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Statistical Mechanics of Lattice Systems

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Abstract. We study the thermodynamic limit for a classical system of particles on a lattice and prove the existence of infinite volume correlation functions for a "large" set of potentials and temperatures.

§ 1. Introduction and Notations

In this article we shall study the statistical mechanics of a classical system on a ν-dimensional lattice Z^ν. We assume that at each lattice point there can be either 0 or 1 particle. We suppose that the particles interact through symmetric translationally invariant many body potentials $\Phi^{(k)}(x_1 \ldots x_k)$. Let $X = \{x_1, \ldots, x_N\}$ be a finite subset of Z^ν, then the potential energy U of N particles located at $x_1, x_2, \ldots, x_N$ is:

$$U(X) = \sum_{k=1}^{\infty} \frac{1}{k!} \sideset{}{^{\neq}}\sum_{(i_1,\ldots,i_k) \in \{1,\ldots,N\}} \Phi^{(k)}(x_{i_1}, \ldots, x_{i_k}) \tag{1}$$

where $\sum^{\neq}$ extends over all k-ples $i_1, \ldots, i_k$ of distinct indices (between 1 and N); in particular $U(\emptyset) = 0$. We shall consider only interactions $\Phi = (\Phi^{(k)})_{k \geqq 1}$ such that

$$\|\Phi\| = \sum_{k=1}^{\infty} \frac{1}{k!} \sideset{}{^{\neq}}\sum_{0 \neq x_1 \ldots x_{k-1} \in Z^\nu} |\Phi^{(k)}(0, x_1, \ldots, x_{k-1})| < +\infty \tag{2}$$

where the second sum extends over all $(k-1)$-ples $x_1, \ldots, x_{k-1}$ of distinct lattice points different from the origin 0 of Z^ν. With respect to the norm (2) the set $\mathscr{B}$ of interactions Φ such that $\|\Phi\| < +\infty$ is a (real) Banach space.

§ 2. Definitions and Inequalities

From (1) and (2) we deduce the following stability property:

$$|U(\{x_1, \ldots, x_N\})| \leqq N \|\Phi\| . \tag{3}$$

We define a subspace $\mathscr{B}'$ of $\mathscr{B}$ by

$$\mathscr{B}' = \{\Phi \in \mathscr{B} : \Phi^{(1)} = 0\} .$$

We may write $\Phi = (-\mu, \Phi')$ for every $\Phi \in \mathscr{B}$ with $\mu = -\Phi^{(1)}$ and $\Phi' \in \mathscr{B}$. We interpret μ as chemical potential and denote by U' the potential energy corresponding to Φ'.

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If β is the inverse temperature, the grand partition function corresponding to the region (finite set) Λ is then, if $N(X)$ is the number of points of X:

$$\Xi(\beta, \mu, \Phi') = \sum_{X \subseteq \Lambda} e^{-\beta U(X)} = \sum_{X \subseteq \Lambda} e^{\beta\mu N(X)} e^{-\beta U'(X)} . \tag{4}$$

It is notationally convenient to define

$$Z_\Lambda(\Phi) = \sum_{X \subseteq \Lambda} e^{-U(X)} \tag{5}$$

and, if $V(\Lambda)$ is the number of points of Λ:

$$P_\Lambda(\Phi) = V(\Lambda)^{-1} \log Z_\Lambda(\Phi) .$$

Then

$$\Xi(\beta, \mu, \Phi') = Z_\Lambda(\beta(-\mu, \Phi'))\,; \quad Z_\Lambda(\Phi^{(1)}, \Phi') = \Xi(1, -\Phi^{(1)}, \Phi') . \tag{6}$$

Proposition 1. *If* $\Phi', \Psi' \in \mathscr{B}'$ *then*

$$Z_\Lambda(\Phi^{(1)} + \|\Psi'\|, \Phi') \leqq Z_\Lambda(\Phi^{(1)}, \Phi' + \Psi') \leqq Z_\Lambda(\Phi^{(1)} - \|\Psi'\|, \Phi') . \tag{7}$$

Let indeed V' be the potential energy corresponding to Ψ' then

$$(\Phi^{(1)} - \|\Phi'\|)\, N(X) + U'(X) \leqq \Phi^{(1)} N(X) + U'(X) + V'(X) \leqq$$
$$\leqq (\Phi^{(1)} + \|\Psi\|)\, N(X) + U'(X) .$$

Where $N(X)$ is the number of points of X. The result then follows taking the exponentials and summing over X.

