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ACTIVE POINTING CONTROL FOR SHORT RANGE FREE-SPACE OPTICAL COMMUNICATION*

ARASH KOMAEE†, P. S. KRISHNAPRASAD†, AND PRAKASH NARAYAN†

Abstract. Maintaining optical alignment between stations of a free-space optical link requires an active pointing mechanism to persistently aim an optical beam toward the receiving station with an acceptable accuracy. This mechanism ensures delivery of maximum optical power to the receiving station in spite of the relative motion of the stations. In the active pointing scheme proposed in the present paper, the receiving station estimates the center of the incident optical beam based on the output of a position-sensitive photodetector. The transmitting station receives this estimate via an independent communication link and uses it to accurately aim at the receiving station. The overall mechanism which implements this scheme can be described in terms of a diffusion process which modulates the rate of a doubly stochastic space-time Poisson process. At the receiving station, observation of the space-time process over a subset of $\mathbb{R}^2$ is provided in order to control the diffusion process. Our goal is to determine a control law, measurable with respect to the history of the space-time process, which minimizes a quadratic cost functional.

1. Introduction.

Free-space optical communication is increasingly regarded as a high-bandwidth power-efficient means for point-to-point communication. The range of applications include fixed-location terrestrial communication [1], communication between mobile robots [2], airborne communication [3], and intersatellite communication [4].

In free-space optical communication using (narrow) laser beams, it is necessary to maintain the alignment of the transmitter and the receiver in spite of their relative motion. This relative motion might be caused by the mobile nature of the stations, mechanical vibration, or accidental shocks. Maintaining alignment is achieved through two operations: spatial tracking and active pointing. Spatial tracking properly adjusts the orientation of the receiver to hold the transmitter within its field of view [5]. Pointing serves to aim the transmitted beam toward the receiver within an acceptable accuracy [5]. While coarse pointing is essential for initiating a free-space optical link, active pointing—a persistent fine pointing operation—is required during data transmission in order to compensate for pointing error caused by relative motion. The last operation attempts to maximize the received optical power during the course

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†The authors are with the Department of Electrical and Computer Engineering and the Institute for Systems Research, University of Maryland, College Park, MD 20742, USA, E-mail: {akomaee, krishna, prakash}@umd.edu

of communication. In this paper, our focus is on active pointing for short range applications.

The one-way optical link under consideration comprises an optical transmitter and an optical receiver which are subject to relative motion. The optical transmitter is equipped with a servo-driven pointing assembly which can control the azimuth and elevation of a transmitting laser source. The optical beam emitted by the laser source has a nonuniform intensity profile which is assumed to be Gaussian [5]. Normally, the aperture of the receiver is smaller than the received optical beam, so that the receiver can collect only a fraction of the optical beam. In order to enlarge this captured fraction, the goal of active pointing is to hold the center of the optical beam at the center of the receiving aperture. The receiver employs a position-sensitive photodetector to measure the intensity profile of the optical beam that strikes its aperture. The output of the photodetector is used to estimate the center of the received optical beam, which is then conveyed to the transmitter through an optical link or a low-bandwidth RF channel. The pointing assembly then adjusts the orientation of the transmitter based on this estimate.

The performance of the proposed active pointing scheme depends significantly on the accuracy of the estimate of the beam center. In order to achieve a good estimate of the beam center, it is necessary that the size of the receiving aperture be comparable with the size of the beam. This requirement limits the application of our method to short distance links.

The rest of this paper is organized as follows. In the next two sections we develop a stochastic model for the overall scheme and state the control problem associated with the model. Based on this model, in Section 4, we formulate and solve the problem of estimating the center of the beam. The results of this section will be used later in Section 5 to discuss the optimal control problem.

2. The Model. The structure of our model follows that introduced in [6] for spatial tracking systems. Although the model in [6] describes a spatial tracking system rather than an active pointing one, since the two systems share similar components, the models for pointing assembly, relative motion, and the photodetector are adopted from [6]. We refer the reader to [6] for a detailed description and justification of the models.

Let the two-dimensional vector θ_t denote the azimuth and elevation angles of the transmitter axis with respect to some fixed coordinate system, where the subscript t indicates time dependence. Similarly, α_t denotes the azimuth and elevation angles of the line-of-sight of the stations (passing through the center of receiving aperture) with respect to the same coordinate system. The pointing error is $\varphi_t = \theta_t - \alpha_t$. We assume that the receiving aperture is held perpendicular to the line-of-sight by means of a spatial tracking system. Then, for a small pointing error, the displacement of the center of optical beam with respect to the center of receiving aperture is given by $y_t = l\varphi_t$, where l is the distance between the stations which is assumed to be a constant.

The pointing assembly is an electro-mechanical system with input vector $u_t \in \mathbb{R}^2$ and output vector $\theta_t \in \mathbb{R}^2$, which correspond, respectively, to the azimuth and elevation angles. The associated stochastic state space model is

$$\begin{aligned} dx_t^p &= A_t^p x_t^p dt + B_t^p u_t dt + D_t^p dw_t^p \\ \theta_t &= C_t^p x_t^p \end{aligned} \tag{1}$$

where $x_t^p \in \mathbb{R}^{n_p}$ is the state vector, $\{w_t^p, t \geqslant 0\}$ is a m_p-dimensional standard Wiener process, and A_t^p, B_t^p, D_t^p, and C_t^p are known uniformly bounded matrices of appropriate dimensions. The use of a linear model for the pointing assembly is justified by the fact that the system operates over small angles during the active pointing regime.

We model α_t by a Gauss-Markov stochastic process [6] described by the state space equations

$$\begin{aligned} dx_t^d &= A_t^d x_t^d dt + D_t^d dw_t^d \\ \alpha_t &= C_t^d x_t^d \end{aligned} \tag{2}$$

with state vector $x_t^d \in \mathbb{R}^{n_d}$, m_d-dimensional standard Wiener process $\{w_t^d, t \geqslant 0\}$, and known uniformly bounded matrices A_t^d, D_t^d, and C_t^d of proper dimensions.

The displacement vector $y_t = l\varphi_t$ is a linear function of x_t^p and x_t^d, so that (1) and (2) can be combined in a compact form:

$$\begin{aligned} dx_t &= A_t x_t dt + B_t u_t dt + D_t dw_t \\ y_t &= C_t x_t \end{aligned} \tag{3}$$

with state vector $x_t \in \mathbb{R}^n$ and m-dimensional standard Wiener process $\{w_t, t \geqslant 0\}$, where $n = n_p + n_d$ and $m = m_p + m_d$. The initial state x_0 is assumed to be Gaussian with mean $\bar{x}_0$ and covariance matrix $\bar{\Sigma}_0$, and independent of $\{w_t, t \geqslant 0\}$.

Let $\Phi_k : \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R}^{k \times k} \to \mathbb{R}^+$ be a Gaussian map defined as

$$\Phi_k (z; \bar{z}, \Theta) = (2\pi)^{-k/2} (\det \Theta)^{-1/2} \exp \left\{ -\frac{1}{2} (z - \bar{z})^T \Theta^{-1} (z - \bar{z}) \right\}.$$

Let r denote the position vector of an arbitrary point on the plane of the receiving aperture with respect to a coordinate system centered at the center of the aperture. Then, for a Gaussian beam centered at $y_t = C_t x_t$, the optical intensity $I_t(r)$ over the plane of the aperture is proportional to

$$I_t(r) \propto \Phi_2 (r; C_t x_t, R_t)$$

where $R_t = R_t^T$ is a 2×2 positive definite matrix describing the shape of the beam. For a circular symmetric beam with constant radius $\varrho > 0$ we have $R_t = \varrho^2 I_2$[1].

Let $\mathcal{A}$ denote the area of the receiving aperture. In a practical system, the optical field over the receiving aperture is focused on a photodetector of small surface area by means of a focusing lens. The photodetector measures the intensity profile of the imaged optical field, which is a scaled-down version of the optical intensity over the receiving aperture. Therefore, we consider the combination of the lens and the photodetector as a virtual photodetector of area $\mathcal{A}$, i.e., we assume that the virtual photodetector provides the observation of the optical intensity over an area $\mathcal{A}$.

The position-sensitive photodetector is a photoelectron converter whose surface is partitioned into small regions. The output of each region counts the number of converted electrons regardless of their location within the region. The photoelectron conversion rate depends linearly on the optical power absorbed by the region. Generally, a photoelectron converter is modeled by a Poisson process with a rate proportional to the impinging optical power [5, 7]. In the present case, where the optical power is a stochastic field, the output of each region is modeled by a doubly stochastic Poisson process. We assume that the receiver employs a high spatial resolution photodetector. Following [6], we use an infinite resolution model for such a sensing device. This idealized model, which is characterized by a doubly stochastic space-time Poisson process, provides a reasonable approximation for a high spatial resolution photodetector.

