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FROBENIUS RECIPROCITY OF DIFFERENTIABLE REPRESENTATIONS

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ABSTRACT. In this note we give the construction of the adjoint and the coadjoint of the restriction functor in the category of differentiable G-modules, where G is a Lie group.

1. **Introduction.** Let G be a Lie group, countable at infinity. A continuous representation λ of G in a complete locally convex space E is *differentiable* if for each $a \in E$ the map $\hat{a}: x \to \lambda(x)a$ of G into E is C^∞, and if the injection $a \to \hat{a}$ of E into $C^\infty(G, E)$ is a topological homeomorphism [**8**]. We then say that E is a differentiable G-module.

There is a natural way of associating a differentiable representation to any continuous, in particular unitary, representation of G. In fact, let ρ be a continuous representation of G on a complete locally convex space F. Let $F_\infty = \{a \in F : \hat{a} \in C^\infty(G, F)\}$. Then F_∞ is a dense ρ-invariant linear subspace of F. The injection $a \to \hat{a}$ sends F_∞ onto a closed subspace of $C^\infty(G, F)$. When F_∞ is equipped with the relative topology of $C^\infty(G, F)$ it becomes a complete locally convex space, and the corresponding subrepresentation λ_∞ of λ on F_∞ is differentiable. If λ is topologically irreducible then λ_∞ is topologically irreducible and conversely. For details and other basic facts concerning differentiable representations see [**8**].

The purpose of the present note is to show that the Frobenius reciprocity theorem is valid in the category of differentiable G-modules. The history of the Frobenius reciprocity theorem is long and interesting. For some recent developments the reader is referred to the work of Bruhat [**1**], Moore [**4**], Rieffel [**5**] and Rigelhof [**6**]. In particular Rigelhof succeeded in constructing an adjoint and a coadjoint for the restriction functor in the category of continuous (locally convex) G-modules.

2. **Construction of the adjoint and the coadjoint.** Let K be a closed subgroup of G, and let F be a differentiable G-module. The restriction $F \to F_K$ is a functor from the category of differentiable G-modules to the category of differentiable K-modules.

Let E be a differentiable K-module and let π be the corresponding representation.

(1) *Coadjoint functor.* Let $\mathscr{E}' = \mathscr{E}'(G)$ denote the space of distributions

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with compact support on G, equipped with the strong topology as the dual of $C^\infty(G)$.

Let E^G denote the space of all continuous K-linear maps of $\mathscr{E}'$ into E, i.e., $E^G = \mathrm{Hom}_K(\mathscr{E}', E)$ and $m \in E^G$ iff

$$m(kS) = km(S); \qquad S \in \mathscr{E}',\ k \in K. \tag{2.1}$$

Here $(kS)(f) = S(fk)$, where $(fk)(x) = f(kx)$, $f \in C^\infty(G)$, $x \in G$. E^G is given the topology of uniform convergence on compact sets and is a complete locally convex space. We define the induced representation π^G of G on E^G by

$$[\pi^G(x)m](S) = m(Sx). \tag{2.2}$$

PROPOSITION 1. *π^G is a differentiable representation of G on E^G.*

(2) *Adjoint functor*. A bilinear map $\omega: \mathscr{E}' \times E \to H$, H a locally convex space is K-balanced if $\omega(Sk, a) = \omega(S, ka)$ for all $S \in \mathscr{E}'$, $k \in K$, $a \in E$. (We let K act to the right on $\mathscr{E}'$ here and write ka for $\pi(k)a$.) Let $B_K(\mathscr{E}', E)$ denote the space of all K-balanced bilinear maps of $\mathscr{E}' \times E$ into C. Let $\chi: \mathscr{E}' \times E \to B_K(\mathscr{E}', E)^*$ be the canonical map:

$$\chi(S, a)b = b(S, a); \qquad b \in B_K(\mathscr{E}', E).$$

χ is K-balanced and bilinear. We let $\mathscr{E}' \otimes_K E$ denote the linear span of the range of χ. Typical elements of $\mathscr{E}' \otimes_K E$ will be written $\sum_{i=1}^n S_i \otimes a_i$. We give $\mathscr{E}' \otimes_K E$ the inductive tensor product topology with respect to the family of bounded subsets of $\mathscr{E}'$ and E [**2**]. Let ${}^GE$ be the completion of $\mathscr{E}' \otimes_K E$ with respect to this topology. We define the representation ${}^G\pi$ of G on ${}^GE$ by

$${}^G\pi(x)(S \otimes a) = xS \otimes a. \tag{2.3}$$

This construction is similar to the one given in [**5**] and [**6**].

PROPOSITION 2. *${}^G\pi$ is a differentiable representation of G on ${}^GE$.*

3. **Main result.** Preserve the notation and assumptions above.

THEOREM 1. *$E \to {}^GE$ is the adjoint and $E \to E^G$ is the coadjoint functor of the restriction functor $F \to F_K$, i.e., there are natural isomorphisms (in the sense of category theory)*:

$$\mathrm{Hom}_G({}^GE, F) \cong \mathrm{Hom}_K(E, F_K); \tag{3.1}$$

$$\mathrm{Hom}_G(F, E^G) \cong \mathrm{Hom}_K(F_K, E). \tag{3.2}$$

Moreover, the adjoint and coadjoint are unique to within equivalence of differentiable G-modules.