Proposition 2. *If* $\Phi \in \mathscr{B}$,

$$\log(1 + e^{-\Phi^{(1)} - \|\Phi'\|}) \leqq P_\Lambda(\Phi) \leqq \log(1 + e^{-\Phi^{(1)} + \|\Phi'\|}) . \tag{8}$$

This is obtained from proposition 1 by taking $\Phi' = 0$.

Proposition 3. *The function* $\Phi \to P_\Lambda(\Phi)$ *is convex on* $\mathscr{B}$.

The proof is standard[1] and will be omitted.

Proposition 4. *If* $\Phi' \in \mathscr{B}'$, *the following inequality holds:*

$$|P_\Lambda(\Phi^{(1)} - \lambda, \Phi') - P_\Lambda(\Phi^{(1)} + \lambda, \Phi')| \leqq 2\lambda \quad \forall\, \Phi^{(1)} \in R, \forall\, \lambda \geqq 0 . \tag{9}$$

This follows from the fact that the derivative of $P_\Lambda(\Phi^{(1)}, \Phi')$ with respect to $\Phi^{(1)}$ is the expectation value of $[-N(X)/V(\Lambda)]$:

$$\frac{d P_\Lambda(\Phi^{(1)}, \Phi')}{d\Phi^{(1)}} = - \frac{\sum_{X \subseteq \Lambda} e^{-U(X)} N(X)}{Z_\Lambda(\Phi)\, V(\Lambda)}$$

and is therefore contained in the interval $[-1, 0]$.

§ 3. Existence and Properties of the Thermodynamic Limit

Let $\mathscr{B}_0 \subset \mathscr{B}$ consist of those Φ which have finite range i. e. $\Phi^{(k)} \equiv 0$ for k sufficiently large and $\Phi^{(k)}(0, x_1, \ldots, x_{k-1})$ vanishes except for a finite number of values of $x_1, \ldots, x_{k-1}$.

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[1] It follows from the convexity criteria for many variables functions and the Schwartz inequality.

Proposition 5. *If $\Phi \in \mathcal{B}_0$, the following limit exists*

$$P(\Phi) = \lim_{\Lambda \to \infty} P_\Lambda(\Phi) \,. \tag{10}$$

This result is well known (see [1], [2], [3], [4], [5]).

In this proposition and in the following Λ may be taken a parallelopiped and $\Lambda \to \infty$ means that each side of Λ tends to ∞. It is also possible to let Λ to go to ∞ in a more general manner (see [6], [7] for a definition of Van-Hove convergence to ∞).

Theorem 1. *If $\Phi \in \mathcal{B}$, the following limit exists*

$$P(\Phi) = \lim_{\Lambda \to \infty} P_\Lambda(\Phi) \tag{11}$$

and satisfies the following properties

i) *if $\Phi', \Psi' \in \mathcal{B}'$ then*

$$P(\Phi^{(1)} + \|\Psi'\|, \Phi') \leqq P(\Phi^{(1)}, \Phi' + \Psi') \leqq P(\Phi^{(1)} - \|\Psi'\|, \Phi') \,; \tag{12}$$

ii) $\log(1 + e^{-\Phi^{(1)} - \|\Phi'\|}) \leqq P(\Phi) \leqq \log(1 + e^{-\Phi^{(1)} + \|\Phi'\|})$; (13)

iii) *the functional $P(\cdot)$ is convex and continuous on the Banach space $\mathcal{B}$.*

Let $\Phi'_n \in \mathcal{B}_0$ be such that $\lim_{n \to \infty} \|\Phi'_n - \Phi'\| = 0$. From proposition 1 we obtain:

$$P_\Lambda(\Phi^{(1)} + \|\Phi' - \Phi'_n\|, \Phi'_n) \leqq P_\Lambda(\Phi) \leqq P_\Lambda(\Phi^{(1)} - \|\Phi' - \Phi'_n\|, \Phi'_n). \tag{14}$$

