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CR MAPPINGS OF FINITE MULTIPLICITY AND EXTENSION OF PROPER HOLOMORPHIC MAPPINGS

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1. Introduction. We shall describe some general theorems about CR mappings between three-dimensional manifolds which, among other results, imply that any proper holomorphic mapping $f: D \to D'$ between pseudoconvex domains in $\mathbf{C}^2$ with real analytic boundaries extends to be holomorphic in a neighborhood of the closure of D (Theorem 8). In case the domain D is strictly pseudoconvex, this result follows from the classical Lewy-Pinčuk reflection principle [**9**, **11**]. In case D' is strictly pseudoconvex, or in case D and D' are given by polynomial defining functions, f extends by [**2**]. In case the proper mapping f is biholomorphic, the extendability has been proved by Baouendi, Jacobowitz, and Treves [**1**]. The general case of a proper holomorphic mapping between weakly pseudoconvex domains which is not biholomorphic is more complicated because branching might occur. We have developed a method in the spirit of [**1**] which allows us to prove extendability at boundary points even if branching occurs (Theorems 3 and 6).

The mapping $f(z,w) = (z^2, w)$ which maps the domain $\mathbf{E} = \{(z,w) \in \mathbf{C}^2: |z|^4 + |w|^2 < 1\}$ onto the unit ball in $\mathbf{C}^2$ has the property that it maps points of type four (in the sense of Kohn [**8**]) in the boundary of $\mathbf{E}$ to points of type two in the boundary of the ball. Furthermore, the local branching order of f at these points is two. We prove that this phenomenon holds in general. If M and M' are abstract three-dimensional CR manifolds, and $H: M \to M'$ a CR mapping, there is a notion of multiplicity of H at $p_0 \in M$, for which the type of p_0 is equal to the multiplicity at p_0 times the type of $H(p_0)$. Theorems 1 and 2 state these results more precisely. Theorems 5 and 7 give applications of these results and of the extendability result (Theorem 3) to CR and proper self-mappings.

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2. Main results and applications. A real smooth manifold M is called a *CR manifold* if there is a subbundle $\mathcal{V}$ of $\mathbf{C}TM$, the complexified tangent bundle of M, satisfying the conditions $[\mathcal{V}, \mathcal{V}] \subset \mathcal{V}$, and $\mathcal{V} \cap \overline{\mathcal{V}} = (0)$; $\mathcal{V}$ is called the CR *bundle* of M.

If M and M' are two CR manifolds with CR bundles $\mathcal{V}$ and $\mathcal{V}'$, a CR mapping from M into M' is a smooth mapping $H: M \to M'$, such that at every $p \in M$

$$H'(\theta) \in \mathcal{V}'_{H(p)}, \qquad \forall \theta \in \mathcal{V}_p, \tag{1}$$

where $\mathcal{V}_p$ is the fiber of $\mathcal{V}$ at p, $\mathcal{V}'_{H(p)}$ is the fiber of $\mathcal{V}'$ at $H(p)$, and $H': \mathbf{C}TM \to \mathbf{C}TM'$ is the differential map of H.

From now on we restrict ourselves to the case: $\dim_{\mathbf{R}} M = \dim_{\mathbf{R}} M' = 3, \dim_{\mathbf{C}} \mathcal{V} = \dim_{\mathbf{C}} \mathcal{V}' = 1$. Let $p_0 \in M$, $p'_0 = H(p_0) \in M'$, and let L, L' be two nonvanishing smooth sections of $\mathcal{V}$ and $\mathcal{V}'$, defined near p_0 and $H(p_0)$ respectively. It follows from (1) that there exists a smooth function λ defined in a neighborhood of p_0 in M such that, for all $p \in M$ near p_0,

$$H'(L_p) = \lambda(p) L'_{H(p)}. \tag{2}$$

DEFINITION. The CR mapping $H: M \to M'$ is of *finite multiplicity* at p_0 if there exists a differential operator of the form

$$S = M_1 \cdots M_j \tag{3}$$

(called *string* of length j) with $M_p = L$ or $\overline{L}$, such that

$$S\lambda(p_0) \neq 0. \tag{4}$$

The mapping is said to be of *multiplicity* k if the shortest string S for which (4) holds is of length $k - 1$.

In particular H is of multiplicity 1 if $\lambda(p_0) \neq 0$, of multiplicity 2 if $\lambda(p_0) = 0$, and either $L\lambda(p_0) \neq 0$ or $\overline{L}\lambda(p_0) \neq 0$, etc.

It is clear that for a CR mapping H to be of multiplicity k is independent of the choice of the vector fields L and L'.

Following Kohn [**8**], the CR manifold M is of *finite type* at p_0 if and only if any smooth complex vector field defined near p_0 is in the Lie algebra spanned by L and $\overline{L}$. It is of *type* m *at* p_0 ($m \geq 2$) if the shortest bracket of L's and $\overline{L}$'s not in the span of L and $\overline{L}$ at p_0, is of length m. (We have used the following convention: the length of $[L, \overline{L}]$ is 2, that of $[L, [L, \overline{L}]]$ is 3, etc.)

