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Large deviations for the Bessel clock

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We show the law of large numbers, the central limit theorem and the large-deviation principle for the Bessel clock $\int_0^t \mathrm{d}s/(R_s^{(\nu)})^2$, where $(R_t^{(\nu)}, t \geqslant 0)$ is a Bessel process of index $\nu > 0$. We also give functional versions of these limit theorems.

Keywords: Bessel processes; Brownian motion; large deviations

1. Introduction

Consider $(B_t, t \geqslant 0)$, a Brownian motion (BM) in $\mathbb{R}^d$, with $d \geqslant 2$, and $B_0 \neq 0$ almost surely (a.s.). It is well known that $(B_t, t \geqslant 0)$ a.s. does not visit 0, hence the increasing process

$$C_t = \int_0^t \frac{\mathrm{d}s}{|B_s|^2}, \qquad t \geqslant 0, \tag{1}$$

is finite. We shall call this process the Bessel clock, since, on the one hand, it involves only the Bessel process $(|B_t|, t \geqslant 0)$ and, on the other hand, $(C_t, t \geqslant 0)$ is the time-change, or 'clock', which appears in the skew product decomposition

$$B_t = |B_t| \theta_{C_t} \tag{2}$$

of $(B_t, t \geqslant 0)$ into its radial part $(|B_t|, t \geqslant 0)$ and its angular part $B_t/|B_t| = \theta_{C_t}$, where $(\theta_u, u \geqslant 0)$ is a BM taking values in S_{d-1}, the unit sphere of $\mathbb{R}^d$, and independent of $(|B_t|, t \geqslant 0)$ (see, for example, Itô and McKean 1974; Revuz and Yor 1999).

More generally, if $(R_t^{(1)}, t \geqslant 0)$ and $(R_t^{(2)}, t \geqslant 0)$ are two independent Bessel processes with respective dimensions d_1 and d_2, and if $d_1 + d_2 \geqslant 2$, and $R_0^{(1)} + R_0^{(2)} > 0$ a.s., then there is the skew product representation

$$R_t^{(1)} = \rho_t Y_{C_t^{(\nu)}}, \tag{3}$$

where

$$C_t^{(\nu)} = \int_0^t \frac{\mathrm{d}s}{\rho_s^2}$$

and $\nu = (d_1 + d_2 - 2)/2$, and where $\rho_t = \{(R_t^{(1)})^2 + (R_t^{(2)})^2\}^{1/2}$ is a Bessel process with dimension $d_1 + d_2$, independent of $(Y_u, u \geqslant 0)$, a so-called Jacobi process with dimensions (d_1, d_2). See, for example, Warren and Yor (1997; 1998) for a discussion of these processes,

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and Hu *et al*. (1999) for some applications to asymptotics of diffusions in (Brownian) random media. See also Karlin and Taylor (1981).

A third natural occurrence of the Bessel clock concerns geometric Brownian motion $(\exp(\beta_t + \nu t);\ t \geqslant 0)$ which may be represented in Lamperti's form as

$$\exp(\beta_t + \nu t) = R^{(\nu)}_{A^{(\nu)}_t}, \tag{4}$$

where $(R^{(\nu)}_u,\ u \geqslant 0)$ is a Bessel process with dimension $d = 2(1+\nu)$ and

$$A^{(\nu)}_t = \int_0^t \mathrm{d}s \exp 2(\beta_s + \nu s). \tag{5}$$

Here, of course, $R^{(\nu)}$ and $A^{(\nu)}$ are not at all independent. In fact, they are closely related and, in particular,

$$(A^{(\nu)}_t > u) = \left(t > C^{(\nu)}_u \equiv \int_0^u \frac{\mathrm{d}s}{(R^{(\nu)}_s)^2} \right), \tag{6}$$

that is to say, the inverse of $A^{(\nu)}$ is the Bessel clock associated with $R^{(\nu)}$. This relationship and the knowledge of the law of $C^{(\nu)}$ obviously yield some results on $A^{(\nu)}_t$; see, for example, Yor (1992) and Geman and Yor (1993) for some applications to the pricing of Asian options.

The previous discussion shows that there is some interest in studying the asymptotic behaviour of $(C^{(\nu)}_t;\ t \to +\infty)$, the Bessel clock associated with a Bessel process of dimension $d > 2$. The following law of large numbers (LLN) and central limit theorem (CLT) are essentially given in Revuz and Yor (1999).

