative) value of τ. We may take, without loss of generality, α and β both integral or both half integral and nonnegative. The representation $P_{\alpha\beta\tau}$ and $P_{\alpha\beta-\tau}$ are conjugate to each other.

For $\alpha > \beta \geqq 0, \tau = 0$ we get a class of representations with the same domain of variation

$$\begin{aligned} P_{\alpha\beta o}: \beta + 1 \leqq f \leqq \alpha + 1 \leqq e < \infty \\ \alpha > \beta > 0\,; \quad i\tau = \gamma - 1 = 0\,. \end{aligned} \tag{II.15}$$

These representations are self-conjugate.

For $\alpha = \beta \geqq 0, \tau^2 > 0$ we get a class of representations which are simply specializations of (II.14) for $\alpha = \beta$; they are distinguished by the fact that f is now kept fixed:

$$\begin{aligned} P_{\alpha\alpha\tau}: f = \alpha + 1 \leqq e < \infty \\ \alpha = \beta \geqq 0\,, \quad \tau^2 > 0\,. \end{aligned} \tag{II.16}$$

The representations $P_{\alpha\alpha\tau}$ and $P_{\alpha\alpha-\tau}$ are not equivalent but conjugate to each other. For $\alpha = \beta \geqq 0, \tau = 0$ we have a specialization of (II.15) for $\alpha = \beta$ with f fixed:

$$\begin{aligned} P_{\alpha\alpha o}: f = \alpha + 1 \leqq e < \infty \\ \alpha = \beta \geqq 0\,; \quad i\tau = \gamma - 1 = 0\,. \end{aligned} \tag{II.17}$$

These representations are self-conjugate.

The representations of the principal series are thus defined by three numbers α, β, τ of which the first two are nonnegative while the third one may take any real value positive negative or zero. The domain of variation of e and f (i.e., the branching rules) depend only on α, β and are:

$$\begin{aligned} P_{\alpha\beta\tau}: \beta + 1 \leqq f \leqq \alpha + 1 \leqq e < \infty \\ \alpha \geqq \beta \geqq 0\,; \quad -\infty < \tau < \infty \\ \alpha - \beta \text{ integral}; \quad 2\beta \text{ integral}\,. \end{aligned} \tag{II.18}$$

For $\tau \neq 0$ the representation $P_{\alpha\beta\tau}$ is not self-conjugate but the conjugate representation is $P_{\alpha\beta-\tau}$. For $\tau = 0$ the representations are self-conjugate.

For $\beta > \alpha \geqq \beta - 1$ there are no representations.

ii) *The Supplementary Series* $\beta = \gamma = 1$. For the case that $\gamma = 1\,(\tau = 0)$ we have consecutive zeros of the polynomials $E(e)$ and $F(f)$. From our previous experience we anticipate new kinds of representations to arise in such cases. The corresponding MAR diagram is given in Fig. 8. We see that the range of variation of e is like in the case of the principal series but there are two distinct ranges for f. There are double zeros at $e = -1$ $f = 0$ and simple zeros at $e = 0$, α and $f = 1$, $\alpha + 1$. The allowed range, $2 = f \leqq \alpha + 1 \leqq e < \infty$ gives a specialization of the self-conjugate representations of the principal series:

$$P_{\alpha 1 0}: 2 = f \leqq \alpha + 1 \leqq e < \infty$$
$$\alpha - 1 \text{ nonnegative integer }.$$

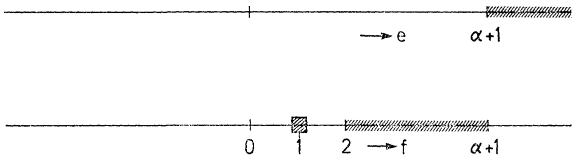

Fig. 8. MAR diagram for the principal series of $SO(5, 1)$

But, in addition, the states with $f = 1$ form a set which cannot be connected to any other states with $f > 1$. We find therefore the following class of representations:

$$S_\alpha: f = 1 \,;\, \alpha + 1 \leqq e < \infty$$
$$\alpha - 1 \text{ nonnegative integer }.$$