The rate of the space-time process which models the output of the photodetector is proportional to the optical intensity $I_t(r)$. Thus, introducing a proportionality constant $\mu_t \geqslant 0$, we express the rate as

$$\lambda_t(r, x_t) = \mu_t \Phi_2(r; C_t x_t, R_t).$$

In a general situation, μ_t is a nonnegative stochastic process representing the random optical fade caused by atmospheric turbulence and the information-bearing signal modulating the optical beam. However, here we simplify the model by assuming that μ_t is deterministic and nonnegative.

The space-time Poisson process, defined over $[0, \infty) \times \mathcal{A}$, characterizes the occurrence of discrete events (e.g., release of a single electron) with a temporal component $t \in [0, \infty)$ and a spatial component $r \in \mathcal{A}$. For $\mathcal{T}$ and $\mathcal{S}$ Borel sets in $[0, \infty)$ and $\mathcal{A}$, respectively, let $N(\mathcal{T} \times \mathcal{S})$ denote the number of points occurring in $\mathcal{T} \times \mathcal{S}$. Define the random variable

$$\rho(\mathcal{T} \times \mathcal{S}) = \int_{\mathcal{T} \times \mathcal{S}} \lambda_t(r, x_t)\, dt dr.$$

[1] I_k is the $k \times k$ identity matrix

Then $N(\mathcal{T}\times\mathcal{S})$ is a conditionally Poisson random variable with conditional probability distribution

$$\Pr\{N(\mathcal{T}\times\mathcal{S})=n|\rho(\mathcal{T}\times\mathcal{S})\}=\frac{e^{-\rho(\mathcal{T}\times\mathcal{S})}\rho^{n}(\mathcal{T}\times\mathcal{S})}{n!}.$$

Moreover, for disjoint $\mathcal{T}_1\times\mathcal{S}_1$ and $\mathcal{T}_2\times\mathcal{S}_2$, the random variables $N(\mathcal{T}_1\times\mathcal{S}_1)$ and $N(\mathcal{T}_2\times\mathcal{S}_2)$, conditioned on $\rho(\mathcal{T}_1\times\mathcal{S}_1)$ and $\rho(\mathcal{T}_2\times\mathcal{S}_2)$ are (conditionally) independent.

With $(\Omega,\mathscr{F},P)$ as the underlying probability space for the stochastic model above, define $\mathscr{B}_t$ as the σ-algebra generated by the space-time process over $[0,t)$. We define the counting process N_t as the number of points that occur during $[0,t)$ over the entire surface of the photodetector regardless of their location, i.e., $N_t=N([0,t)\times\mathcal{A})$.

3. Problem Statement. The central objective of an active pointing system is to maintain the centroid of the optical beam as close as possible to the center of the photodetector. This control task can be interpreted as one of minimizing y_t with respect to some appropriate norm. We adopt the quadratic form

$$J=\mathrm{E}\left[\int_0^T\left(x_t^TQ_tx_t+u_t^TP_tu_t\right)dt+x_T^TSx_T\right] \tag{4}$$

with $P_t=P_t^T>0$, $Q_t=Q_t^T\geqslant 0$, and $S=S^T\geqslant 0$, as the cost functional. For purpose of active pointing, a reasonable choice is $Q_t=C_t^TC_t$, $P_t=I_2$, and $S=0$.

We say u_t is an admissible control if u_t is $\mathscr{B}_t$-measurable and the solution to (3) is well-defined. Based on the cost functional (4), the control problem can be defined as: *Subject to state space equation* (3)*, find the admissible control u_t that minimizes the cost functional* (4)*.*

An intermediate step for solving the control problem is to obtain the posterior density $p_{x_t}(x|\mathscr{B}_t)$. In the next section we discuss this problem and develop an approximation for this posterior density. Employing this approximation, we propose a solution for the optimal control problem.

4. Estimation Problem. Let $(t_{k-1},t_k]$ be the interval between the $(k-1)^{th}$ and k^{th} occurrences of the space-time process, and let r_k be the location of k^{th} occurring point. For a function $b_t(r,\xi_t)$ that is continuous in r and left-continuous in t and ξ_t, the stochastic differential equation

$$d\xi_t=\int_{\mathcal{A}}b_t(r,\xi_t)\,N(dt\times dr)$$

is defined such that $d\xi_t=0$ during $(t_{k-1},t_k]$ and ξ_t encounters a jump of $b_{t_k}(r_k,\xi_{t_k})$ at $t=t_k$.

For the model of Section 2, Rhodes and Snyder [8] derived a stochastic partial differential equation describing the time-evolution of the posterior density $p_{x_t}(x|\mathscr{B}_t)$. This equation is expressed by

$$dp_{x_t}(x|\mathscr{B}_t) = \mathcal{L}\{p_{x_t}(x|\mathscr{B}_t)\}\,dt + p_{x_t}(x|\mathscr{B}_t)\int_{\mathcal{A}}\left(\lambda_t(r,x)\,\hat{\lambda}_t^{-1}(r) - 1\right)N(dt\times dr)$$
$$- p_{x_t}(x|\mathscr{B}_t)\int_{\mathcal{A}}\left(\lambda_t(r,x) - \hat{\lambda}_t(r)\right)dr\,dt \tag{5}$$

where $\hat{\lambda}_t(r) = \mathrm{E}[\lambda_t(r,x_t)\,|\mathscr{B}_t]$ and $\mathcal{L}\{\cdot\}$ is the forward Kolmogorov operator associated with (3) and defined as

$$\mathcal{L}\{\cdot\} = -\sum_{i=1}^{n}\partial\left[(A_t x + B_t u_t)(\cdot)\right]_i/\partial x^i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\partial^2\left[D_t D_t^T(\cdot)\right]_{ij}/\partial x^i\partial x^j.$$

Also, for special case $\mathcal{A} = \mathbb{R}^2$, it was shown in [8] that the solution of (5) is Gaussian with conditional mean $\hat{x}_t$ and conditional covariance Σ_t which satisfy the stochastic differential equations

$$d\hat{x}_t = A_t\hat{x}_t dt + B_t u_t dt + \int_{\mathbb{R}^2} M_t(r - C_t\hat{x}_t)\,N(dt\times dr) \tag{6}$$

$$d\Sigma_t = A_t\Sigma_t dt + \Sigma_t A_t^T dt + D_t D_t^T dt - M_t C_t \Sigma_t dN_t \tag{7}$$

with initial states $\hat{x}_0 = \bar{x}_0$ and $\Sigma_0 = \bar{\Sigma}_0$. In these equations, we have $M_t = \Gamma_t(\Sigma_t)$ where $\Gamma_t(\cdot)$ is defined as

$$\Gamma_t(\Sigma) = \Sigma C_t^T\left(C_t\Sigma C_t^T + R_t\right)^{-1}.$$

The aperture of a practical receiver has finite area so that the ideal condition $\mathcal{A} = \mathbb{R}^2$ is not feasible. In practice, where $\mathcal{A} \neq \mathbb{R}^2$, the filtering problem associated with (5) is infinite-dimensional. However, when $\mathcal{A}$ is large enough compared with the size of the optical beam, a finite-dimensional approximation is reasonable. The fact that $p_{x_t}(x|\mathscr{B}_t)$ is Gaussian for $\mathcal{A} = \mathbb{R}^2$ motivates us to consider a Gaussian approximation for $p_{x_t}(x|\mathscr{B}_t)$ when $\mathcal{A} \neq \mathbb{R}^2$. In the reminder of this section, we develop a method to determine the mean and covariance matrix of such a Gaussian approximation. The cumulant generating function associated with $p_{x_t}(x|\mathscr{B}_t)$ plays a central role in this development.

The conditional cumulant generating function of x_t given $\mathscr{B}_t$ is defined as

$$\psi_t(s) = \ln \mathrm{E}\left[\exp\left(j\omega^T x_t\right)|\mathscr{B}_t\right]\Big|_{j\omega=s},$$

and can be expanded in terms of conditional cumulants $\kappa_t^{i_1 i_2\cdots i_j}$ [9] as

$$\psi_t(s) = \sum_{j=1}^{\infty}\sum_{\mathscr{I}_j^n}\frac{1}{j!}\,\kappa_t^{i_1 i_2\cdots i_j}s_{i_1}s_{i_2}\cdots s_{i_j} \tag{8}$$

where $\mathscr{I}_j^n = \{1, 2, \ldots, n\}^j$ and $s = (s_1, s_2, \ldots, s_n)$. Note that $\hat{x}_t$ and Σ_t are represented in terms of the first and second order cumulants as $\hat{x}_t = \left(\kappa_t^1, \kappa_t^2, \ldots, \kappa_t^n\right)$ and $\Sigma_t = [\kappa_t^{ij}]$. The time-evolution of $\psi_t(\cdot)$ is described by a partial differential integral equation derived from (5) and is stated next.