On the other hand $Z_\Lambda(\Phi^{(1)}, \Phi')$ is a decreasing function of $\Phi^{(1)}$ so that:

$$P_\Lambda(\Phi^{(1)} + \|\Phi' - \Phi'_n\|, \Phi'_n) \leqq P_\Lambda(\Phi^{(1)}, \Phi'_n) \leqq P_\Lambda(\Phi^{(1)} - \|\Phi' - \Phi'_n\|, \Phi'_n) \tag{15}$$

from proposition 4, the difference between extreme terms in (14) and (15) is bounded by $2\|\Phi' - \Phi'_n\|$, hence

$$|P_\Lambda(\Phi) - P_\Lambda(\Phi^{(1)}, \Phi'_n)| \leqq 2\|\Phi' - \Phi'_n\| \tag{16}$$

from this it follows that

$$\lim_{n \to \infty} P_\Lambda(\Phi^{(1)}, \Phi'_n) = P_\Lambda(\Phi) \tag{17}$$

uniformly in Λ. On the other hand by proposition 5

$$\lim_{\Lambda \to \infty} P_\Lambda(\Phi^{(1)}, \Phi'_n) = P(\Phi^{(1)}, \Phi'_n) \,. \tag{18}$$

The existence of the limits (17) and (18) and the uniformity of (17) imply the existence of

$$\lim_{\Lambda \to \infty} P_\Lambda(\Phi) = \lim_{\Lambda \to \infty} \lim_{n \to \infty} P_\Lambda(\Phi^{(1)}, \Phi'_n) \,. \tag{19}$$

This proves (11), (i) follows then from proposition 1; (ii) from proposition 2; the convexity of $P(\cdot)$ implies its continuity in $\Phi^{(1)}$ and then by (i) its continuity in Φ follows.

Remark. From the above theorem we have the existence of

$$\beta\, p(\beta, \mu, \Phi') = \lim_{\Lambda\to\infty} V(\Lambda)^{-1} \log \Xi(\beta, \mu, \Phi') \tag{20}$$

where p is the thermodynamique pressure:

$$p(\beta, \mu, \Phi') = \beta^{-1} P(\beta(-\mu, \Phi'))\,. \tag{21}$$

§ 4. Existence of Correlation Functions

Let Λ be a finite subset of Z^ν, $\Phi \in \mathscr{B}$, the correlation function is defined by:

$$\varrho_{\Phi\Lambda}(X) = Z_\Lambda(\Phi)^{-1} \sum_{\substack{Y \subseteq \Lambda \\ Y\cap X = \emptyset}} e^{-U(X\cup Y)} \tag{22}$$

if $X \subseteq \Lambda$ and $\varrho_{\Phi\Lambda}(X) = 0$ otherwise. By averaging over translations we get

$$\bar{\varrho}_{\Phi\Lambda}(\{x_1 \ldots x_n\}) = V(\Lambda)^{-1} \sum_{X\in Z^\nu} \varrho_{\Phi\Lambda}(x_1 + x, \ldots, x_n + x) \tag{23}$$

so that if $\Psi \in \mathscr{B}$ with corresponding potential energy V:

$$\begin{aligned}
&\sum_{n\geq 1} \frac{1}{n!} \sum_{0 \neq x_2 \ldots x_n \in Z^\nu} \bar{\varrho}_{\Phi\Lambda}(0, x_2 \ldots x_n)\, \Phi^{(n)}(0, x_2 \ldots x_n)\\
&= V(\Lambda)^{-1} \sum_{X\neq\emptyset} \varrho_{\Phi\Lambda}(X)\, \Psi(X) = Z_\Lambda(\Phi)^{-1} V(\Lambda)^{-1} \sum_{\substack{\Phi\neq X;\, Y\subseteq\Lambda \\ X\cap Y=\emptyset}} e^{-U(X\cup Y)}\, \Psi(X)\\
&= Z_\Lambda(\Phi)^{-1} V(\Lambda)^{-1} \sum_{X\subseteq\Lambda} e^{-U(X)}\, V(X)\,.
\end{aligned} \tag{24}$$

Let $T \subset \mathscr{B}$ be the set of Φ such that the graph of P has a unique tangent plane at Φ, i. e. there exists a unique α_Φ in the dual $\mathscr{B}^*$ of $\mathscr{B}$ such that for all $\Psi \in \mathscr{B}$

$$P(\Phi + \Psi) \geqq P(\Phi) - \alpha_\Phi(\Psi) \tag{25}$$

we note that $\alpha_\Phi(\Psi)$ can be interpreted as the functional derivative of $P(\Phi)$ in the direction Ψ [8].