We can find, for the supplementary series, one more possibility: take $\alpha = 0$. In this case the range of variation of e, f are:

$$S_0: f = 1 \,; \quad 1 \leqq e < \infty \,.$$

We may thus write the full set:

$$\begin{aligned} S_\alpha: f = 1 \,; \quad & \alpha + 1 \leqq e < \infty \\ \alpha = 0, 1, 2, \ldots \quad & \beta = \gamma = 1 \,. \end{aligned} \tag{II.19}$$

This new class of self-conjugate representations constitutes the supplementary series. The e, f values are all integral and by virtue of the $SO(5)$ branching rules, $f = 1$ implies that $y = 0$; hence the supplementary series furnishes all the "symmetric tensor" representations of $SO(5, 1)$ with α labelling the rank.

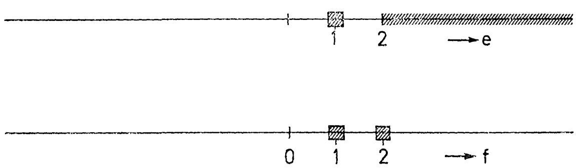

Fig. 9. MAR diagram for the identity representation of $SO(5, 1)$

iii) *The Identity Representation* $\alpha = \beta = \gamma = 1$. For the principal series we found that $\alpha = \beta$ implies that the range of variation of f was limited to a single value $f = \alpha + 1$. We now consider the special case when $\alpha = \beta = \gamma = 1$. There are double zeros at $e = -1$ and $f = 0$ and simple zeros at $e = 0, 1$ and $f = 1, 2$. Searching for the allowed domains of e and f consistent with $e \geqq f \geqq 1$ we find first of all a specialization of the selfconjugate representations of the principal series:

$$P_{110} : 2 = f \leqq e < \infty .$$

Next we find a special representation of the supplementary series:

$$S_1 : f = 1 ; \quad 2 \leqq e < \infty .$$

Finally, since the states with $e = 1$ are not connected with states with $e > 1$, we have the new representation:

$$I : f = 1 ; \quad e = 1 . \tag{II.20}$$

This representation is the one-dimensional identity representation.

iv) *The Exceptional Series* $0 < \beta < 1$, $\gamma = 1$. In the principal series of representations we made use of the vanishing of $F(f)$ at $f = \beta$ to obtain the range of variation $\beta + 1 \leqq f \leqq \alpha + 1$. And for the supplementary series we made use of the vanishing of $F(f)$ at $f = 0$ and $f = \beta = 1$ to show that states with $f = 1$ formed an isolated set. However, for $\gamma = 1$, $F(f)$ has a double zero at $f = 0$ and we could make use of this zero to obtain an allowed range $1 \leqq f \leqq \alpha + 1$ *provided* the zero at $f = \beta$ does not interfere to spoil this. But this is assured as long as β lies in the open interval $0 < \beta < 1$. We obtain in this manner the representations of the

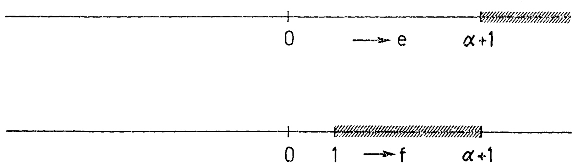

Fig. 10. MAR diagram for the exceptional series of $SO(5, 1)$

exceptional series:

$$E_{\alpha\beta} : 1 \leqq f \leqq \alpha + 1 \leqq e < \infty$$

$$\alpha \text{ nonnegative integral} \tag{II.21}$$

$$0 < \beta < 1 ; \quad \gamma = 1 .$$

In these representations e and f are always integral and these representations are self-conjugate. The case $\beta = 0$ has already been identified as the representation $P_{\alpha 0 0}$ of the principal series which is, in turn, the self-conjugate member of the special class $P_{\alpha 0 \tau}$.