THEOREM 4.1. *Let $\psi_t(\cdot)$ be the conditional cumulant generating function of x_t given $\mathscr{B}_t$ where x_t is the solution of* (3) *and $\mathscr{B}_t$ is defined in Section 2. Then, the time-evolution of $\psi_t(\cdot)$ is described by*

$$\begin{aligned} d\psi_t(s) &= s^T \left(A_t \frac{\partial \psi_t(s)}{\partial s} + B_t u_t \right) dt + \frac{1}{2}\, s^T D_t D_t^T s\, dt \\ &\quad + \int_{\mathcal{A}} \big(\ln \beta_t(r,s) - \ln \beta_t(r,0)\big) N(dt \times dr) - \int_{\mathcal{A}} \big(\beta_t(r,s) - \beta_t(r,0)\big) dr dt \end{aligned} \tag{9}$$

where $\beta_t(\cdot,\cdot)$ is defined as

$$\beta_t(r,s) = \exp\{-\psi_t(j\omega)\} \, \mathrm{E}\left[\exp\left(j\omega^T x_t\right) \lambda_t(r, x_t) | \mathscr{B}_t\right] \Big|_{j\omega = s}. \tag{10}$$

Moreover, if the Fourier transform of $\lambda_t(r,\cdot)$,

$$\Lambda_t(r, j\omega) = \int_{\mathbb{R}^n} \lambda_t(r,x) \exp\left(-j\omega^T x\right) dx$$

exists, $\beta_t(\cdot,\cdot)$ can be expressed as

$$\beta_t(r,s) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \Lambda_t(r, j\nu) \exp\big\{\psi_t(j\nu + s) - \psi_t(s)\big\} d\nu. \tag{11}$$

Proof. See Appendix A.1. □

The time-evolution of the cumulants is described by a (generally infinite) set of nonlinear stochastic differential equations driven by the space-time point process $N(\mathcal{T} \times \mathcal{S})$. This set of equations can be derived from (9) by matching the coefficients of corresponding $s_{i_1} s_{i_2} \cdots s_{i_j}$ on the two sides of (9). We can usually suppose that the first few cumulants approximate $p_{x_t}(x|\mathscr{B}_t)$ with an acceptable precision. This means that the infinite set of equations can be approximated by a finite-dimensional one.

Regarding this approach, two issues should be addressed. First, we need to compute $\beta_t(\cdot,\cdot)$ in terms of the cumulants via equations (10) or (11) and expansion (8), which is not straightforward for an arbitrary number of cumulants. Second, when we truncate (8) to a limited number of terms, the corresponding approximation for $p_{x_t}(x|\mathscr{B}_t)$ might not be a valid probability density function, i.e., it might be negative for some x. When we limit the expansion (8) to the first and second order terms (Gaussian approximation), these difficulties are avoided. In this case, $\beta_t(\cdot,\cdot)$ can be easily calculated and the truncated expansion leads to a valid probability density.

In Appendix A.2, we have used the method above to approximate $p_{x_t}(x|\mathscr{B}_t)$ with a Gaussian probability density. It is shown there that the mean $\tilde{x}_t$ and covariance matrix $\tilde{\Sigma}_t$ of this Gaussian approximation are solution to the stochastic differential equations

$$d\tilde{x}_t = A_t\tilde{x}_t dt + B_t u_t dt + \int_{\mathcal{A}} \tilde{M}_t\left(r - C_t\tilde{x}_t\right) N\left(dt \times dr\right) - \mu_t h_t\big(\tilde{x}_t, \tilde{\Sigma}_t\big) dt \tag{12}$$

$$d\tilde{\Sigma}_t = A_t\tilde{\Sigma}_t dt + \tilde{\Sigma}_t A_t^T dt + D_t D_t^T dt - \tilde{M}_t C_t \tilde{\Sigma}_t dN_t + \mu_t H_t\big(\tilde{x}_t, \tilde{\Sigma}_t\big) dt \tag{13}$$

with initial state $\tilde{x}_0 = \bar{x}_0$ and $\tilde{\Sigma}_0 = \bar{\Sigma}_0$. Here, $\tilde{M}_t = \Gamma_t\big(\tilde{\Sigma}_t\big)$, and $h_t\left(\cdot,\cdot\right) : \mathbb{R}^n \times \mathbb{R}^{n\times n} \to \mathbb{R}^n$ and $H_t\left(\cdot,\cdot\right) : \mathbb{R}^n \times \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ are defined as

$$h_t\left(x, \Sigma\right) = \int_{\mathcal{A}} \Gamma_t\left(\Sigma\right)\left(r - C_t x\right) \Phi_2\left(r; C_t x, C_t \Sigma C_t^T + R_t\right) dr \tag{14}$$

$$H_t\left(x, \Sigma\right) = \int_{\mathcal{A}} \Gamma_t\left(\Sigma\right)\left(C_t \Sigma C_t^T + R_t - \left(r - C_t x\right)\left(r - C_t x\right)^T\right) \Gamma_t^T\left(\Sigma\right) \cdot \Phi_2\left(r; C_t x, C_t \Sigma C_t^T + R_t\right) dr. \tag{15}$$

Note that $\tilde{x}_t$ and $\tilde{\Sigma}_t$ are approximations of $\hat{x}_t$ and Σ_t, not their exact values.

REMARK 4.1. *Equations* (14) *and* (15) *imply that as* $\mathcal{A} \to \mathbb{R}^2$, $h\left(\cdot,\cdot\right) \to 0$ *and* $H\left(\cdot,\cdot\right) \to 0$, *then as a consequence, the approximate estimator* (12), (13) *tends to exact estimator* (6), (7). *In this sense, we can say that* (12), (13) *is an asymptotically optimal estimator.*

5. Control Problem. We exploit the results of the previous section in solving the control problem as Theorem 5.1 below. Before stating this result, we fix some notation. Let $\Sigma = [\sigma_{ij}]$ denote a symmetric $n\times n$ matrix and $f\left(\Sigma\right)$ be a scalar function of Σ. Assume that the partial derivatives of $f\left(\Sigma\right)$ with respect to the elements of Σ exist. We denote by $\partial f\left(\Sigma\right)/\partial\Sigma$ a $n\times n$ symmetric matrix $F\left(\Sigma\right) = \left[F_{ij}\left(\Sigma\right)\right]$ such that $F_{ii} = \partial f/\partial\sigma_{ii}$ and $F_{ij} = \left(1/2\right)\partial f/\partial\sigma_{ij}$ for $i\neq j$. Let $g_t\left(x, \Sigma\right)$ be a scalar function of $x \in \mathbb{R}^n$ and $n\times n$ symmetric matrix Σ. Assume that the partial derivatives of $g_t\left(x, \Sigma\right)$ with respect to x and Σ exist. Define the linear operator $\mathcal{L}_t\left\{\cdot\right\}$ as

$$\begin{aligned}\mathcal{L}_t\left\{g_t\left(x, \Sigma\right)\right\} = &\int_{\mathcal{A}} \Big(g_t\big(x + \Gamma_t\left(\Sigma\right)\left(r - C_t x\right), \Sigma - \Gamma_t\left(\Sigma\right) C_t \Sigma\big) - g_t\left(x, \Sigma\right)\Big) \\ &\cdot \Phi_2\left(r; C_t x, C_t \Sigma C_t^T + R_t\right) dr \\ &- \left(\partial g_t\left(x, \Sigma\right)/\partial x\right)^T h_t\left(x, \Sigma\right) + \operatorname{tr}\left\{\left(\partial g_t\left(x, \Sigma\right)/\partial\Sigma\right) H_t\left(x, \Sigma\right)\right\}.\end{aligned} \tag{16}$$

Finally, we use $\|\cdot\|_{P_t}^2$ to denote $\left(\cdot\right)^T P_t\left(\cdot\right)$.