Theorem 2. *If $\Phi \in T$ then if V is the potential energy associated with any $\Psi \in \mathscr{B}$ the limit*

$$\lim_{\Lambda\to\infty} Z_\Lambda(\Phi)^{-1} V(\Lambda)^{-1} \sum_{X\subseteq\Lambda} e^{-U(X)}\, V(X) = \alpha_\Phi(\Psi) \tag{26}$$

exists and defines an element $\alpha_\Phi \in \mathscr{B}^*$; the following limit therefore exists:

$$\lim_{\Lambda\to\infty} \bar{\varrho}_{\Phi,\Lambda}(X) = \bar{\varrho}_\Phi(X) \tag{27}$$

and defines the infinite volume correlation function $\bar{\varrho}_\Phi$.

For finite Λ, the function $P_\Lambda(\cdot)$ has a unique tangent plane at any $\Phi \in \mathscr{B}$ corresponding to $\alpha_{\Phi\Lambda} \in \mathscr{B}^*$:

$$\alpha_{\Phi,\Lambda}(\Psi) = Z_\Lambda(\Phi)^{-1} V(\Lambda)^{-1} \sum_{X\subseteq\Lambda} e^{-U(X)}\, V(X)\,. \tag{28}$$

From (3) it is clear that $|\alpha^{(\Psi)}_{\Phi\Lambda}| \leqq \|\Psi\|$, i. e. $\|\alpha_{\Phi\Lambda}\| \leqq 1$. Let A be a total sequence in $\mathscr{B}$ ($\mathscr{B}$ is separable), one can choose a sequence $\Lambda_n \to \infty$ as $n \to \infty$ such that $\alpha_{\Phi,\Lambda_n}(\Psi)$ converges for every $\Psi \in A$. Since $\|\alpha_{\Phi\Lambda_n}\| \leqq \leqq 1$, $\alpha_{\Phi\Lambda_n}$ converges weakly.

Let $(\Phi + \Psi, \xi)$ be a point strictly above the graph of P in $\mathscr{B} \times R$, then for large Λ, $(\Phi + \Psi, \xi)$ is above the graph of P_Λ and therefore of $\alpha_{\Phi\Lambda}$: in particular if α_Φ is the limit of $\alpha_{\Phi\Lambda_n}$

$$\xi = P(\Phi) - \alpha_\Phi(\Psi) + \alpha_\Phi(\Phi) \tag{29}$$

is the equation of a tangent plane to P at Φ. If $\Phi \in T$, the tangent plane is unique, therefore

$$\operatorname*{weak\,lim}_{\Lambda\to\infty} \alpha_{\Phi\Lambda} = \alpha_\Phi\,. \tag{30}$$

Remark. If $-\left.\dfrac{dP(\Phi+\lambda\Psi)}{d\lambda}\right|_{\lambda=0} = \alpha_\Phi(\Psi)$ exists for a certain Ψ then

$$\lim_{\Lambda\to\infty} \alpha_{\Phi\Lambda}(\Psi) = \alpha_\Phi(\Psi)\,. \tag{31}$$

We note also that the existence of $\left.\dfrac{dP(\Phi+\lambda\Psi)}{d\lambda}\right|_{\lambda=0}$ for Ψ in a total set is a necessary and sufficient condition for the existence of a unique tangent plane at Φ.

These results follow by inspection of the proof of the above theorem.

We conclude with the following:

Theorem 3. i) *The set T contains a countable intersection of dense open subsets of $\mathscr{B}$ and therefore is dense (Baire theorem* [9]*).*

ii) *There exists a dense subset T' of $\mathscr{B}'$ such that for $\Phi' \in T'$ and almost every $(\beta, \mu) \in R_+ \times R$ the point $\beta(-\mu, \Phi') \in T$.*

i) follows by inspection of the proof of reference [10].

Let e_n be a base of normalized vectors on the space $\mathscr{B}'$ [11]. Let $\Phi'_{(0)}$ be an arbitrary point of $\mathscr{B}'$ and let $\{C_n\}$ be a sequence of positive numbers such that $\sum_{n=1}^{\infty} C_n < +\infty$. Let K be the set

$$K = \{\Phi' \in \mathscr{B}' : |\Phi'_n - \Phi'_{(0)n}| < C_n\} \tag{32}$$

where Φ'_n, $\Phi'_{(0)n}$ are the components of Φ', $\Phi'_{(0)}$ along e_n.