Theorem 1.1. *Let* $(R^{(\nu)}_t,\ t \geqslant 0)$ *be a Bessel process starting from* $R^{(\nu)}_0 \neq 0$ *a.s., with dimension* $d > 2$*, i.e.* $\nu > 0$*. Then we have*

$$\frac{1}{\log t} C^{(\nu)}_t \underset{t\to+\infty}{\longrightarrow} \frac{1}{d-2} = \frac{1}{2\nu} \quad \textit{a.s. and in } L^p; \tag{7}$$

$$\sqrt{\log t}\left(\frac{1}{\log t} \int_0^t \frac{\mathrm{d}s}{(R^{(\nu)}_s)^2} - \frac{1}{d-2} \right) \underset{t\to+\infty}{\overset{law}{\longrightarrow}} N, \tag{8}$$

where N *is centred Gaussian with variance* $\sigma^2 = 1/2\nu^3$.

Proof. These results are stated respectively as Exercise (4.23), Chapter IV and Exercise (3.20), Chapter X in Revuz and Yor (1999). In fact, in these exercises $(R_t,\ t \geqslant 0)$ denotes the radial part of the BM in $\mathbb{R}^d$, $d > 2$, but the proofs easily extend to Bessel processes (as defined and studied in Revuz and Yor 1999, Chapter XI) with dimension $d > 2$. To give a few details, we show how (8) follows from (7). In this proof, we skip the exponent ν for the sake of clarity. From Itô's formula,

$$\log R_t = \log R_0 + \int_0^t \frac{\mathrm{d}\beta_s}{R_s} + \frac{d-2}{2} \int_0^t \frac{\mathrm{d}s}{R_s^2}; \tag{9}$$

hence

$$\sqrt{\log t}\left(\frac{d-2}{2\log t}\int_0^t \frac{ds}{R_s^2} - \frac{1}{2}\right)$$

has the same asymptotic law as

$$\frac{1}{\sqrt{\log t}}\int_0^t \frac{d\beta_s}{R_s} = \gamma_{s_t} \tag{10}$$

where

$$S_t = \frac{C_t^{(\nu)}}{\log t}$$

and where $(\gamma_u,\ u \geqslant 0)$ is a (Dubins–Schwarz) BM obtained after 'log-scaling'. Finally, from (7) the right-hand term of equality (10) converges in law towards

$$\gamma_{1/2\nu} \overset{law}{=} \frac{1}{\sqrt{2\nu}}\gamma_1.$$

$\square$

Since Bessel processes form the core of this paper, we devote the following section to their most important properties. The remainder of the paper is organized as follows. In Section 3, we complement the result (7) in Theorem 1.1 with a large-deviation principle (LDP). In Section 4, we give functional versions of the LLN, CLT and LDP for $\{C_t^{(\nu)}\}$. Finally, in the Appendix, we show, informally, how to find the rate function of the one-dimensional LDP, by means of a contraction technique.

Some related work on large deviations for exponential functionals of BM can be found in Zani (2000, Chapter 5).

2. A few facts about Bessel processes

The Bessel process $(R_t^{(\nu)},\ t \geqslant 0)$ of index ν is an $\mathbb{R}^+$-valued diffusion with infinitesimal generator $\mathscr{L}^{(\nu)}$ given by

$$\mathscr{L}^{(\nu)} f(x) = \frac{1}{2} f''(x) + \frac{2\nu+1}{2x} f'(x), \qquad f \in \mathscr{C}_b^2(]0, +\infty]). \tag{11}$$

Let $r > 0$ and let $P_r^{(\nu)}$ denote the law of $(R_t^{(\nu)},\ t \geqslant 0)$ starting from $R_0^{(\nu)} = r$ on $\mathscr{C}(\mathbb{R}_+, \mathbb{R}_+)$, the set of continuous functions from $\mathbb{R}_+$ to $\mathbb{R}_+$. Let $(R_t,\ t \geqslant 0)$ be the canonical process on $\mathscr{C}(\mathbb{R}_+, \mathbb{R}_+)$, and $\mathscr{R}_t = \sigma\{R_s,\ s \leqslant t\}$ the canonical filtration. From Girsanov's theorem, for $\nu \geqslant 0$, the mutual absolute continuity relation holds:

$$P_{r|\mathscr{R}_t}^{(\nu)} = \left(\frac{R_t}{r}\right)^{\nu} \exp\left\{\frac{-\nu^2}{2}\int_0^t \frac{ds}{R_s^2}\right\} P_{r|\mathscr{R}_t}^{(0)}. \tag{12}$$

If $\nu < 0$, the Bessel process reaches 0 a.s.; and for $\nu < 0$, we need to modify (12) as follows:

$$P^{(\nu)}_{r|\mathscr{R}_t \cap (t<T_0)} = \left(\frac{R_t}{r}\right)^{\nu} \exp\left\{\frac{-\nu^2}{2}\int_0^t \frac{\mathrm{d}s}{R_s^2}\right\} P^{(0)}_{r|\mathscr{R}_t}, \tag{13}$$

where $T_0 = \inf\{u > 0;\ R_u = 0\}$.