THEOREM 5.1. *Let $x \in \mathbb{R}^n$ and Σ be a $n \times n$ symmetric matrix. Suppose that $g_t(x,\Sigma)$ is the backward solution of the partial differential equation*

(17)
$$-\frac{\partial}{\partial t} g_t(x,\Sigma) = \left(\frac{\partial}{\partial x} g_t(x,\Sigma)\right)^T A_t x - \frac{1}{4}\left(\frac{\partial}{\partial x} g_t(x,\Sigma)\right)^T B_t P_t^{-1} B_t^T \left(\frac{\partial}{\partial x} g_t(x,\Sigma)\right) + \operatorname{tr}\left\{\left(\frac{\partial}{\partial \Sigma} g_t(x,\Sigma)\right)\left(A_t\Sigma + \Sigma A_t^T + D_t D_t^T\right) + Q_t \Sigma\right\} + x^T Q_t x + \mu_t \mathcal{L}_t\{g_t(x,\Sigma)\}$$

with boundary condition $g_T(x,\Sigma) = x^T S x$. Then the cost functional (4) can be expressed as

(18)
$$J = g_0(\tilde{x}_0, \tilde{\Sigma}_0) + \mathrm{E}\left[\int_0^T \delta_t dt\right] + \mathrm{E}\left[\int_0^T \left\| u_t + \frac{1}{2} P_t^{-1} B_t^T \left(\frac{\partial}{\partial \tilde{x}_t} g_t(\tilde{x}_t, \tilde{\Sigma}_t)\right)\right\|_{P_t}^2 dt\right],$$

where

$$\delta_t = \int_{\mathbb{R}^n} \left\{ x^T Q_t x + \int_{\mathcal{A}} \left(g_t\left(\tilde{x}_t + \tilde{M}_t(r - C_t\tilde{x}_t), \tilde{\Sigma}_t - \tilde{M}_t C_t \tilde{\Sigma}_t\right) - g_t(\tilde{x}_t, \tilde{\Sigma}_t)\right) \lambda_t(r,x)\, dr \right\}$$
(19)
$$\cdot \left(p_{x_t}(x|\mathscr{B}_t) - \tilde{p}_{x_t}(x|\mathscr{B}_t)\right) dx$$

is the error term resulting from replacing the posterior density $p_{x_t}(x|\mathscr{B}_t)$ by its Gaussian approximation $\tilde{p}_{x_t}(x|\mathscr{B}_t)$.

Proof. See Appendix A.3. □

The first term in the right side of (18) clearly does not depend on u_t and so is not involved in the minimization. While the hard-to-compute error term δ_t in (18) depends on u_t, it is supposed to be small. Therefore, in minimizing (18), we ignore δ_t and only minimize the third term. We note that the minimum of the third term is 0 and is achieved when u_t is given by

$$u_t^* = -\frac{1}{2} P_t^{-1} B_t^T \left(\frac{\partial}{\partial \tilde{x}_t} g_t(\tilde{x}_t, \tilde{\Sigma}_t)\right). \tag{20}$$

Then the cost associated with u_t^* will be

$$J^* = g_0(\bar{x}_0, \bar{\Sigma}_0) + \int_0^T \mathrm{E}\left[\delta_t|_{u_t=u_t^*}\right] dt.$$

When $\mathcal{A} = \mathbb{R}^2$, the solution to (17) can be simplified. This is stated as the following theorem which confirms that the optimal control in (20) is consistent with that obtained for $\mathcal{A} = \mathbb{R}^2$ by Rhodes and Snyder [8]. This shows that the approximation tends to the exact solution as $\mathcal{A}$ tends to $\mathbb{R}^2$.

THEOREM 5.2. *When $\mathcal{A} = \mathbb{R}^2$, the backward solution of the partial differential equation* (17) *with boundary condition $g_T(x, \Sigma) = x^T S x$ can be expressed as*

$$g_t(x, \Sigma) = x^T K_t x + f_t(\Sigma) \tag{21}$$

where K_t is the solution to the Riccati equation

$$\dot{K}_t = -K_t A_t - A_t^T K_t + K_t B_T P_t^{-1} B_t^T K_t - Q_t \tag{22}$$

with $K_T = S$, and $f_t(\Sigma)$ is the backward solution to the partial differential equation

$$\begin{aligned} -\frac{\partial}{\partial t} f_t(\Sigma) = \operatorname{tr}\left\{ \left(\frac{\partial}{\partial \Sigma} f_t(\Sigma) \right) \left(A_t \Sigma + \Sigma A_t^T + D_t D_t^T \right) + Q_t \Sigma \right\} \\ + \mu_t \big(f_t(\Sigma - \Gamma_t(\Sigma) C_t \Sigma) - f_t(\Sigma) \big) + \mu_t \operatorname{tr}\{\Gamma_t(\Sigma) C_t \Sigma K_t\} \end{aligned} \tag{23}$$

with boundary condition $f_T(\Sigma) = 0$.

Proof. See Appendix A.4. □

We see from (20) and (21) that when $\mathcal{A} = \mathbb{R}^2$, the optimal control is given by

$$u_t^* = -P_t^{-1} B_t^T K_t \hat{x}_t \tag{24}$$

with associated optimal cost

$$J^* = \bar{x}_0^T K_0 \bar{x}_0 + f_0\left(\bar{\Sigma}_0\right). \tag{25}$$

While the optimal control (24) has been obtained by Rhodes and Snyder [8], the value of the corresponding optimal cost (25) is newly obtained here.

6. Conclusion. Optimal control and estimation problems associated with a diffusion process modulating the rate of a doubly stochastic space-time Poisson process are discussed. It is assumed that the observation of the space-time process over a subset of $\mathbb{R}^2$ is used for the purpose of estimation and control. Our prime motivation for examining this optimal control problem is its application to active pointing in free-space optical communication. The main contributions of the paper are a suboptimal estimator for state of the diffusion process and a suboptimal control associated with a quadratic cost functional. Further, it is shown that when the observation is available over all of $\mathbb{R}^2$, our results tend toward the results of Rhodes and Snyder [8] for the latter case.

Appendix A. Proof of the Theorems.

A.1. Proof of Theorem 4.1. The Fourier transform of (5) is given by [8]

$$\begin{aligned} d\phi_t(j\omega) = \mathrm{E}\left[\exp\left(j\omega^T x_t\right) \left(j\omega^T (A_t x_t + B_t u_t) - \frac{1}{2}\omega^T D_t D_t^T \omega \right) \Big| \mathscr{B}_t \right] dt \\ + \int_{\mathcal{A}} \mathrm{E}\left[\exp\left(j\omega^T x_t\right) \left(\lambda_t(r, x_t) \hat{\lambda}_t^{-1}(r) - 1 \right) \Big| \mathscr{B}_t \right] N(dt \times dr) \\ - \int_{\mathcal{A}} \mathrm{E}\left[\exp\left(j\omega^T x_t\right) \left(\lambda_t(r, x_t) - \hat{\lambda}_t(r) \right) \Big| \mathscr{B}_t \right] dr dt. \end{aligned} \tag{26}$$

Let $t_1 < t_2 < t_3 < \cdots$ be the occurrence times of the space-time process $N(\mathcal{T} \times \mathcal{S})$. During the interval $(t_k, t_{k+1}),\ k = 0, 1, 2, \ldots$, the first integral in the right side of (26) is zero, thus we can write (26) as

$$d\exp\{\psi_t(j\omega)\} = \mathrm{E}\left[\exp\left(j\omega^T x_t\right)\left(j\omega^T\left(A_t x_t + B_t u_t\right) - \frac{1}{2}\,\omega^T D_t D_t^T \omega\right)\Big|\mathscr{B}_t\right] dt$$
$$- \int_{\mathcal{A}} \mathrm{E}\left[\exp\left(j\omega^T x_t\right)\left(\lambda_t(r, x_t) - \hat{\lambda}_t(r)\right)\big|\mathscr{B}_t\right] dr dt. \tag{27}$$

Continuing, we write

$$\exp\{\psi_t(j\omega)\}\, d\psi_t(j\omega) = \exp\{\psi_t(j\omega)\}\left[j\omega^T\left(A_t \frac{\partial\psi_t(j\omega)}{\partial j\omega} + B_t u_t\right) - \frac{1}{2}\,\omega^T D_t D_t^T \omega\right] dt$$
$$- \int_{\mathcal{A}} \mathrm{E}\left[\exp\left(j\omega^T x_t\right)\left(\lambda_t(r, x_t) - \hat{\lambda}_t(r)\right)\big|\mathscr{B}_t\right] dr dt, \tag{28}$$

using the identity

$$\mathrm{E}\left[x_t \exp\left(j\omega^T x_t\right)|\mathscr{B}_t\right] = \frac{\partial}{\partial j\omega}\,\mathrm{E}\left[\exp\left(j\omega^T x_t\right)|\mathscr{B}_t\right] = \frac{\partial\psi_t(j\omega)}{\partial j\omega}\exp\{\psi_t(j\omega)\}.$$