Let us consider the space $R_+ \times R \times K$ of the variables $(\beta, \Phi^{(1)}, \Phi')$ as a topological space with the topology product of the natural topologies on R_+ and R and the relative topology on K as a subset of $\mathscr{B}'$ (it is easy to see that this topology on K is identical with the product topology on K considered as $\prod_{n=1}^{\infty} I_n$ where $I_n = (-C_n, +C_n)$).

Let us introduce on $R_+ \times R$ a normalized measure $g(d\beta\, d\Phi^{(1)})$ equivalent to the Lebesgue measure and on K the measure $\gamma(d\Phi') = \prod_{n=1}^{\infty} \dfrac{d\Phi'_n}{2C_n}$. Let $\mu = g \times \gamma$ be the product measure of g and γ defined on (the Borel sets of) $R_+ \times R \times K$. It is convenient to introduce the vector $e_0 = (1, 0) \in \mathscr{B}$.

Now the set B_n of points $(\beta, \Phi^{(1)}, \Phi') \in R_+ \times R \times K$ where the derivative $\frac{d}{d\lambda} P(\beta(\Phi + \lambda e_n))$ does not exist is a Borel set of $R_+ \times R \times K$ since

$$B_n = \bigcup_{k=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{|s|,|t|>N} C^n_{kst}, \quad n = 0, 1, \ldots \tag{33}$$

where k, N are positive integers and s, t are integers and

$$C^n_{kst} = \left\{(\beta, \Phi^{(1)}, \Phi') \in R_+ \times R \times K : \left| \frac{P(\beta(\Phi + 1/te_n)) - P(\beta\Phi)}{1/t} - \right.\right. \\ \left.\left. - \frac{P(\beta(\Phi + 1/se_n)) - P(\beta\Phi)}{1/s} \right| \geqq \frac{1}{k} \right\}. \tag{34}$$

Applying the pointwise Fubini-Jessen theorem [12] we get

$$\mu(B_n) = \int_{R_+ \times R \times P} \chi_{B_n}(s)\, \mu(ds) \\ = \lim_{M\to\infty} \int \cdots \int \chi_{B_n}(\beta, \Phi^{(1)}, \Phi'_1, \ldots, \Phi'_M, \bar{\Phi}'_{M+1}, \ldots)\, \mu_M(d\beta\, d\Phi^{(1)} \ldots d\Phi'_M) \tag{35}$$

where χ_{B_n} is the caracteristic function of B_n, $\mu_M(d\beta \ldots d\Phi'_M) = g(d\beta\, d\Phi^{(1)}) \times \prod_{m=1}^{M} \left(\frac{d\Phi'_m}{2C_m}\right)$ and $\bar{\Phi}' = \sum_{m=1}^{\infty} \bar{\Phi}'_m e_m$ is a suitable point of K. But as soon as $M > n$ the integral in the r. h. s. of (35) is zero because of the ordinary Fubini theorem and the well known fact that a convex function depending on one variable is differentiable except for a denumerable set of points. Hence $\mu(B_n) = 0$ and then $\mu\left(\bigcup_{n=0}^{\infty} B_n\right) = 0$. Let D be the complement in K of $\bigcup_{n=0}^{\infty} B_n$ then, as a consequence of the definition of B_n and of the remark following theorem 2, at every point of D there is a unique tangent plane. Furthermore $\mu(D) = 1$.

From

$$1 = \mu(D) = \int_{R_+ \times R \times K} \chi_D(\beta, \Phi^{(1)} \Phi')\, g(d\beta\, d\Phi^{(1)})\, \gamma(d\Phi') \\ = \int_K \gamma(d\Phi') \int_{R_+ \times R} \chi_D(\beta, \Phi^{(1)}, \Phi')\, g(d\beta\, d\Phi^{(1)}) \tag{36}$$

and from the fact that all measures are normalized we get

$$\int_{R_+ \times R} \chi_D(\beta, \Phi^{(1)}, \Phi')\, g(d\beta\, d\Phi^{(1)}) = 1 \tag{37}$$

for Φ' γ-almost everywhere in K. Then (ii) follows from the equivalence to the Lebesque measure of g and from the arbitrarity of the "center" Φ'_0 of K and of the dimensions $\{C_n\}$ of K.

Theorems 2 and 3 specify in which sense the set of $\Phi \in \mathscr{B}$ and $\beta > 0$ for which the infinite volume correlations functions exist and are unique is large.

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