Moreover, we shall need a slight extension of the family of Bessel processes, for dimension δ varying with time, $\delta(t) = 2(1+\nu(t))$, and for ν a simple function given by

$$\nu(t) = \sum_{i=1}^{n} \nu_i \mathbb{1}_{]t_i, t_{i+1}]}(t),$$

and the ν_i are real numbers. The construction of such processes is done step by step on each time inverval: we still denote the corresponding law $P^{\nu(.)}$. It is easily seen that the absolute continuity property extends to this family; we leave the computation of the Radon–Nikodym densities to the reader.

3. Large-deviation result

Write

$$S_t^{(\nu)} = \frac{1}{\log t} C_t^{(\nu)}.$$

We state an LDP for $(S_t^{(\nu)},\ t \geqslant 0)$, and we show how this translates to $(A_t^{(\nu)},\ t \to +\infty)$ via property (6).

Theorem 3.1. *Let* $(R_t^{(\nu)},\ t \geqslant 0)$ *be a Bessel process starting from* $R_0^{(\nu)} \neq 0$ *a.s., with dimension* $d > 2$. *Set* $\nu = d/2 - 1$. *Then* $\{S_t^{(\nu)}\}$ *satisfies an LDP with speed* $(\log t)$ *and good rate function*

$$\Lambda_\nu^*(x) = \frac{(2\nu x - 1)^2}{8x}, \qquad x \geqslant 0. \tag{14}$$

In particular, for any $s > 0$,

$$\frac{1}{\log t} P\left(S_t^{(\nu)} > \frac{1}{2\nu} + s\right) \underset{t\to+\infty}{\longrightarrow} -\Lambda_\nu^*\left(\frac{1}{2\nu} + s\right).$$

An immediate consequence is the following corollary:

Corollary 3.2. *Let* $(\beta_u;\ u \geqslant 0)$ *be a real-valued BM starting from* 0, $\nu > 0$ *and* $A^{(\nu)}$ *given by* (5). *Then, for any* $a > 0$,

$$\frac{1}{v}\log P\left(\frac{1}{v}\log A_v^{(\nu)} < 2\nu - a\right) \underset{v\to+\infty}{\longrightarrow} -\frac{a^2}{8}.$$

Proof of Corollary 3.2. This is immediate from Theorem 3.1 and relation (6).

Proof of Theorem 3.1. With the notation introduced in Section 2 we obtain, for $\phi \leqslant \nu^2/2$,

$$\mathrm{E}_r^{(\nu)}(\mathrm{e}^{\phi C_t}) = \mathrm{E}_r^{(\nu_\phi)}\left(\frac{R_t}{r}\right)^{2s(\phi)}, \tag{15}$$

where

$$s(\phi) = \frac{1}{2}(\nu - \nu_\phi)$$

and

$$\nu_\phi = \sqrt{\nu^2 - 2\phi}.$$

From the scaling property,

$$\mathrm{E}_r^{(\nu_\phi)}(R_t)^{2s(\phi)} = \mathrm{E}_{r/\sqrt{t}}^{(\nu_\phi)}(R_1)^{2s(\phi)} t^{s(\phi)}.$$

We compute the normalized cumulant generating function of C_t:

$$\Lambda_\nu^t(\phi) := \frac{1}{\log t} \log \mathrm{E}_r^{(\nu)}(\mathrm{e}^{\phi C_t}) = s(\phi) + \frac{1}{\log t} \log\left[\mathrm{E}_{r/\sqrt{t}}^{(\nu_\phi)}(R_1)^{2s(\phi)}\right] - \frac{\log r^{2s(\phi)}}{\log t}.$$

Moreover,

$$\mathrm{E}_{r/\sqrt{t}}^{(\nu_\phi)}(R_1)^{2s(\phi)} \to \mathrm{E}_0^{(\nu_{(\phi)})}(R_1)^{2s(\phi)}$$

as $t \to +\infty$, hence

$$\Lambda_t^\nu(\phi) \to \Lambda_\nu(\phi) \equiv s(\phi).$$

Since s is an essentially smooth function on its domain, we can apply the Gärtner–Ellis theorem (see Dembo and Zeitouni 1998). This yields the LDP, with rate function $\Lambda_\nu^* = s^*$, the Legendre dual of s. □

Remark 3.3. Similar results hold for $\int_a^t \mathrm{d}s/R_s^2$, where $a > 0$. The only change is that r should be replaced everywhere in the above proof by $R_a \neq 0$ a.s. Asymptotically, this makes no difference.