We can now state our first result.

THEOREM 1. *Let $H: M \to M'$ be a CR mapping from M into M'. If H is of multiplicity k at $p_0 \in M$ ($1 \leq k < \infty$), and M' is of type m' at $p'_0 = H(p_0)$ ($2 \leq m' < \infty$), then:*

(i) *M is of type m at p_0 with $m = km'$.*

(ii) *If S is a string of the form* (3) *with length $\leq k - 1$, and with at least one $M_p = \overline{L}$, then $S\lambda(p_0) = 0$; in particular $L^{k-1}\lambda(p_0) \neq 0$.*

(iii) $H'(\mathbf{C}T_{p_0}M) \not\subset \mathcal{V}'_{H(p_0)} \oplus \overline{\mathcal{V}'}_{H(p_0)}$.

Note that if $H'_{p_0}: T_{p_0}M \to T_{H(p_0)}M'$ is the differential map at p_0, then (iii) implies that

$$H'_{p_0} \neq \{0\}.$$

If $k = 1$, it follows from (2), (4) (with $S = 1$), and (iii) that H'_{p_0} is an isomorphism; therefore H is a local diffeomorphism from a neighborhood of p_0 onto a neighborhood of $H(p_0)$. This fact, when H is a boundary value of a holomorphic mapping from a pseudoconvex domain in $\mathbf{C}^2$ to another, follows from Derridj [**5**].

The proof of Theorem 1 uses recursive arguments with repeated applications of the following two observations.

First, suppose A and B are vector fields on M for which there exist smooth functions $\alpha_j(u)$, $\beta_j(u)$ on M, $1 \leq j \leq r$, and smooth vector fields A'_j and B'_j on M' such that

$$H'(A_u) = \sum_j \alpha_j(u) A'_{j,H(u)}, \qquad H'(B_u) = \sum_j \beta_j(u) B'_{j,H(u)};$$

then

$$H'([A,B]_u) = \sum_j (A\beta_j)(u) B'_{j,H(u)} - \sum_j (B\alpha_j)(u) A'_{j,H(u)} + \sum_{p,q} (\alpha_p \beta_q)(u)[A'_p, B'_q]_{H(u)}.$$

Second, if E_1 and E_2 are two commutators of L and $\overline{L}$ of length n_1 and n_2 respectively then

$$[E_1, E_2] = aL + b\overline{L} + \sum_\alpha c_\alpha C^\alpha,$$

where each C^α is a commutator of L and $\overline{L}$ of length $|\alpha| < n_1 + n_2$, and a, b, c_α are smooth functions on M.

One of the crucial steps in the proof of Theorem 1 consists of proving that if S_1 and S_2 are two strings of the form (3) with length $k-1$, having the same number of L's, then

$$S_1\lambda(p_0) = S_2\lambda(p_0).$$

We say that the mapping $H\colon M \to M'$ is *flat* at p_0 if all partial derivatives of H of any order vanish at p_0.

The following result is a consequence of Theorem 1.

THEOREM 2. *Let $H\colon M \to M'$ be a CR mapping and $p_0 \in M$. If M and M' are of finite type at p_0 and $p'_0 = H(p_0)$ respectively, then the following conditions are equivalent:*

(a) *H is of finite multiplicity at p_0.*

(b) $H'_{p_0} \neq \{0\}$.

(c) *H is not flat at p_0.*

Examples show that the conclusions of Theorems 1 and 2 are no longer valid if the finite type conditions are dropped in the assumptions.

Our main analyticity result is the following.

THEOREM 3. *Let M, M' be two real analytic CR manifolds, H a smooth CR mapping from M into M', and $p_0 \in M$. If H is of finite multiplicity at p_0*

and M' of finite type at $p'_0 = H(p_0)$, then H is real analytic in a neighborhood of p_0.

The proof of Theorem 3 uses the general approach of [**1**]. Since M and M' are real analytic, they can be considered as embedded in $\mathbf{C}^2$, where the variables are denoted by z, w. We assume that $p_0 = H(p_0) = 0$, and that M and M' are respectively given locally by

$$\operatorname{Im} w = \varphi(z, \bar{z}, \operatorname{Re} w), \qquad \operatorname{Im} w = \psi(z, \bar{z}, \operatorname{Re} w),$$

with $\varphi(z, 0, \operatorname{Re} w) = \psi(z, 0, \operatorname{Re} w) = 0$. The mapping H is then locally given by a pair of CR functions (f, g) defined on M and satisfying

$$\frac{g - \bar{g}}{2i} = \psi\left(f, \bar{f}, \frac{g + \bar{g}}{2}\right). \tag{5}$$

As in [**1**], it suffices to show that f and g are real analytic with respect to $\operatorname{Re} w$, uniformly in z.

