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RESEARCH ANNOUNCEMENTS

AN INVARIANT SUBSPACE THEOREM[1]

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A fundamental problem in the study of operators on Hilbert space is to determine which operators have nontrivial invariant subspaces. If H is a complex Hilbert space then a compact subset K of $\mathbf{C}$ is a *spectral set* for T if K contains the spectrum of T, $\sigma(T)$, and

$$\|f(T)\| \leqslant \max\{ |f(z)| : z \in \sigma(T)\}$$

for every rational function f with poles off K. In this note it is shown that any operator for which the spectrum is a spectral set has a nontrivial invariant subspace.

In [6] von Neumann introduced the notion of spectral set and showed that if T has $\|T\| = 1$ then the closed unit disc, $\mathbf{D}^-$, is a spectral set for T. For this reason any operator whose spectrum is a spectral set is called a *von Neumann* operator. Hence if $\|T\| = 1$ and $\sigma(T) = \mathbf{D}^-$ then T is a von Neumann operator. If T or T^* is a subnormal operator then T is a von Neumann operator. Thus the result of this note generalizes the recent result of Scott Brown [1] that every subnormal operator has an invariant subspace, although the proof relies heavily on his techniques. We wish here to thank him for an early manuscript containing his results.

THEOREM. *Any von Neumann operator has a nontrivial invariant subspace.*

The rest of this note is a much compressed proof of this fact. All notation used is as in [3]. R_k denotes the ultraweak (= weak star) closure of the rational functions in T with poles off K.

Let T be a von Neumann operator on complex separable Hilbert space and

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assume:

(R) T has no reducing subspaces

(AP) $\sigma_p(T) = \sigma_c(T) = \square$.

DEFINITION. A compact set $K \subset \mathbf{C}$ is D-spectral for T if K is a spectral set and $R(K)$ is a Dirichlet algebra on ∂K.

LEMMA 1. *If K is a D-spectral for T then the natural contraction Φ_k: $R(K) \longrightarrow \mathcal{R}_K$ extends to a norm contractive algebra homomorphism Φ_K: $H^\infty(\partial K) \longrightarrow \mathcal{R}_K$ which is continuous when domain and range are equipped with their weak* topologies.*

If K is D-spectral and K^0 has one component then K^0 is simply connected. Let ϕ_K denote the conformal map from K^0 onto $\mathbf{D}$. By Lemma 4.3 in [5], $\phi_K \in H^\infty(\partial K)$. The operator $\Phi_K(\phi_K)$ is central to our efforts.

LEMMA 2. *If K is D-spectral and K^0 has one component then,* $\sigma(\Phi_K(\phi_K)) \cap \mathbf{D} = \phi_K(\sigma(T) \cap K)$.

This is an easy consequence of (AP).

LEMMA 3. *There exists a compact $K \subset \mathbf{C}$ with the properties*: (P1) *K is D-spectral for T*; (P2) *K^0 has one component*; (P3) $\partial\mathbf{D} \subset \sigma(\Phi_K(\phi_K))$.

SKETCH OF PROOF. Define for each countable ordinal α a compact set as follows:

(1) K_1 = the polynomially convex hull of $\sigma(T)$.

(ii) If α is a limit ordinal then $K_\alpha = \bigcap_{\beta<\alpha} K_\beta$.

(iii) $K_{\alpha+1} = K_\alpha \backslash [\phi_\alpha^{-1}(V_\alpha) \cup U_\alpha]$ where U_α is the union of the components of K_α^0 that miss $\sigma(T)$ and V_α is defined in terms of K_α as follows. $R(K_\alpha)$ is by transfinite induction, a Dirichlet algebra. It follows that if $L_\alpha = K_\alpha \backslash U_\alpha$ then L_α is D-spectral for T. Condition (R) implies, via Theorem 2.4 in [4], that L_α^0 has one component. Let $\phi_\alpha(T) = \Phi_{L_\alpha}(\phi_{L_\alpha})$ and $\phi_\alpha = \phi_{L_\alpha}$.

The V_α in (iii) above is the union of the components, U, of $\mathbf{D}\backslash\phi_\alpha(\sigma(T) \cap L_\alpha^0)$ with the property that $\partial U \cap \partial\mathbf{D}$ contains a nontrivial arc, I, with the property that $I \cap \sigma(\phi_\alpha(T)) = \square$. The K of Lemma 3 may now be obtained by setting $K = \bigcap K_\alpha$. The proof that $\mathcal{R}(K_\alpha)$ is a Dirichlet algebra and K has the desired properties follows by a modification of [5].

LEMMA 4. *For the K of Lemma* 3,

$$\|h\|_{H^\infty(K)} = \sup\{|h(z)|: z \in K^0 \cap \sigma(T)\}$$

all $h \in H^\infty(\partial K)$.

SKETCH OF PROOF. Otherwise, by a geometric construction with $\sigma(\Phi_K(\phi_K))$ (see Lemma 3.1 in [1]),

(i) $\sigma(\Phi_K(\phi_K)$ contains an exposed arc or

(ii) by deleting an open "sliver" from K one obtains J, admissible for T, and with the property that $R(J)$ has two nontrivial Gleason parts meeting $\sigma(T)$.

If (i) occurs then one can obtain a growth condition on the resolvent of $\Phi_K(\phi_K)$ near the exposed arc that contradicts (P3) of Lemma 3. (ii) cannot occur by Theorem 2.4 in [4].

LEMMA 5. *For the K of Lemma 3, Φ_K is an isometric weak $*$ homeomorphism onto $\mathcal{R}_K$.*

Lemmas 4 and 5 imply that $A = \Phi_K(\phi_K)$ satisfies the hypotheses of Theorem 4.1 in [2] and hence A has a nontrivial invariant subspace. Choose a sequence of polynomials such that $p_n \longrightarrow \phi_K^{-1} w^*$ in $H^\infty(\partial D)$. Then $p_n(A) \longrightarrow T w^*$ in $L(H)$ so that T has a nontrivial invariant subspace.

REFERENCES

1. S. Brown, *Some invariant subspaces for subnormal operators* (preprint).

2. S. Brown, B. Chevreau and C. Pearcy, *An invariant subspace theorem* (preprint).

3. J. B. Conway and R. F. Olin, *A functional calculus for subnormal operators*. II, Mem. Amer. Math. Soc. No. 184 (1977).

4. R. G. Lautzenheiser, *Spectral sets, reducing subspaces, and function algebras*, Thesis, Indiana University, 1973.

5. D. E. Sarason, *Weak-star density of polynomials*, J. Reine Angew. Math. **252** (1972), 1–15.

6. J. von Neumann, *Eine spektral theorie für all-gemeine operatoren eines unitären Raumes*, Math. Nachr. **4** (1951), 258–281.

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