4. Functional results

In this section, we consider the functional version of Theorems 1.1 and 3.1. Let $(R_t^{(\nu)},\ t \geqslant 0)$ be a d-dimensional Bessel process starting from $R_0^{(\nu)} \neq 0$ a.s. Denote by $\{Z_n^{(\nu)}\}$ the family of random functions defined on [0, 1] by

$$u \to Z_n^{(\nu)}(u) = \frac{1}{n}\int_1^{\mathrm{e}^{nu}} \frac{\mathrm{d}s}{(R_s^{(\nu)})^2}. \tag{16}$$

Let $\mathscr{C}_0([0, 1])$ be the set of continuous functions from [0, 1] to $\mathbb{R}$ such that $f(0) = 0$, endowed with the supremum norm topology. We have the following theorem.

Theorem 4.1.

(i) *The family* $\{Z_n^{(\nu)}\}_n$ *converges a.s. on* $\mathscr{C}_0([0, 1])$ *towards* $u/(2\nu)$.
(ii) *The family* $\{\sqrt{n}(Z_n^{(\nu)} - \mathrm{E}(Z_n^{(\nu)}))\}$ *converges in distribution in* $\mathscr{C}_0([0, 1])$ *towards*

$$\left(\frac{1}{2\nu}\gamma_u,\ 0 \leqslant u \leqslant 1\right),$$

where $(\gamma_u,\ u \geqslant 0)$ *is a Brownian motion.*

For the proofs of Theorem 4.1 and 4.2, ν is fixed, and we make no further mention of this in the notation.

Proof. (i) We know that $Z_n(u) \to u/2\nu$ for any fixed $u \in [0, 1]$. This is a family of increasing functions on a compact, converging pointwise (outside of a set of measure 0) to a continuous function. Hence from Dini's theorem, the convergence is uniform (a.s.).

(ii) From formula (9),

$$\sqrt{n}\left(\frac{d-2}{2n}\int_1^{\mathrm{e}^{nu}} \frac{\mathrm{d}s}{R_s^2} - \frac{u}{2}\right) + \frac{1}{\sqrt{n}}\int_1^{\mathrm{e}^{nu}} \frac{\mathrm{d}\beta_s}{R_s} = \frac{1}{\sqrt{n}}\log\left(\frac{R_{\mathrm{e}^{nu}}}{\mathrm{e}^{nu/2}}\right). \tag{17}$$

Denote by $H_n(u)$ the above right-hand term. Then

$$\sup_{u \leqslant A}|H_n(u)| = \frac{1}{\sqrt{n}}\sup_{1 \leqslant t \leqslant \mathrm{e}^{nA}}\left|\log\left(\frac{R_t}{\sqrt{t}}\right)\right|, \tag{18}$$

which tends to 0 as $n \to \infty$, using on the one hand the law of the iterated logarithm, which takes care of the large values of R_t, and on the other hand the Dvoretsky–Erdős law (see, for example, Itô and McKean 1974) which takes care of the probability of R_t being bounded away from 0. Furthermore

$$\frac{1}{\sqrt{n}}\int_1^{\mathrm{e}^{nu}} \frac{\mathrm{d}\beta_s}{R_s} = \gamma^{(n)}_{A_u^{(n)}} \tag{19}$$

where

$$A_u^{(n)} = \frac{1}{n}\int_1^{\mathrm{e}^{nu}} \frac{\mathrm{d}s}{R_s^2}.$$

Now let us estimate, for any $\varepsilon > 0$,

$$p_n = P\left(\sup_{u \leqslant T}|\gamma^{(n)}_{A_u^{(n)}} - \gamma^{(n)}_{cu}| \geqslant \varepsilon\right),$$

where $c = 1/(2\nu)$. We can bound

$$p_n \leqslant P\left(\sup_{u \leqslant T}|\gamma^{(n)}_{A_u^{(n)}} - \gamma^{(n)}_{cu}| \geqslant \varepsilon;\ \sup_{u \leqslant T}|A_u^{(n)} - cu| \leqslant \delta\right) + P\left(\sup_{u \leqslant T}|A_u^{(n)} - cu| \geqslant \delta\right)$$