Multiplying both sides of (28) by $\exp\{-\psi_t(j\omega)\}$ and substituting $\beta_t(r, j\omega)$ from (10) into the resulting equation, we obtain

$$d\psi_t(j\omega) = j\omega^T\left(A_t \frac{\partial\psi_t(j\omega)}{\partial j\omega} + B_t u_t\right) dt - \frac{1}{2}\,\omega^T D_t D_t^T \omega\, dt - \int_{\mathcal{A}} \big(\beta_t(r, j\omega) - \beta_t(r, 0)\big) dr dt. \tag{29}$$

The discontinuity at $t = t_k$ is treated as follows. Let r_k be the spatial component of the event occurring at t_k. Then, from (26) we find

$$\phi_{t_k^+}(j\omega) - \phi_{t_k^-}(j\omega) = \mathrm{E}\left[\exp\big(j\omega^T x_{t_k^-}\big)\left(\lambda_{t_k^-}(r_k, x_{t_k^-})\hat{\lambda}_{t_k^-}^{-1}(r_k) - 1\right)\Big|\mathscr{B}_{t_k^-}\right],$$

which can be simplified as

$$\phi_{t_k^+}(j\omega) = \mathrm{E}\left[\exp\big(j\omega^T x_{t_k^-}\big)\lambda_{t_k^-}(r_k, x_{t_k^-})\big|\mathscr{B}_{t_k^-}\right]\hat{\lambda}_{t_k^-}^{-1}(r_k).$$

Multiplying both sides of this equation by $\exp\{-\psi_{t_k^-}(j\omega)\}$ and taking logarithms, we obtain

$$\psi_{t_k^+}(j\omega) - \psi_{t_k^-}(j\omega) = \ln\Big(\exp\{-\psi_{t_k^-}(j\omega)\}\mathrm{E}\left[\exp\big(j\omega^T x_{t_k^-}\big)\lambda_{t_k^-}(r_k, x_{t_k^-})\big|\mathscr{B}_{t_k^-}\right]\Big) - \ln\hat{\lambda}_{t_k^-}(r_k)$$
$$= \ln\beta_{t_k^-}(r_k, j\omega) - \ln\beta_{t_k^-}(r_k, 0).$$

Combining this with (29) and replacing $j\omega$ by s, we obtain (9).

From definition of $\beta_t(r, j\omega)$ (10), we have

$$\beta_t(r, j\omega) = \exp\{-\psi_t(j\omega)\} \int_{\mathbb{R}^n} p_{x_t}(x|\mathscr{B}_t) \exp\left(j\omega^T x\right) \lambda_t(r, x)\, dx. \tag{30}$$

Note that $p_{x_t}(x|\mathscr{B}_t)$ is the Fourier transform of $\exp\{\psi_t(j\omega)\}$, and so we can write

$$p_{x_t}(x|\mathscr{B}_t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \exp\{\psi_t(j\nu)\} \exp\left(-j\nu^T x\right) d\nu.$$

Upon substituting this into (30) and interchanging the order of integration[2], we obtain

$$\beta_t(r, j\omega) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \exp\big\{\psi_t(j\nu) - \psi_t(j\omega)\big\} \int_{\mathbb{R}^n} \lambda_t(r, x) \exp\left\{-j(\nu - \omega)^T x\right\} dx d\nu.$$

Replacing the second integral above by $\Lambda_t(r, j\nu - j\omega)$ and changing the variable of integration ν with $\nu + \omega$, we get

$$\beta_t(r, j\omega) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \Lambda_t(r, j\nu) \exp\big\{\psi_t(j\nu + j\omega) - \psi_t(j\omega)\big\} d\nu.$$

Finally we obtain (11) upon replacing $j\omega$ with s. □

A.2. Derivation of (12), (13). We first state a technical lemma from [10] which will be used later in deriving (12) and (13). For sake of completeness, we repeat below the proof from [10].

LEMMA A.1. *Let $z_k, \bar{z}_k \in \mathbb{R}^k$, $z_l \in \mathbb{R}^l$, and Θ_k and Θ_l be respectively $k \times k$ and $l \times l$ positive definite matrices. Assume that G is any $l \times k$ matrix. Then we have*

$$\int_{\mathbb{R}^k} \Phi_k(z_k; \bar{z}_k, \Theta_k)\, \Phi_l(z_l; G z_k, \Theta_l)\, dz_k = \Phi_l\left(z_l; G\bar{z}_k, \Theta_l + G\Theta_k G^T\right). \tag{31}$$

Proof. Denoting the Fourier transform of the left side of (31) by $F_l(\omega_l)$, we can write

$$\begin{aligned} F_l(\omega_l) &= \int_{\mathbb{R}^k} \Phi_k(z_k; \bar{z}_k, \Theta_k) \exp\left(j\omega_l^T G z_k - \tfrac{1}{2}\,\omega_l^T \Theta_l \omega_l\right) dz_k \\ &= \exp\left(j\omega_l^T G\bar{z}_k - \tfrac{1}{2}\,\omega_l^T G\Theta_k G^T \omega_l\right) \exp\left(-\tfrac{1}{2}\,\omega_l^T \Theta_l \omega_l\right) \\ &= \exp\left(j\omega_l^T G\bar{z}_k - \tfrac{1}{2}\,\omega_l^T \left(\Theta_l + G\Theta_k G^T\right) \omega_l\right) \end{aligned}$$

Taking inverse Fourier transform of the expression above, we get the right side of (31). □

The probability density function associated with the truncated expansion $\tilde{\psi}_t(s) = s^T \tilde{x}_t + \frac{1}{2}\, s^T \tilde{\Sigma}_t s$ is Gaussian with mean $\tilde{x}_t$ and covariance matrix $\tilde{\Sigma}_t$. With this approximate probability density function and with $\lambda_t(r, x_t) = \mu_t \Phi_2(r; C_t x_t, R_t)$, the approximation of $\beta_t(\cdot,\cdot)$ is given by

$$\tilde{\beta}_t(r, s) = \exp\left\{-\tilde{\psi}_t(s)\right\} \int_{\mathbb{R}^n} \Phi_n\big(x; \tilde{x}_t, \tilde{\Sigma}_t\big) \exp\left(s^T x\right) \mu_t \Phi_2(r; C_t x, R_t)\, dx.$$

[2] This interchange is permissible since for any fixed t, the integrand is continuous in x and ν.

A simple calculation yields that

$$\exp\left\{-\tilde{\psi}_t(s)\right\}\Phi_n\big(x;\tilde{x}_t,\tilde{\Sigma}_t\big)\exp\big(s^T x\big)=\Phi_n\big(x;\tilde{x}_t+\tilde{\Sigma}_t s,\tilde{\Sigma}_t\big).$$

Then, using Lemma A.1, we get

$$\tilde{\beta}_t(r,s)=\mu_t\Phi_2\left(r;C_t\tilde{x}_t+C_t\tilde{\Sigma}_t s,C_t\tilde{\Sigma}_t C_t^T+R_t\right),$$

which leads to

$$\begin{aligned}\ln\tilde{\beta}_t(r,s)-\ln\tilde{\beta}_t(r,0)&=s^T\tilde{\Sigma}_t C_t^T\left(C_t\tilde{\Sigma}_t C_t^T+R_t\right)^{-1}(r-C_t\tilde{x}_t)\\&\quad-\frac{1}{2}s^T\tilde{\Sigma}_t C_t^T\left(C_t\tilde{\Sigma}_t C_t^T+R_t\right)^{-1}C_t\tilde{\Sigma}_t s\end{aligned}\tag{32}$$

and

$$\begin{aligned}\tilde{\beta}_t(r,s)-\tilde{\beta}_t(r,0)=\mu_t\Phi_2&\left(r;C_t\tilde{x}_t,C_t\tilde{\Sigma}_t C_t^T+R_t\right)\\&\cdot\left\{s^T\tilde{\Sigma}_t C_t^T\left(C_t\tilde{\Sigma}_t C_t^T+R_t\right)^{-1}(r-C_t\tilde{x}_t)\right.\\&+\frac{1}{2}\left[s^T\tilde{\Sigma}_t C_t^T\left(C_t\tilde{\Sigma}_t C_t^T+R_t\right)^{-1}(r-C_t\tilde{x}_t)\right]^2\\&\left.-\frac{1}{2}s^T\tilde{\Sigma}_t C_t^T\left(C_t\tilde{\Sigma}_t C_t^T+R_t\right)^{-1}C_t\tilde{\Sigma}_t s+O\left(\|s\|^3\right)\right\}.\end{aligned}\tag{33}$$

We combine (32), (33), and (9), and match the coefficients of $s^T(\cdot)$ and $s^T(\cdot)s$ from both sides to obtain (12), (13). □