We have here $\lambda = L\bar{f}$, where the function λ is as in (2). By Theorem 1 we have $L^k \bar{f}(0) \neq 0$ and $L^j \bar{f}(0) = 0$, $0 \leq j < k$.

An important part of the proof consists of repeatedly applying L to (5). In addition to the arguments used in [**1**] (where $k = 1$), the following result in one complex variable is crucial.

LEMMA. *Let a be a positive number and R the domain in $\mathbf{C}$ defined by $|\xi| < a$, $0 < \eta < a$, with $\varsigma = \xi + i\eta$. Let u, v be two functions defined in $\overline{R}$ and satisfying:*

(i) *$u, v \in C^\infty(\overline{R})$ and u, v are holomorphic in R,*

(ii) *$h(\xi) = u(\xi)/v(\xi) \in C^\infty([-a, a])$,*

(iii) *there exists a positive integer p and, for $0 \leq j \leq p - 1$, functions $a_j \in C^\infty(\overline{R})$, holomorphic in R, $a_j(0) = 0$, such that*

$$(h(\xi))^p + a_{p-1}(\xi)(h(\xi))^{p-1} + \cdots + a_0(\xi) = 0, \quad \text{in } [-a, a].$$

Then h extends holomorphically to R as $u(\varsigma)/v(\varsigma)$, and $u/v \in C^\infty(R \cup (-a, a))$.

For real analytic CR mappings, we have the following result, which gives a justification for the definition of finite multiplicity.

THEOREM 4. *Let M, M' be two real analytic CR manifolds of finite type at p_0 and p'_0 respectively, M connected, and $H: M \to M'$ a real analytic CR mapping with $H(p_0) = p'_0$.*

(i) *If H is not of finite multiplicity at p_0, then H is constant, i.e. $H(M) = p'_0$.*

(ii) *If H is of multiplicity k at p_0 $(1 \leq k < \infty)$, then p'_0 is an interior point of $H(M)$. More precisely, for every U, a sufficiently small neighborhood of p_0 in M, there exists V, an open neighborhood of p'_0 in M' such that $V \subset H(U)$. In addition there is a finite number of real analytic curves $\gamma_1, \ldots, \gamma_r$ contained in V such that, for every $p' \in V \backslash (\bigcup_{j=1}^r \gamma_j \cup \{p'_0\})$, there exist exactly k points*

$p_1, \ldots, p_k \in U$ satisfying $H(p_j) = p'$, $1 \le j \le k$, with H of multiplicity 1 at each p_j.

Theorems 3 and 4 together with an argument involving the iteration of H, and the use of properties of real analytic sets, yield the following global result.

THEOREM 5. *Let M be a real analytic compact CR manifold and $H: M \to M$ a smooth CR mapping. If M is of finite type at each point and H of finite multiplicity at each point of M, then H is of multiplicity one at each point. Therefore H is a local analytic diffeomorphism.*

Several of the previous results have applications to holomorphic extendability of proper maps in domains in $\mathbf{C}^2$. Indeed it is well known that the boundary value of a holomorphic mapping is a CR mapping, and, when the boundaries are real analytic, the question of holomorphic extendability of a CR mapping reduces to that of its real analyticity. We give here some of these applications.

THEOREM 6. *Let D and D' be two open bounded sets of $\mathbf{C}^2$ with real analytic boundaries. Let $F: D \to D'$ be a proper holomorphic mapping with $F \in C^\infty(\overline{D})$. If F is nowhere flat on ∂D then F extends holomorphically to a neighborhood of $\overline{D}$. More precisely there exist D_1 and D_1', two open bounded neighborhoods of $\overline{D}$ and $\overline{D}'$ respectively, with real analytic boundaries, such that F extends as a holomorphic proper mapping from D_1 into D_1'.*

Using an argument due to Pinčuk [**10**] and Theorems 5 and 6 we obtain the following, which generalizes a result of Bedford and Bell [**3**].

THEOREM 7. *Let D be an open bounded set in $\mathbf{C}^2$ with real analytic boundary and F a proper holomorphic self-mapping of D. If $F \in C^\infty(\overline{D})$ and F is nowhere flat on ∂D then F extends as a biholomorphism from an open neighborhood of $\overline{D}$ onto another.*

In the pseudoconvex case, using a generalized form of an argument due to Fornaess [**7**] the nowhere flatness can be dropped in Theorem 6. Then making use of Theorem 6 and the result of Bell and Catlin [**4**], and Diederich and Fornaess [**6**], we obtain

THEOREM 8. *Let D and D' be two bounded pseudoconvex domains in $\mathbf{C}^2$ with real analytic boundaries. If F is a proper holomorphic map from D into D' then F extends as a proper holomorphic mapping from a neighborhood of $\overline{D}$ to a neighborhood of $\overline{D}'$.*

Complete proofs will be published elsewhere.

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