REMARK. The construction of the isomorphism in (3.1) rests upon the preliminary result that $\omega: (S, a) \to \lambda(S)a$ is a hypocontinuous bilinear map of $\mathscr{E}' \times F$ into F. Here $\lambda(S)$ denotes the distribution form of the representation λ. ω is also K-balanced, and therefore any $A \in \mathrm{Hom}_K(E, F_k)$ determines a continuous linear map $A': \mathscr{E}' \otimes_K E \to F$ such that

$$A' \sum S_i \otimes a_i = \sum \lambda(S_i)Aa_i = \sum \omega(S_i, Aa_i).$$

The map $A \to A'$ defines (3.1).

For (3.2) let $A \in \mathrm{Hom}_K(F_K, E)$, and let $a \in F$. Define $(A'a)(S) = A\lambda(S)a$. Then $A'a$ belongs to $\mathrm{Hom}_K(\mathscr{E}', E) = E^G$ and $A': a \to A'a$ belongs to $\mathrm{Hom}_G(F, E^G)$. The map $A \to A'$ defines (3.2).

Finally, both (3.1) and (3.2) are topological isomorphisms (with respect to standard topologies).

4. **Realizations.** In this section we give alternative descriptions of the G-modules E^G and ${}^GE$. As may be expected E^G may be realized as a space of E-valued C^∞-functions. Let $C_K^\infty(G, E)$ denote the space of C^∞-functions $f: G \to E$ satisfying

(4.1) $$f(kx) = \pi(k)f(x); \qquad k \in K, x \in G.$$

We give $C_K^\infty(G, E)$ the relative topology from $C^\infty(G, E)$, and let G act as the right regular representation on $C_K^\infty(G, E)$. This makes $C_K^\infty(G, E)$ into a differentiable G-module, and we have

PROPOSITION 3. *The differentiable G-modules E^G and $C_K^\infty(G, E)$ are equivalent.*

The proof is more or less straightforward, based on the isomorphisms $\mathrm{Hom}(\mathscr{E}', E) \cong C^\infty(G) \hat{\otimes} E \cong C^\infty(G, E)$ ($C^\infty(G)$ is a reflexive nuclear space; see [7].)

It does not appear possible to realize ${}^GE$ as a space of functions. There is, however, another representation of ${}^GE$, in a particular case, which throws some light on the connection between ${}^GE$ and E^G.

Let $C_c^\infty(G)$ denote the space of complex-valued C^∞-functions on G with compact support, equipped with the usual topology [7], [**8**]. Let $\mathrm{Hom}_K^0(C_c^\infty(G), E)$ be the space of all continuous linear maps $m: C_c^\infty(G) \to E$ which satisfy

(4.2) $$\operatorname{supp} m \subseteq CK \quad \text{for some compact subset } C \text{ of } G;$$

(4.3) $$m(k\varphi) = \delta(k)^{-1}km(\varphi) \quad \text{where } k \in K,\ \varphi \in C_c^\infty(G), \text{ and}$$
$$\delta \text{ is the modular function of } K.$$

We topologize $\mathrm{Hom}_K^0(C_c^\infty(G), E)$ as follows. For each compact set C, let $\mathrm{Hom}_K^C(C_c^\infty(G), E)$ be the subspace of those m's that have their support in CK. We give this space the relative topology as a subspace of $\mathrm{Hom}(C_c^\infty(G), E)$. $\mathrm{Hom}_K^0(C_c^\infty(G), E)$ is then given the inductive limit topology from the family of spaces $\mathrm{Hom}_K^C(C_c^\infty(G), E)$ as C runs through the collection of compact subsets of G. We make $\mathrm{Hom}_K^0(C_c^\infty(G), E)$ into a differentiable G-module by the action

$$(xm)\varphi = m(\varphi x); \qquad \varphi \in C_c^\infty(G),\ x \in G. \tag{4.4}$$

PROPOSITION 4. *If E is the dual of a reflexive Fréchet space then the differentiable G-modules ${}^G E$ and $\mathrm{Hom}_K^0(C_c^\infty(G), E)$ are equivalent.*

The proof of this result rests upon another proposition, stated below. Let $\mathrm{Hom}^0(C_c^\infty(G), E)$ be the space of all continuous linear maps m: $C_c^\infty(G) \to E$ with compact support. We equip this space with the natural inductive topology. Let K act to the right on the space by

$$(mk)\varphi = \delta(k)k^{-1}m(k\varphi). \tag{4.5}$$

Then define

$$m\#\varphi = \int_K (mk)\varphi \, dk; \qquad k \in K,\ \varphi \in C_c^\infty(G). \tag{4.6}$$

PROPOSITION 5. *The map $m \to m^\#$ is linear, continuous, and open of* $\mathrm{Hom}^0(C_c^\infty(G), E)$ *onto* $\mathrm{Hom}_K^0(C_c^\infty(G), E)$.

REMARK. This result is true when E is a complete locally convex space.

5. **Concluding remarks.** Bruhat [1] has given another definition of differentiably induced representations, and has given a version of the Frobenius reciprocity theorem in terms of intertwining forms. Full proofs of the results above, a discussion of the relationship to Bruhat's work, and other results (inducing in stages, etc.) will be given elsewhere.

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