From Bienaymé–Chebyshev,

$$p_n \leqslant \frac{1}{\varepsilon^2} \mathrm{E}\left[\sup_{u,v \leqslant 2T}\left(\frac{\gamma_u^{(n)} - \gamma_v^{(n)}}{(v-u)^{1/3}}\right)^2\right]\delta^{2/3} + P\left(\sup_{u \leqslant T}|A_u^{(n)} - cu| \geqslant \delta\right). \tag{20}$$

Let us fix $\theta > 0$. Since the first term on the right-hand side of (20) does not depend on n, we can fix δ small enough to have this term smaller than $\theta/2$ (see, for example, Stroock and Varadhan 1979). Now, since $A_u^n \to u/(2\nu)$, there exists N large enough such that, for any $n \geqslant N$, the second right-hand side term of (20) is smaller than $\theta/2$. This proves that $p_n \to 0$ as $n \to \infty$. □

We also give a functional version of the previous LDP. The paths of Z_n are increasing and continuous. Therefore they belong to $\mathscr{D}$, the space of cadlag functions on [0, 1]. Let $\mathscr{X}$ be the set of functions $f[0, 1] \to \mathbb{R}$ such that $f(0) = 0$, with the topology of pointwise convergence. Let, for $\phi \in \mathscr{D}$

$$\mathrm{I}(\phi) = \begin{cases} \displaystyle\int_0^1 \Lambda_\nu^*(\dot{\phi}_1(s))\mathrm{d}s + \frac{\nu^2}{2}\phi_2(1) & \text{if } \phi \text{ is non-decreasing, } \phi(0) = 0, \\ +\infty & \text{otherwise,} \end{cases}$$

where ϕ_1 and ϕ_2 are the absolutely continuous and singular components of ϕ (see de Acosta 1994; Lynch and Sethuraman 1987).

The singular part in the action functional comes from the fact that the underlying log-Laplace transform is not finite everywhere; see formula (26).

Theorem 4.2. *The family* $\{Z_n^{(\nu)}\}$ *satisfies an LDP in* $\mathscr{X}$ *with speed n and good rate function* $I_{\mathscr{X}}$ *which coincides with I on* $\mathscr{D}$.

Proof. We proceed in three steps, detailed below. First, we state an LDP for a finite-dimensional vector $(Z_n(u_1), Z_n(u_2), \ldots, Z_n(u_p))$ for any p-tuple $(u_1, \ldots, u_p)$. Then a projective limit argument leads to a functional LDP in $\mathscr{X}$, endowed with the pointwise convergence topology. A small improvement on this argument gives the result of the theorem. □

First step: Finite-dimensional result. Let $p \in \mathbb{N}$, and let $\mathbf{u}$: $0 < u_1 < u_2 < \ldots < u_p \leqslant 1$ be a subdivision of [0, 1]. We set $t_i = \mathrm{e}^{nu_i}$, and we look for an LDP for the family $\{(Z_n(u_1), Z_n(u_2), \ldots, Z_n(u_p))\}_{n\in\mathbb{N}}$.

Theorem 4.3. *The family* $\{(Z_n(u_1), Z_n(u_2), \ldots, Z_n(u_p))\}_{n\in\mathbb{N}}$ *satisfies an LDP with speed n and good rate function defined by*

$$I_{\mathbf{u}}(y_1, y_2, \ldots, y_p) = \begin{cases} \displaystyle\sum_{k=1}^{p}(u_k - u_{k-1})\Lambda_\nu^*\left(\frac{y_k - y_{k-1}}{u_k - u_{k-1}}\right) & \textit{if for every } i \in \{1, \ldots, p\}, \ y_i > y_{i-1}, \\ +\infty & \textit{otherwise,} \end{cases} \tag{21}$$

where $u_0 = y_0 = 0$.

Proof. Let $(\phi_1, \phi_2, \ldots, \phi_p) \in \mathbb{R}^p$. We compute the Laplace transform $\mathrm{E}_r^\nu(\exp\{\sum \phi_k \tilde{C}_{t_k}\})$, where $t_k = \mathrm{e}^{nu_k}$, $k \geqslant 1$, $t_0 = \mathrm{e}^{nu_0} = 1$, and $\tilde{C}_{t_k=} = C_{t_k} - C_{t_0}$. From Girsanov,

