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MIXED HODGE STRUCTURES ON HOMOTOPY GROUPS

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In [**3**] Deligne defined mixed Hodge structures (M.H.S.'s) and showed that the cohomology of every algebraic variety over $\mathbf{C}$ has a natural M.H.S. Morgan [**12**], using Sullivan's minimal models, showed that the rational homotopy Lie algebra and rational homotopy type of every *smooth* variety have natural M.H.S.'s. In this note we announce an extension of mixed Hodge theory to arbitrary varieties and homotopy fibers of morphisms between varieties. The latter is a major step in extending asymptotic Hodge theory to homotopy groups and periods of iterated integrals. The bar construction and Kuo-Tsai Chen's iterated integrals [**1**] provide the link between Hodge theory and homotopy groups. Some of the results announced have been distributed in preprint form [**7**]. Proofs of the results stated will be published elsewhere.

Because the higher homotopy groups of a non-nilpotent topological space are inaccessible to rational homotopy theory, we make the following definition. The *homotopy Lie algebra* of a pointed topological space (X,x) is the graded Lie algebra $\mathfrak{g}_\bullet(X,x)$ where $\mathfrak{g}_0(X,x)$ is the Malcev Lie algebra associated with $\pi_1(X,x)$ and, when $k \geq 1$,

$$\mathfrak{g}_k(X,x) = \begin{cases} \pi_{k+1}(X,x)\otimes \mathbf{Q} & \text{if } (X,x) \text{ is nilpotent,} \\ 0 & \text{otherwise.} \end{cases}$$

The class of nilpotent spaces includes simply connected spaces and topological groups. There is a Hurewicz homomorphism

$$\mathfrak{g}_k(X,x) \to H_{k+1}(V,\mathbf{Q}).$$

THEOREM 1. *If (V,x) is a pointed algebraic variety, then the homotopy Lie algebra of (V,x) has a M.H.S. that is functorial with respect to morphisms of pointed varieties and such that*

(a) *the bracket is a morphism of M.H.S.'s.*

(b) *The Hurewicz homomorphism is a morphism of M.H.S.'s. Moreover, if (V,W,x) is a pair of simply connected varieties, then $\pi_\bullet(V,W,x)$ has a natural M.H.S. and the long exact sequence of the pair is a long exact sequence of M.H.S.'s.*

If (V,x) is simply connected, then the M.H.S. on $\pi_k(V,x)$ does not depend on the basepoint x. However, if V is not simply connected, this is not the case.

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Denote the integral group ring of a group G by $\mathbf{Z}G$. The augmentation ideal J is the kernel of the augmentation $\mathbf{Z}G \to \mathbf{Z}: g \to 1$. One can also show that if (V, x) is a pointed variety, then for each s, the truncated group ring

$$\mathbf{Z}\pi_1(V, x)/J^s$$

has a M.H.S.

The following theorem is an improvement of a result of the author by M. Pulte [**13**].

THEOREM 2. *Suppose that X and Y are smooth projective curves and that $x \in X$ and $y \in Y$. Suppose that*

$$\Phi: \mathbf{Z}\pi_1(X, x)/J^3 \to \mathbf{Z}\pi_1(Y, y)/J^3$$

is a ring isomorphism that commutes with the augmentations. If Φ induces an isomorphism of M.H.S.'s, then for all but at most two exceptional points x of X, there exists a biholomorphism $\phi: X \to Y$ such that $\phi(x) = y$.

Presumably these exceptional points do not exist. The previous result is related to the study of certain algebraic 1-cycles in $\mathrm{Jac}(X)$ via the work of Bruno Harris [**11**]. The details of this connection will appear in [**13**].

It is sometimes convenient to have a M.H.S. on the rational homotopy type of a variety.

THEOREM 3 (*Cf.* [**12**] *when V is smooth*). *If (V, x) is a pointed variety, then the rational Lie algebra model and rational minimal model of (V, x) have (not necessarily unique) M.H.S.'s.*

Using this result, one can find nontrivial restrictions on the rational homotopy types of projective varieties.

THEOREM 4. *There exists a simply connected finite CW-complex X whose integral cohomology ring is isomorphic to that of a simply connected projective variety, but X does not have the rational homotopy type of any projective variety.*

THEOREM 5. *If X is a complete (e.g. projective) variety whose rational cohomology ring satisfies Poincaré duality, then X is formal.*

There are many such varieties that are not smooth. The main result of [**4**] follows from Theorem 5. The following interesting corollary of Theorem 5 and a result from [**9**] was pointed out to me by S. Halperin.