A.3. Proof of Theorem 5.1. Our proof consists of the following four steps. Step I: Using properties of conditional expectation, it is easy to show that

$$\mathrm{E}\left[x_t^T Q_t x_t\right]=\mathrm{E}\Big[\hat{x}_t^T Q_t\hat{x}_t+\mathrm{tr}\left\{Q_t\Sigma_t\right\}\Big].$$

Then the cost functional (4) can be expressed as

$$J=\mathrm{E}\left[\int_0^T\left(\tilde{x}_t^T Q_t\tilde{x}_t+\mathrm{tr}\left\{Q_t\tilde{\Sigma}_t\right\}+u_t^T P_t u_t+\delta_t^1\right)dt+x_T^T S x_T\right]\tag{34}$$

where δ_t^1 is defined as

$$\delta_t^1=\mathrm{tr}\left\{Q_t\left(\hat{x}_t\hat{x}_t^T-\tilde{x}_t\tilde{x}_t^T+\Sigma_t-\tilde{\Sigma}_t\right)\right\}.$$

Step II: For $t\geqslant 0$ and for any positive ϵ, $\Delta N_t\triangleq N_{t+\epsilon}-N_t$ is a conditionally Poisson random variable with stochastic rate

$$\bar{\lambda}_t^\epsilon=\int_t^{t+\epsilon}\int_{\mathcal{A}}\lambda_\tau(r,x_\tau)\,dr d\tau.$$

Thus, using the law of total probability, we can write

$$\begin{aligned}\Pr\left\{\Delta N_t = 1\middle|\mathscr{B}_t\right\} &= \mathrm{E}\left[\Pr\left\{\Delta N_t = 1|\bar{\lambda}_t^{\epsilon}\right\}|\mathscr{B}_t\right] \\ &= \mathrm{E}\left[\bar{\lambda}_t^{\epsilon}\exp\left(-\bar{\lambda}_t^{\epsilon}\right)|\mathscr{B}_t\right] \\ &= \epsilon q_t + O\left(\epsilon^2\right) \end{aligned} \tag{35}$$

where q_t is defined as

$$q_t = \int_{\mathcal{A}} \mathrm{E}\left[\lambda_t\left(r, x_t\right)|\mathscr{B}_t\right] dr. \tag{36}$$

In a similar manner, we can show that

$$\begin{aligned}\Pr\left\{\Delta N_t = 0\middle|\mathscr{B}_t\right\} &= 1 - \epsilon q_t + O\left(\epsilon^2\right) \\ \Pr\left\{\Delta N_t \geqslant 2\middle|\mathscr{B}_t\right\} &= O\left(\epsilon^2\right).\end{aligned} \tag{37}$$

Let the random vector $R \in \mathbb{R}^2$ denote the location of a single event occurring during $[t, t+\epsilon)$. We show that

$$p_R\left(r|\Delta N_t = 1, \mathscr{B}_t\right) = \frac{\mathrm{E}\left[\lambda_t\left(r, x_t\right)|\mathscr{B}_t\right]}{q_t} I_{\mathcal{A}}\left(r\right) + O\left(\epsilon\right) \tag{38}$$

where $I_{\mathcal{A}}(r) = 1$ if $r \in \mathcal{A}$ and $I_{\mathcal{A}}(r) = 0$ otherwise. For this purpose, let $\mathcal{D}(r) \subset \mathcal{A}$ denote a square with side length Δr and centered at $r \in \mathcal{A}$. Defining $\mathscr{T} = [t, t+\epsilon)$ and using Bayes' rule, we can write

$$\begin{aligned}p_R\left(r|\Delta N_t = 1, \mathscr{B}_t\right) &= \lim_{\Delta r \to 0} \Delta r^{-2} \Pr\left\{R \in \mathcal{D}\left(r\right)|\Delta N_t = 1, \mathscr{B}_t\right\} \\ &= \lim_{\Delta r \to 0} \Delta r^{-2} \Pr\left\{N\left(\mathscr{T} \times \mathcal{D}\left(r\right)\right) = 1|\Delta N_t = 1, \mathscr{B}_t\right\} \\ &= \lim_{\Delta r \to 0} \frac{1}{\Delta r^2} \cdot \frac{\Pr\left\{N\left(\mathscr{T} \times \mathcal{D}\left(r\right)\right) = 1, N\left(\mathscr{T} \times \mathcal{A}\right) = 1|\mathscr{B}_t\right\}}{\Pr\left\{\Delta N_t = 1|\mathscr{B}_t\right\}}.\end{aligned} \tag{39}$$

Note that the event of $N(\mathscr{T} \times \mathcal{D}(r)) = 1$ and $N(\mathscr{T} \times \mathcal{A}) = 1$ is equivalent to the event of $N(\mathscr{T} \times \mathcal{D}(r)) = 1$ and $N(\mathscr{T} \times (\mathcal{A} - \mathcal{D}(r))) = 0$. Therefore, defining $\mathscr{X}_t = \{x_\tau |\, \tau \in \mathscr{T}\}$ and using the law of total probability and properties of a space-time Poisson process, we get

$$\begin{aligned}&\Pr\left\{N\left(\mathscr{T} \times \mathcal{D}\left(r\right)\right) = 1, N\left(\mathscr{T} \times \mathcal{A}\right) = 1|\mathscr{B}_t\right\} \\ &\quad = \mathrm{E}\Big[\Pr\left\{N\left(\mathscr{T} \times \mathcal{D}\left(r\right)\right) = 1, N\left(\mathscr{T} \times \left(\mathcal{A} - \mathcal{D}\left(r\right)\right)\right) = 0\,|\mathscr{X}_t, \mathscr{B}_t\right\}\Big|\mathscr{B}_t\Big] \\ &\quad = \mathrm{E}\Big[\Pr\left\{N\left(\mathscr{T} \times \mathcal{D}\left(r\right)\right) = 1|\mathscr{X}_t\right\}\Pr\left\{N\left(\mathscr{T} \times \left(\mathcal{A} - \mathcal{D}\left(r\right)\right)\right) = 0\,|\mathscr{X}_t\right\}\Big|\mathscr{B}_t\Big] \\ &\quad = \mathrm{E}\left[\int_{\mathscr{T} \times \mathcal{D}(r)} \lambda_\tau\left(s, x_\tau\right) d\tau ds\left(1 - O\left(\epsilon\Delta r^2\right)\right)\middle|\mathscr{B}_t\right] \\ &\quad = \epsilon\Delta r^2 \mathrm{E}\left[\lambda_t\left(r, x_t\right)|\mathscr{B}_t\right] + O\left(\epsilon\Delta r^3\right) + O\left(\epsilon^2\Delta r^2\right).\end{aligned} \tag{40}$$

Substituting (35) and (40) into (39), we obtain (38).

Let $\tilde{p}_{x_t}(x|\mathscr{B}_t)$ be the Gaussian approximation of $p_{x_t}(x|\mathscr{B}_t)$. Then using Lemma A.1, we can write

$$\begin{aligned} \mathrm{E}\left[\lambda_t(r,x_t)\,|\mathscr{B}_t\right] &= \int_{\mathbb{R}^n} \tilde{p}_{x_t}(x|\mathscr{B}_t)\,\lambda_t(r,x)\,dx \\ &\quad + \int_{\mathbb{R}^n} \Big(p_{x_t}(x|\mathscr{B}_t) - \tilde{p}_{x_t}(x|\mathscr{B}_t)\Big)\lambda_t(r,x)\,dx \\ &= \mu_t \Phi_2\big(r; C_t\tilde{x}_t, C_t\tilde{\Sigma}_t C_t^T + R_t\big) \\ &\quad + \int_{\mathbb{R}^n} \Big(p_{x_t}(x|\mathscr{B}_t) - \tilde{p}_{x_t}(x|\mathscr{B}_t)\Big)\lambda_t(r,x)\,dx. \end{aligned} \tag{41}$$

Step III: Let $g_t(x,\Sigma)$ be a scalar function of $x \in \mathbb{R}^n$ and $n \times n$ symmetric matrix Σ. Assume that the partial derivatives of $g_t(x,\Sigma)$ with respect to t, x and Σ exist. Using the law of total probability we can write

$$\mathrm{E}\left[g_{t+\epsilon}\big(\tilde{x}_{t+\epsilon},\tilde{\Sigma}_{t+\epsilon}\big)\big|\mathscr{B}_t\right] = \sum_{k=0}^{\infty} \mathrm{E}\left[g_{t+\epsilon}\big(\tilde{x}_{t+\epsilon},\tilde{\Sigma}_{t+\epsilon}\big)\big|\mathscr{B}_t, \Delta N_t = k\right] \Pr\left\{\Delta N_t = k|\mathscr{B}_t\right\}.$$