$$\mathrm{E}_r^\nu\left(\exp\left\{\sum_{k=1}^{p} \phi_k \tilde{C}_{t_k}\right\}\right) = \mathrm{E}_r^\nu \mathrm{E}_{R_{t_0}}^0\left(\exp\left\{\sum_{k=1}^{p-1} \phi_k \tilde{C}_{t_k} - \psi_p \tilde{C}_{t_p}\right\}\left(\frac{R_{t_p}}{R_{t_0}}\right)^\nu\right) \tag{22}$$

where
$$\psi_p = \frac{\nu^2}{2} - \phi_p.$$

Furthermore, the right-hand term of (22) can be written

$$\mathrm{E}_r^\nu\left(\mathrm{E}_{R_{t_0}}^0\left(\exp\left\{\sum_{k=1}^{p-1} \phi_k \tilde{C}_{t_k} - \psi_p \tilde{C}_{t_{p+1}}\right\}\left(\frac{R_{t_p}}{R_{t_0}}\right)^\nu \left(\frac{R_{t_p}}{R_{t_{p-1}}}\right)^{-\sqrt{2\psi_p}}\left(\frac{R_{t_p}}{R_{t_{p-1}}}\right)^{\sqrt{2\psi_p}} \exp\{-\psi_p(\tilde{C}_{t_p} - \tilde{C}_{t_{p-1}})\}\right)\right). \tag{23}$$

Proceeding in this manner, step by step, we can state that

$$\mathrm{E}_r^\nu\left(\exp\left\{\sum_{k=1}^{p} \phi_k \tilde{C}_{t_k}\right\}\right) = \mathrm{E}_r^{\nu(.)}(R_{t_0}^{\alpha_0} R_{t_1}^{\alpha_1} R_{t_2}^{\alpha_2} \ldots R_{t_p}^{\alpha_p}), \tag{24}$$

where $\nu(t)$ is a time-varying parameter for the changed Bessel process (see Section 2).

Let $\nu_0 = \nu_{p+1} = 0$ and for any $1 \leqslant k \leqslant p$,

$$v_k = \sum_{i=k}^{p} \phi_i, \tag{25}$$

and let

$$\mathscr{D}_\Lambda = \{(\phi_1, \phi_2, \ldots, \phi_p); \forall k \in \{1, \ldots, p\}, v_k \leqslant \nu^2/2\}. \tag{26}$$

Now if $g(v) = \sqrt{\nu^2 - 2v}$, then

$$\nu(t) = g(v_k) \qquad \text{for } t \in (t_{k-1}, t_k],\ 1 \leqslant k \leqslant p,\ t_{-1} = 0.$$

The exponents α_k are given by

$$\alpha_k = g(v_{k+1}) - g(v_k) \qquad v_{p+1} = 0.$$

As in the previous section, we compute the normalized cumulant generating function. From the scaling property,

$$\begin{aligned}
\Lambda_n^{\mathbf{u}}(\phi_1, \phi_2, \ldots, \phi_p) &= \frac{1}{n} \log \mathrm{E}_r^\nu\left(\exp\left\{\sum \phi_i C_{t_i}\right\}\right) \\
&= \frac{1}{n} \log \mathrm{E}_r^{\nu(.)}(R_{t_0}^{\alpha_0} R_{t_1}^{\alpha_1} R_{t_2}^{\alpha_2} \ldots R_{t_{p-1}}^{\alpha_{p-1}} \mathrm{E}_{R_{t_{p-1}}}^{\nu(t_p)}(R_{t_p - t_{p-1}})^{\alpha_p}) \\
&= \frac{1}{n} \log (t_p - t_{p-1})^{\alpha_p/2} + \frac{1}{n} \log \mathrm{E}_r^{\nu(.)}(R_{t_0}^{\alpha_0} R_{t_1}^{\alpha_1} R_{t_2}^{\alpha_2} \ldots R_{t_{p-1}}^{\alpha_{p-1}} \mathrm{E}_{R_{t_{p-1}}/\sqrt{t_p - t_{p-1}}}^{\nu(t_p)}(R_{t_0}^{\alpha_p})).
\end{aligned}$$

It is easily seen that

$$\frac{1}{n}\log(t_p - t_{p-1})^{\alpha_p/2} = \frac{1}{n}\log(\mathrm{e}^{nu_p} - \mathrm{e}^{nu_{p-1}})^{\alpha_p/2} \to \frac{\alpha_p u_p}{2}.$$

Hence

$$\Lambda_n^{\mathbf{u}} \to \Lambda_{\mathbf{u}},$$

where

$$\Lambda_{\mathbf{u}}(\phi_1, \phi_2, \ldots, \phi_p) = \begin{cases} \dfrac{1}{2}\displaystyle\sum_{k=1}^{p} \alpha_k u_k & \text{on } \mathscr{D}_\Lambda \\ +\infty & \text{otherwise.} \end{cases}$$

