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THE ASYMPTOTIC JOINT DISTRIBUTION OF WINDINGS OF PLANAR BROWNIAN MOTION

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Let $Z = (Z(t), t \geq 0)$ be a complex valued Brownian motion starting at z_0, and consider the real valued winding processes $\theta_1, \theta_2, \ldots, \theta_n$, where $\theta_j(t)$ is the continuous total angle wound by Z about z_j up to time t, and z_0, $z_1, \ldots, z_n$ are $n+1$ distinct points in the complex plane $\mathbf{C}$. Let $C_1, \ldots, C_n$, B, B^∞ be $n+2$ mutually independent one-dimensional processes, where the first n are standard Cauchy processes and the last 2 are standard Brownian motions.

Let

$$\begin{aligned} T &= \inf\{t\colon |B_t| = 1\}, \\ U &= \text{local time at 0 of } B \text{ up to time } T, \\ W_j &= C_j(U) + B^\infty(T), \qquad j = 1, \ldots, n. \end{aligned}$$

THEOREM 1. *As $t \to \infty$, the n-tuple $(2\theta_j(t)/\log t,\ 1 \leq j \leq n)$ converges in distribution to $(W_j,\ 1 \leq j \leq n)$.*

The joint Fourier transform of the distribution of $(W_1, \ldots, W_n)$ is readily computed to be

$$E \exp\left(i \sum_j \lambda_j W_j\right) = \left[\cosh \sigma + \left(\sum_j |\lambda_j|\right) \frac{\sinh \sigma}{\sigma}\right]^{-1} \tag{1}$$

where $\sigma = \sum_j \lambda_j$. Thus for each j the marginal distribution of W_j is Cauchy with parameter 1, and we recover from Theorem 1 the result of Spitzer **[S]** for windings about a single point. Theorem 1 is an immediate corollary of Theorem 2 below, which is an extension of Theorem 7 of Messulam and Yor **[MY]**. To state this result, fix radii $0 < r_j < \infty$, $j = 1, \ldots, n$, let D_j^0 be the open disc centered at z_j with radius r_j, and let D_j^∞ be the complement of D_j^0 in $\mathbf{C}$. For $^* = 0$ or ∞, define

$$\theta_j^*(t) = \int_0^t d\theta_j(s)\, 1(Z_s \in D_j^*).$$

THEOREM 2. *As $t \to \infty$, the n-tuple of pairs $(2\theta_j^0(t)/\log t, 2\theta_j^\infty(t)/\log t; 1 \leq j \leq n)$ converges in distribution to $(C_j(U), B^\infty(T); 1 \leq j \leq n)$.*

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A key step in the proof of Theorem 2 is the use of the pinching method of D. Williams [**W**] to show that the asymptotic behavior of

(2) $$2\theta_j^*(t)/\log t \quad \text{as } t \to \infty$$

is identical to that of

(3) $$\theta_j^*(T_R)/\log R \quad \text{as } R \to \infty,$$

where $T_R = \inf\{t\colon |Z_t| = R\}$.

The identical limiting behavior of (2) and (3) should be contrasted with the *different* limiting behavior described by Lyons and McKean [**LM**] for

(4) $$\theta_j^*(\tau_u)/u \quad \text{as } u \to \infty,$$

where τ_u is the inverse of an additive functional A_t with finite representing measure α. Let $C^\infty(\frac{1}{2})$ be a Cauchy random variable with parameter $\frac{1}{2}$ which is independent of the independent and identically distributed random variables $C_1(\frac{1}{2}), \dots, C_n(\frac{1}{2})$. The following theorem is the extension to windings about n points of the result given by Lyons and McKean [**LM**] for the asymptotic behavior of (4) in the case $n = 2$.

THEOREM 3. *As $u \to \infty$, the n-tuple of pairs $((\alpha(\mathbf{C})/4\pi u)[\theta_j^0(\tau_u), \theta_j^\infty(\tau_u)];$ $1 \le j \le n)$ converges in distribution to $([C_j(\frac{1}{2}), C^\infty(\frac{1}{2})]; 1 \le j \le n)$.*

In particular, comparison of the Fourier transform (1) with the corresponding transform for $(C_j(\frac{1}{2}) + C^\infty(\frac{1}{2}), 1 \le j \le n)$ reveals the striking result that while for each j the marginal limit distribution for $2\theta_j(t)/\log t$ is the same Cauchy distribution as for $\alpha(\mathbf{C})\theta_j(\tau_u)/4\pi u$, the *joint* limiting distributions of these quantities are *different.*

An important ingredient in our proof of these results is the independence of various radial and angular parts of the Brownian motion as described in Theorem 4 below. We derive this theorem using Tanaka's formula and the theorem of Knight [**K**] representing orthogonal continuous martingales in terms of independent Brownian motions.

For each $j = 1, \dots, n$, $^* = 0$ or ∞, introduce the additive functional

$$U_j^*(t) = \int_0^t 1(Z_s \in D_j^*)|Z_s - z_j|^{-2}\, ds$$

and the clock

$$t_j^*(u) = \inf\{t\colon U_j^*(t) = u\},$$

and define time changed processes

$$\beta_j^*(u) = \theta_j^*(t_j^*(u)), \qquad \rho_j^*(u) = \log|Z(t_j^*(u)) - z_j|.$$

THEOREM 4. (i) *For each $j = 1, \dots, n$, $^* = 0$ or ∞, β_j^* is a Brownian motion, ρ_j^* is a Brownian motion with reflection at* $\log r_j$, *and these two processes are independent.*

(ii) *If the domains $D_1^0, \dots, D_{n-1}^0, D_n^\infty$ are disjoint, then the $2n$ processes $\beta_1^0, \rho_1^0, \dots, \beta_{n-1}^0, \rho_{n-1}^0, \beta_n^\infty, \rho_n^\infty$ are mutually independent.*

For the proof of Theorems 2 and 3, it is an important observation that the origins z_j for the angles and the radii r_j separating θ_j^∞ and θ_j^0 play no rôle in the limit. We prove this using estimates relating martingales to their increasing processes (see, e.g., Burkholder [**B**]). This allows reduction to the case when the domains $D_1^0, \dots, D_{n-1}^0, D_n^\infty$ are disjoint. In this case we analyse the angles indexed by local times on the n circles, making use of the independence results of Theorem 4 and the ratio ergodic theorem for additive functionals (see, e.g., Ito and McKean [**IM**, §17]).

The Cauchy processes arise from the fact that if τ_{ju}^* denotes the inverse local time process on the boundary circle of D_j^*, then $(\theta_j^*(\tau_{ju}), u \geq 0)$ is a Cauchy process.

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