Substituting $\Pr\{\Delta N_t = k|\mathscr{B}_t\}$ from (35) and (37) into the previous expression, and using the law of total probability again, we find

$$\begin{aligned} \mathrm{E}\left[g_{t+\epsilon}\big(\tilde{x}_{t+\epsilon},\tilde{\Sigma}_{t+\epsilon}\big)\big|\mathscr{B}_t\right] &= \mathrm{E}\left[g_{t+\epsilon}\big(\tilde{x}_{t+\epsilon},\tilde{\Sigma}_{t+\epsilon}\big)\big|\mathscr{B}_t, \Delta N_t = 0\right](1-\epsilon q_t) \\ &+ \mathrm{E}\left[\mathrm{E}\left[g_{t+\epsilon}\big(\tilde{x}_{t+\epsilon},\tilde{\Sigma}_{t+\epsilon}\big)\big|\mathscr{B}_t, \Delta N_t = 1, R = r\right]\big|\mathscr{B}_t, \Delta N_t = 1\right]\epsilon q_t + O\left(\epsilon^2\right). \end{aligned}$$

Inserting (36) and (38) above and rearranging terms, we obtain

$$\begin{aligned} \mathrm{E}\left[g_{t+\epsilon}\big(\tilde{x}_{t+\epsilon},\tilde{\Sigma}_{t+\epsilon}\big)\big|\mathscr{B}_t\right] &= \mathrm{E}\left[g_{t+\epsilon}\big(\tilde{x}_{t+\epsilon},\tilde{\Sigma}_{t+\epsilon}\big)\big|\mathscr{B}_t, \Delta N_t = 0\right] \\ &\quad + \epsilon \int_{\mathcal{A}} \bigg(\mathrm{E}\left[g_{t+\epsilon}\big(\tilde{x}_{t+\epsilon},\tilde{\Sigma}_{t+\epsilon}\big)\big|\mathscr{B}_t, \Delta N_t = 1, R = r\right] \\ &\quad - \mathrm{E}\left[g_{t+\epsilon}\big(\tilde{x}_{t+\epsilon},\tilde{\Sigma}_{t+\epsilon}\big)\big|\mathscr{B}_t, \Delta N_t = 0\right]\bigg) \\ &\quad \cdot \mathrm{E}\left[\lambda_t(r,x_t)\,|\mathscr{B}_t\right] dr + O\left(\epsilon^2\right). \end{aligned} \tag{42}$$

Conditioned on $\mathscr{B}_t$ and $\Delta N_t = 0$, (12) and (13) can be solved during $[t, t+\epsilon)$ to obtain

$$\begin{aligned} \tilde{x}_{t+\epsilon} &= \tilde{x}_t + \epsilon A_t\tilde{x}_t + \epsilon B_t u_t - \epsilon\mu_t h_t\big(\tilde{x}_t,\tilde{\Sigma}_t\big) + O\left(\epsilon^2\right) \\ \tilde{\Sigma}_{t+\epsilon} &= \tilde{\Sigma}_t + \epsilon A_t\tilde{\Sigma}_t + \epsilon\tilde{\Sigma}_t A_t^T + \epsilon D_t D_t^T + \epsilon\mu_t H_t\big(\tilde{x}_t,\tilde{\Sigma}_t\big) + O\left(\epsilon^2\right). \end{aligned} \tag{43}$$

Also, conditioned on $\mathscr{B}_t$, $\Delta N_t = 1$, and $R = r$, we can write

$$\begin{aligned} \tilde{x}_{t+\epsilon} &= \tilde{x}_t + \tilde{M}_t(r - C_t\tilde{x}_t) + O(\epsilon) \\ \tilde{\Sigma}_{t+\epsilon} &= \tilde{\Sigma}_t - \tilde{M}_t C_t\tilde{\Sigma}_t + O(\epsilon). \end{aligned} \tag{44}$$

Inserting (43) and (44) into (42), and linearizing with respect to ϵ, we obtain

$$\begin{aligned}
\mathrm{E}\left[g_t\big(\tilde{x}_{t+\epsilon},\tilde{\Sigma}_{t+\epsilon}\big)\big|\mathscr{B}_t\right] &= g_t\big(\tilde{x}_t,\tilde{\Sigma}_t\big) \\
&+ \epsilon\frac{\partial}{\partial t}\,g_t\big(\tilde{x}_t,\tilde{\Sigma}_t\big) + \epsilon\left(\frac{\partial}{\partial\tilde{x}_t}\,g_t\big(\tilde{x}_t,\tilde{\Sigma}_t\big)\right)^T\Big(A_t\tilde{x}_t + B_t u_t - \mu_t h_t\big(\tilde{x}_t,\tilde{\Sigma}_t\big)\Big) \\
&+ \epsilon\,\mathrm{tr}\left\{\left(\frac{\partial}{\partial\tilde{\Sigma}_t}\,g_t\big(\tilde{x}_t,\tilde{\Sigma}_t\big)\right)\Big(A_t\tilde{\Sigma}_t + \tilde{\Sigma}_t A_t^T + D_t D_t^T + \mu_t H_t\big(\tilde{x}_t,\tilde{\Sigma}_t\big)\tilde{M}_t^T\Big)\right\} \\
&+ \epsilon\int_{\mathcal{A}}\Big[g_t\Big(\tilde{x}_t + \tilde{M}_t\left(r - C_t\tilde{x}_t\right),\tilde{\Sigma}_t - \tilde{M}_t C_t\tilde{\Sigma}_t\Big) - g_t\big(\tilde{x}_t,\tilde{\Sigma}_t\big)\Big] \\
&\qquad\cdot\mathrm{E}\left[\lambda_t\left(r,x_t\right)|\mathscr{B}_t\right]dr + O\left(\epsilon^2\right).
\end{aligned}$$

We substitute $\mathrm{E}\left[\lambda_t\left(r,x_t\right)|\mathscr{B}_t\right]$ from (41) into the expression above and use the linear operator $\mathcal{L}_t\left\{\cdot\right\}$ defined by (16) to obtain the simplified form

$$\begin{aligned}
\text{(45)}\quad \mathrm{E}\left[g_t\big(\tilde{x}_{t+\epsilon},\tilde{\Sigma}_{t+\epsilon}\big)\big|\mathscr{B}_t\right] &= g_t\big(\tilde{x}_t,\tilde{\Sigma}_t\big) + \epsilon\frac{\partial}{\partial t}\,g_t\big(\tilde{x}_t,\tilde{\Sigma}_t\big) \\
&+ \epsilon\left(\frac{\partial}{\partial\tilde{x}_t}\,g_t\big(\tilde{x}_t,\tilde{\Sigma}_t\big)\right)^T\left(A_t\tilde{x}_t + B_t u_t\right) \\
&+ \epsilon\,\mathrm{tr}\left\{\left(\frac{\partial}{\partial\tilde{\Sigma}_t}\,g_t\big(\tilde{x}_t,\tilde{\Sigma}_t\big)\right)\Big(A_t\tilde{\Sigma}_t + \tilde{\Sigma}_t A_t^T + D_t D_t^T\Big)\right\} \\
&+ \epsilon\mu_t\mathcal{L}_t\Big\{g_t\big(\tilde{x}_t,\tilde{\Sigma}_t\big)\Big\} + \epsilon\delta_t^2 + O\left(\epsilon^2\right)
\end{aligned}$$

where the error term δ_t^2 is defined as

$$\begin{aligned}
\delta_t^2 = \int_{\mathbb{R}^n}\int_{\mathcal{A}}&\Big[g_t\Big(\tilde{x}_t + \tilde{M}_t\left(r - C_t\tilde{x}_t\right),\tilde{\Sigma}_t - \tilde{M}_t C_t\tilde{\Sigma}_t\Big) - g_t\big(\tilde{x}_t,\tilde{\Sigma}_t\big)\Big] \\
&\cdot\Big(p_{x_t}\left(x|\mathscr{B}_t\right) - \tilde{p}_{x_t}\left(x|\mathscr{B}_t\right)\Big)\lambda_t\left(r,x\right)drdx.
\end{aligned}$$

Define the nonlinear operator $\mathcal{K}_t\left\{\cdot\right\}$ by

$$\begin{aligned}
\mathcal{K}_t\left\{g_t\left(x,\Sigma\right)\right\} &= \frac{\partial}{\partial t}\,g_t\left(x,\Sigma\right) + \left(\frac{\partial}{\partial x}\,g_t\left(x,\Sigma\right)\right)^T A_t x \\
&- \frac{1}{4}\left(\frac{\partial}{\partial x}\,g_t\left(x,\Sigma\right)\right)^T B_t P_t^{-1} B_t^T\left(\frac{\partial}{\partial x}\,g_t\left(x,\Sigma\right)\right) \\
&+ \mathrm{tr}\left\{\left(\frac{\partial}{\partial\Sigma}\,g_t\left(x,\Sigma\right)\right)\left(A_t\Sigma + \Sigma A_t^T + D_t D_t^T\right) + Q_t\Sigma\right\} \\
&+ x^T Q_t x + \mu_t\mathcal{L}_t\left\{g_t\left(x,\Sigma\right)\right\}.
\end{aligned}$$