We now determine the Fenchel-Legendre dual of $\Lambda_{\mathbf{u}}$ defined on $\mathbb{R}^p$ by

$$\Lambda_{\mathbf{u}}^*(\mathbf{y}) = \sup_{\phi\in\mathbb{R}^p}\{\mathbf{y}\cdot\phi - \Lambda_{\mathbf{u}}((\phi)\},$$

where $\mathbf{y} = (y_1, y_2, \ldots, y_p)$. Tedious but easy calculations lead to

$$\Lambda_{\mathbf{u}}^* = I_{\mathbf{u}},$$

where $I_{\mathbf{u}}$ is given by expression (21). □

Second step: Projective limit. We now define a projective system so that we can transport the discrete result to a continuous one. Let J denote the collection of all ordered finite subsets of [0, 1]. A partial order by inclusion can be defined on J as follows: for any $\mathbf{u} = \{0 < u_1 \leqslant u_2 \leqslant \ldots \leqslant u_p \leqslant 1\}$ and $\mathbf{v} = \{0 < v_1 \leqslant v_2 \leqslant \ldots \leqslant v_q \leqslant 1\}$, $\mathbf{u} \leqslant \mathbf{v}$ if and only if, $\forall i \leqslant p$, $u_i = v_{h(i)}$ for some applications h: $\mathbb{N} \to \mathbb{N}$. Set the natural projection $p_{\mathbf{u},\mathbf{v}}$:

$$p_{\mathbf{u},\mathbf{v}}\colon \mathbb{R}^{|\mathbf{u}|} \to \mathbb{R}^{|\mathbf{v}|}, \qquad \mathbf{u} \leqslant \mathbf{v}.$$

Denote $\mathscr{Y}_{\mathbf{u}} = \mathbb{R}^{|\mathbf{u}|}$ and let $\tilde{\mathscr{X}}$ be the projective limit of $(\mathscr{Y}_{\mathbf{u}}, p_{\mathbf{u},\mathbf{v}})_{\mathbf{u}\leqslant\mathbf{v}\in\mathbf{J}}$. For $f \in \mathscr{X}$ and $\mathbf{u} = \{u_1, u_2, \ldots, u_p\} \in J$, let

$$p_{\mathbf{u}}(f) = (f(u_1), f(u_2), \ldots, f(u_p)).$$

Actually $\tilde{\mathscr{X}}$ can be identified with $\mathscr{X}$ as follows: for any $f \in \mathscr{X}$, $(p_{\mathbf{u}}(f))_{\mathbf{u}\in J} \in \tilde{\mathscr{X}}$ since $p_{\mathbf{u}}(f) = p_{\mathbf{u},\mathbf{v}}(p_{\mathbf{v}}(f))$ for any $\mathbf{u} \leqslant \mathbf{v}$. Conversely, for any $(x_{\mathbf{u}})_{\mathbf{u}\in J}$ of $\tilde{\mathscr{X}}$, we can associate $f \in \mathscr{X}$ such that $p_{\mathbf{u}}(f) = x_{\mathbf{u}}$, i.e. $f(t_i) = x_{t_i}$. The projective topology on $\tilde{\mathscr{X}}$ coincides with the pointwise topology on $\mathscr{X}$. Note that the $\mathscr{Y}_{\mathbf{u}}$ are Hausdorff spaces and that the $I_{\mathbf{u}}$ of Theorem 4.3 are good rate functions. Hence, we can apply the Dawson–Gärtner theorem (see Theorem 4.6.1 of Dembo and Zeitouni 1998): the family $\{Z_n\}$ satisfies an LDP with rate n and good rate function

$$I_{\mathscr{X}}(\phi) = \sup_{\mathbf{u}\in J} I_{\mathbf{u}}(p_{\mathbf{u}}(\phi)).$$

Third step: Conclusion. If ϕ is not increasing, $\mathrm{I}_{\mathscr{X}}(\phi) = +\infty$. If ϕ is increasing and cadlag, $\mathrm{I}_{\mathscr{X}}(\phi) = \mathrm{I}(\phi)$ as a consequence of Lynch and Sethuraman (1987) page 617. This ends the proof of Theorem 4.2. □

Appendix

The rate function of Theorem 3.1 can be found, informally, by the contraction principle (see Dembo and Zeitouni 1998). Let us consider for the sake of simplicity the case $\delta = d$ integer, $d > 2$.