It is to be noted that for $\beta = 1/2, \gamma = 1$, we have an exceptional representation $E_{\alpha 1/2}$ in which the parameter α is integral so that e, f take on only integral values. We have already found a representation of the principal series with α half integral and $\beta = 1/2$ with $\gamma = 1 + i\tau$ for which e and f take on only half-integral values independent of the value of $\tau (-\infty < \tau < \infty)$ and in particular for $\tau = 0$.

All representations of the exceptional series are self-conjugate. We note that, here also, the values of these three Casimir operators do not label the representation uniquely: $P_{\alpha 1 0}$ and S_α have the same Casimir operators; $P_{1 1 0}$, S_1 and I have the same canishing Casimir operators. To distinguish between different representations with the same Casimir operators we should use nonanalytic labels like the domain of variation of e, f. As in the case of representations of $SO(4, 1)$, the representations with the same Casimir operators have matrix elements which are the same analytic function; and they are all associated with the same UIR having the same values of the Casimir operators for the compact group $SO(6)$.

All the UIR with integral e, f are representations of $SO(5, 1)$ and those with half integral e, f are those of the covering group.

For the convenience of the reader, we list here all the representations of $O(5, 1)$ [13], giving in each class the spectrum of representations of $O(5)$ that appear (the trivial identity representation is omitted).

$$P_{\alpha\beta\tau} : \alpha \geqq \beta \geqq 0 ; \quad \gamma = 1 + i\tau ; \quad \alpha - \beta = \text{integer} ; \quad -\infty < \tau < \infty ;$$
$$\beta + 1 \leqq f \leqq \alpha + 1 \leqq e < \infty .$$
$$S_\alpha : \alpha \geqq 0 ; \quad \beta = \gamma = 1 ; \quad \alpha = \text{integer} ;$$
$$f = 1 ; \quad \alpha + 1 \leqq e < \infty$$
$$E_{\alpha\beta} : \alpha \geqq 0 ; \quad 0 < \beta < 1 ; \quad \gamma = 1 ; \quad \alpha = \text{integer} ;$$
$$1 \leqq f \leqq \alpha + 1 \leqq e < \infty .$$

[In $P_{\alpha\beta\tau}$, α and β are simultaneously integral or simultaneously half-odd integral.] In any given representation, the matrix elements of the generator A_{56} linking one representation of $O(5)$ to another are given by the transcription (II.10) applied to the matrix elements (II.3).

Discussion

With increasing use of noncompact groups in particle physics, it would be of interest to be able to deal with their representations in a direct and straightforward manner. Even when mathematical discussions of the unitary representations of certain noncompact groups exist, different families of representation are handled differently. This is in direct contrast with the uniform treatment presented in this paper. We believe that the method of MAR brings the theory of unitary representations of noncompact groups down to the same level of familiarity as the unitary representations of compact groups.

Elsewhere we have studied several other applications of MAR to other noncompact groups [11]. The major technical obstacle to the detailed study of all simple groups is the problem of multiplicity: the same representation of a labelling subgroup occurring more than once. But this is a major obstacle to the representation theory for compact groups as well.

The method of MAR can be traced back to the work of DIRAC [4] and HARISH-CHANDRA [14] on expansor and exspinor representations of the Lorentz group. Dirac pointed out that the notion of a tensor could be extended to tensors of complex rank with an infinite number of components. The generalization of Dirac's discovery relates different representations of the same Lie Algebra. There is another idea due to WEYL [2[which relates a representation of one Lie algebra to a representation (of the same dimension) of a different Lie algebra with the same complex extension. This involves multiplication of suitable elements by appropriate complex numbers; and is often referred to as the Weyl trick. The method of MAR may be thought of as a synthesis of the Dirac principle and the Weyl trick.

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