Then, (45) can be rewritten as

$$\begin{aligned}
\mathrm{E}\left[g_{t+\epsilon}\big(\tilde{x}_{t+\epsilon},\tilde{\Sigma}_{t+\epsilon}\big)\big|\mathscr{B}_t\right] &= g_t\big(\tilde{x}_t,\tilde{\Sigma}_t\big) + \epsilon\left\|u_t + \frac{1}{2}\,P_t^{-1}B_t^T\left(\frac{\partial}{\partial\tilde{x}_t}\,g_t\big(\tilde{x}_t,\tilde{\Sigma}_t\big)\right)\right\|_{P_t}^2 + \epsilon\delta_t^2 \\
&- \epsilon\left(\tilde{x}_t^T Q_t\tilde{x}_t + \mathrm{tr}\Big\{Q_t\tilde{\Sigma}_t\Big\} + u_t^T P_t u_t\right) + \epsilon\mathcal{K}_t\Big\{g_t\big(\tilde{x}_t,\tilde{\Sigma}_t\big)\Big\} + O\left(\epsilon^2\right).
\end{aligned}$$

Now, let $g_t(\cdot,\cdot)$ be the backward solution of the partial differential equation (17) with boundary condition $g_T(x,\Sigma) = x^T S x$. This implies that $\mathcal{K}_t\{g_t(\tilde{x}_t,\tilde{\Sigma}_t)\} = 0$. Under this condition, we take expectation from the equation above to get

$$\begin{aligned} \mathrm{E}\left[g_{t+\epsilon}(\tilde{x}_{t+\epsilon},\tilde{\Sigma}_{t+\epsilon})\right] &= \mathrm{E}\left[g_t(\tilde{x}_t,\tilde{\Sigma}_t)\right] \\ &\quad + \epsilon\mathrm{E}\left[\left\| u_t + \frac{1}{2} P_t^{-1} B_t^T \left(\frac{\partial}{\partial \tilde{x}_t} g_t(\tilde{x}_t,\tilde{\Sigma}_t)\right)\right\|_{P_t}^2\right] + \epsilon\mathrm{E}\left[\delta_t^2\right] \\ &\quad - \epsilon\mathrm{E}\left[\tilde{x}_t^T Q_t \tilde{x}_t + \mathrm{tr}\left\{Q_t\tilde{\Sigma}_t\right\} + u_t^T P_t u_t\right] + O\left(\epsilon^2\right). \end{aligned} \tag{46}$$

Step IV: We partition the interval $[0,T)$ into K subintervals $[t_k,t_{k+1})$, $k = 0,1,\ldots,K-1$, where $t_0 = 0$, $t_K = T$, and $t_{k+1} - t_k \triangleq \epsilon_k > 0$. Recalling that $x_T^T S x_T = g_{t_K}(\tilde{x}_{t_K},\tilde{\Sigma}_{t_K})$, we approximate the cost functional (34) by the finite sum

$$J \simeq J_K = \sum_{k=0}^{K-1} \epsilon_k \mathrm{E}\left[\tilde{x}_{t_k}^T Q_{t_k}\tilde{x}_{t_k} + \mathrm{tr}\left\{Q_{t_k}\tilde{\Sigma}_{t_k}\right\} + u_{t_k}^T P_{t_k} u_{t_k} + \delta_{t_k}^1\right] + \mathrm{E}\left[g_{t_K}(\tilde{x}_{t_K},\tilde{\Sigma}_{t_K})\right].$$

This finite sum can be rearranged as

$$\begin{aligned} J_K = &\sum_{k=0}^{K-2} \epsilon_k \mathrm{E}\left[\tilde{x}_{t_k}^T Q_{t_k}\tilde{x}_{t_k} + \mathrm{tr}\left\{Q_{t_k}\tilde{\Sigma}_{t_k}\right\} + u_{t_k}^T P_{t_k} u_{t_k} + \delta_{t_k}^1\right] + \mathrm{E}\left[\delta_{t_{K-1}}^1\right] \\ &+ \epsilon_{K-1}\mathrm{E}\left[\tilde{x}_{t_{K-1}}^T Q_{t_{K-1}}\tilde{x}_{t_{K-1}} + \mathrm{tr}\left\{Q_{t_{K-1}}\tilde{\Sigma}_{t_{K-1}}\right\} + u_{t_{K-1}}^T P_{t_{K-1}} u_{t_{K-1}}\right] \\ &+ \mathrm{E}\left[g_{t_K}(\tilde{x}_{t_K},\tilde{\Sigma}_{t_K})\right]. \end{aligned}$$

In the right side above, we replace $\mathrm{E}\left[g_{t_K}(\tilde{x}_{t_K},\tilde{\Sigma}_{t_K})\right]$ by the right side of (46). With minor manipulations, and upon defining $\delta_t = \delta_t^1 + \delta_t^2$ according to (19), we find that

$$\begin{aligned} J_K = &\sum_{k=0}^{K-2} \epsilon_k \mathrm{E}\left[\tilde{x}_{t_k}^T Q_{t_k}\tilde{x}_{t_k} + \mathrm{tr}\left\{Q_{t_k}\tilde{\Sigma}_{t_k}\right\} + u_{t_k}^T P_{t_k} u_{t_k} + \delta_{t_k}^1\right] \\ &+ \mathrm{E}\left[g_{t_{K-1}}(\tilde{x}_{t_{K-1}},\tilde{\Sigma}_{t_{K-1}})\right] \\ &+ \epsilon_{K-1}\mathrm{E}\left[\left\| u_{t_{K-1}} + \frac{1}{2} P_{t_{K-1}}^{-1} B_{t_{K-1}}^T \left(\frac{\partial}{\partial \tilde{x}_{t_{K-1}}} g_{t_{K-1}}(\tilde{x}_{t_{K-1}},\tilde{\Sigma}_{t_{K-1}})\right)\right\|_{P_{t_{K-1}}}^2\right] \\ &+ \epsilon_{K-1}\mathrm{E}\left[\delta_{t_{K-1}}\right] + O\left(\epsilon_{K-1}^2\right). \end{aligned}$$

Repeating this procedure for $k = K-2, K-3, \ldots, 1, 0$, we obtain

$$\begin{aligned} J_K = &\mathrm{E}\left[g_{t_0}(\tilde{x}_{t_0},\tilde{\Sigma}_{t_0})\right] + \sum_{k=0}^{K-1} \epsilon_k \mathrm{E}\left[\delta_{t_k}\right] \\ &+ \sum_{k=0}^{K-1} \epsilon_k \mathrm{E}\left[\left\| u_{t_k} + \frac{1}{2} P_{t_k}^{-1} B_{t_k}^T \left(\frac{\partial}{\partial \tilde{x}_{t_k}} g_{t_k}(\tilde{x}_{t_k},\tilde{\Sigma}_{t_k})\right)\right\|_{P_{t_k}}^2\right] + \sum_{k=0}^{K-1} O\left(\epsilon_k^2\right). \end{aligned}$$

Finally, we take the limit of J_K as $K \to \infty$ and $\max \epsilon_k \to 0$ to obtain (18). □

A.4. Proof of Theorem 5.2. For $\mathcal{A} = \mathbb{R}^2$ and $g_t(x, \Sigma)$ given by (21), we can show that

$$\mathcal{L}_t \{g_t(x, \Sigma)\} = f_t\big(\Sigma - \Gamma_t(\Sigma) C_t \Sigma\big) - f_t(\Sigma) + \mathrm{tr}\{\Gamma_t(\Sigma) C_t \Sigma K_t\}$$

which is clearly not dependent on x. Therefore, (17) can be decomposed into two independent equations: the partial differential equation (23) with boundary condition $f_T(\Sigma) = 0$ and the following equation

$$-\frac{\partial\left(x^T K_t x\right)}{\partial t} = \left(\frac{\partial\left(x^T K_t x\right)}{\partial x}\right)^T A_t x - \frac{1}{4}\left(\frac{\partial\left(x^T K_t x\right)}{\partial x}\right)^T B_t P_t^{-1} B_t^T \left(\frac{\partial\left(x^T K_t x\right)}{\partial x}\right) + x^T Q_t x$$

with boundary condition $x^T K_T x = x^T S x$. This equation holds for any arbitrary x if and only if K_t satisfies (22) with terminal condition $K_T = S$. □

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