On the one hand, large deviations on S_t can be carried to large deviations on as $(1/t)\int_0^t |X_s|^{-2}\,\mathrm{d}s$, via the classical transformation $X_s = \mathrm{e}^{-s/2}R(\mathrm{e}^s)$, where X_s is the d-dimensional stationary Ornstein–Uhlenbeck process with infinitesimal generator L_d given by

$$L_d\phi(x) = \tfrac{1}{2}(\Delta\phi(x) - \langle\nabla\phi(x), x\rangle).$$

On the other hand, let us consider the level 2 LDP for X_s, i.e. the LDP on the occupation measure $(1/t)\int_0^t \delta_{X_s}\mathrm{d}s$. From Donsker and Varadhan (1975a; 1975b; 1976; 1983), the rate function governing this LDP is

$$I_2(\mu) := \tfrac{1}{2}\int_{\mathbb{R}^d} |\nabla f|^2\,\mathrm{d}m_0 = -\int_{\mathbb{R}^d} f(x)L_d f(x)\mathrm{d}m_0(x), \tag{27}$$

where m_0 is the standard Gaussian measure, μ is a probability on $\mathbb{R}^d$ and $\mathrm{d}\mu/\mathrm{d}m_0 = f^2$. The use of this result – and the classical transformation above – in large-deviation theory has been studied by Heck (1998) (see also March and Seppäläinen 1997).

Let us now consider a contraction. If $g \in \mathscr{C}_b(\mathbb{R}^d, \mathbb{R})$, the mapping $\mu \mapsto \int g\,\mathrm{d}\mu$ is continuous, and by the contraction principle, the LDP for $(1/t)\int_0^t g(X_s)\mathrm{d}s$ has rate function

$$I_1(a) = \inf\left\{ I_2(\mu);\ \int g\mathrm{d}\mu = a\right\}. \tag{28}$$

If g is a radial function, say $g(x) = \mathbf{g}(|x|^2)$, we may look for radial f, say $f(x) = \mathbf{f}(|x|^2)$. Let us denote by $\mathbf{m}_0$ the gamma distribution, the image of m_0 by the mapping $x \mapsto |x|^2$.

Since $|\nabla f(x)^2|^2 = 4|x|^2\mathbf{f}'(|x|^2)^2$, the variational problem (28) becomes

$$I_1(a) = \inf\left\{ -\int_0^\infty \mathbf{f}(\rho)\mathbf{L}_d\mathbf{f}(\rho)\mathrm{d}\mathbf{m}_0(\rho);\ \int \mathbf{f}^2(\rho)\mathbf{g}(\rho)\mathrm{d}\mathbf{m}_0 = a;\ \int \mathbf{f}^2(\rho)\mathrm{d}\mathbf{m}_0(\rho) = 1\right\}, \tag{29}$$

where $\mathbf{L}_d$ is the Laguerre infinitesimal generator, $\mathbf{L}_d\mathbf{f}(\rho) = 2\rho\mathbf{f}''(\rho) + (d-\rho)\mathbf{f}'(\rho)$. This generator is the image of the radial part of L_d by the mapping $r \to r^2$. From the Lagrange multipliers method, we find the differential equation satisfed by $\mathbf{f}$:

$$-\mathbf{L}_d\mathbf{f}(\rho) + \alpha\mathbf{f}(\rho)\mathbf{g}(\rho) + \beta\mathbf{f}(\rho) = 0. \tag{30}$$

We wish to apply this scheme to obtain large deviations on $\int_0^t |X_s|^{-2}\mathrm{d}s$. We try $\mathbf{g}(\rho) = \rho^{-1}$, which is far from continuous. In this case the partial differential equation (30) becomes

$$-\mathbf{L}_d\mathbf{f}(\rho) + \alpha\rho^{-1}\mathbf{f}(\rho) + \beta\mathbf{f}(\rho) = 0. \tag{31}$$

The only solution of (31) with the constraints as in (29) is

$$\mathbf{f}_a(\rho) = C_a\rho^{s_a},$$

with

$$s_a = \frac{1}{4a} - \frac{\nu}{2} \quad \text{and} \quad C_a = 2^{-s_a}\sqrt{\frac{\Gamma(\nu+1)}{\Gamma(2s_a+\nu+1)}}, \qquad \nu = \frac{d-2}{2}.$$

For such an $\mathbf{f}_a$,

$$I_2(f\,\mathrm{d}m_0) = \frac{(2\nu a - 1)^2}{8a},$$

and we check

$$I_2(f\,\mathrm{d}m_0) = \Lambda_\nu^*(a).$$

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