COROLLARY 6. *Suppose that V is a simply connected complete variety. If $\pi_k(V)$ is torsion when k is sufficiently large, then V is formal.*

In the homotopy category, every continuous map $f: X \to Y$ can be canonically replaced by a fibration $\pi: E_f \to Y$ where E_f is homotopy equivalent to X. The fiber $\pi^{-1}(y)$ is called the *homotopy fiber* of f over y and will be denoted by $E_f(y)$. If Y is path connected, then all homotopy fibers are homotopy equivalent. For example, if Y is an Eilenberg-Mac Lane space $K(\pi, 1)$, then the homotopy fiber of f is the total space of the covering of X determined by the kernel of $\pi_1(X) \to \pi_1(Y)$.

Set $X_y = f^{-1}(y)$. There is a natural inclusion $X_y \to E_f(y)$, which is a weak equivalence if f is a fibration.

THEOREM 7. *Suppose that $f: X \to Y$ is a morphism of algebraic varieties. If $E_f(y)$ is path connected and $\pi_1(Y, y)$ acts unipotently on $H^{\bullet}(E_f(y); \mathbf{Q})$, then the cohomology and the rational homotopy Lie algebra (and types) of $E_f(y)$ have natural M.H.S.'s. Furthermore, the restriction map*

$$H^{\bullet}(E_f(y)) \to H^{\bullet}(X_y)$$

and the monodromy representation

$$H^{\bullet}(E_f(y)) \otimes \mathfrak{g}_0(Y, y) \to H^{\bullet}(E_f(y))$$

are morphisms of M.H.S.'s. If X and Y are simply connected, then the long exact sequence of homotopy groups associated with the fibration $E_f(y) \to X \to Y$ is a sequence of M.H.S.'s.

This theorem is true for spaces more general than algebraic varieties. Loosely speaking, X and Y may be replaced by any topological spaces whose cohomology and homotopy have natural M.H.S.'s. In [**5**] it is shown that the cohomology and homotopy of a deleted tubular neighbourhood of a subvariety of a smooth variety have natural M.H.S.'s.

If $f: X \to \Delta$ is a local degeneration of projective varieties, then after a base change if necessary, we may assume that the monodromy representation is unipotent. Theorem 7, combined with Steenbrink's work [**14**], then yields the next result.

THEOREM 8. *If $f: X \to \Delta$ is a degeneration of projective varieties with unipotent monodromy, then the M.H.S. on the homotopy fiber of $X^* \to \Delta^*$ is the limit M.H.S. on the cohomology of the generic fiber of f. Moreover, if $\sigma: \Delta \to X$ is a section, then there is a limit M.H.S. on the homotopy Lie algebra of $(X_t, \sigma(t))$ when $t \neq 0$ such that the natural map*

$$\mathfrak{g}_{\bullet}(X_0, \sigma(0)) \to \mathfrak{g}_{\bullet}(X_t, \sigma(t))_{\mathrm{lim}}$$

is a morphism of M.H.S.'s.

One can show that the Hodge filtration on the homotopy Lie algebra varies holomorphically with (X, x) and satisfies Griffiths transversality. Also, in joint work with Zucker [**8**], we show that the variation of M.H.S. over a variety V whose fiber over x is $\mathfrak{g}_0(V, x)$ classifies all variations of M.H.S. over V with unipotent monodromy, a result conjectured by Deligne.

9. Sketch of proofs. The first step is to show that if $A^{\bullet}$ is a mixed Hodge complex (M.H.C.) that is also a d.g.a. (a multiplicative M.H.C.) and if $M^{\bullet}$, $N^{\bullet}$ are M.H.C.'s that are right and left $A^{\bullet}$ modules respectively, then Chen's reduced bar construction $\overline{B}(M^{\bullet}, A^{\bullet}, N^{\bullet})$ [**2**] is also a M.H.C. For Theorem 1, we need to show that if $A^{\bullet}$ is a multiplicative M.H.C. that is augmented, then the usual bar construction $B(A^{\bullet})$ and its indecomposables $QB(A^{\bullet})$ are also M.H.C.'s.

The second step is to construct for each algebraic variety V a commutative multiplicative M.H.C. whose rational part has the same homotopy type as the Thom-Sullivan rational de Rham complex of V. This is done by taking $A^{\bullet}$ to be a collection of compatible forms on the geometric realization of a hypercovering of V.

Theorems 1 and 7 now follow from results in [**1, 6, 10**] and standard results from algebraic topology.

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