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Variational principles for nonpotential operators, by V. M. Filippov. Transl. Math. Monographs, vol. 77, Amer. Math. Soc. Providence, RI, 1989, 239 pp., $99.00. ISBN 0-8218-4529-2

Variational methods reduce the problem of solving linear or nonlinear operator equations to an equivalent problem of determining the stationary values of a suitable functional. If a functional can be found such that its derivative, in some sense, is the operator whose equation is to be solved, then the points at which the functional achieves its extreme values coincide with the solutions of the given operator equation. The usefulness of variational methods is based on the fact that in general it is easier to obtain existence and approximation results for the stationary values of a functional than for the solution of operator equations.

Following the publication of Hilbert's fundamental paper *Über das Dirichletsche Prinzip* [5] and the work of Friedrichs [3], Levi [6], Zaremba [12], and others, the development of variational principles and methods for partial differential and abstract equations became widespread and received theoretical justification in mathematical literature. Today there are a number of excellent books and monographs devoted to variational methods for solving abstract and differential equations (i) $A(x) = f$ in suitable Hilbert spaces when A is a densely defined symmetric operator if (i) is a linear equation, or A is a **potential operator** (i.e. its Gateaux derivative is symmetric) if (i) is a nonlinear equation. However, there does not appear to be a single book devoted exclusively to the variational principle concerning the solvability of equations (i) when A is nonsymmetric (if linear) or nonpotential (if nonlinear), although there is a large number of research papers dealing with variational approach to specific equations, scattered over various mathematics and physics journals and university reports, some of which are not easily (if at all) accessible. The aim of Fillipov's book is to fill this gap and, as will be indicated below, he has accomplished this useful and important task very well. In fact, his book not only includes a very comprehensive survey of the above results, but it is essentially devoted to the study of a general and unified variational method in the solvability of abstract and differential equations (i), involving a general class of linear B-symmetric and B-positive operators A (i.e. A is symmetrizable by means of some auxiliary operator B) and the corresponding class of nonpotential nonlinear operators whose Gateaux derivative A'_x (with $A'_x = A$ if A is linear) is B-symmetric and B-positive. The study of approximate solvability of such a class of linear differential equations was initiated by Kravchuk and Krylov in the 1920s and the study of specific symmetrizable differential equations and linear transformations was later continued by many authors, including Friedrichs, Kharazov, Lax, Lions, Marchuk, Silberstein, Vladimorov, and others (see the monograph for exhaustive bibliography in this field). The systematic study of the solvability of equations involving linear B-symmetric and B-positive operators was carried out by Martyniuk, Petryshyn, and Shalov in the 1960s and, later on, the nonlinear case by Petryshyn [7], whose theory of Friedrich's type solvable extensions of densely defined nonlinear operators allowed the extension of the linear theory of B-positive definite operators to the nonlinear case. The approximate solvability of nonlinear equations involving nonpotential operators A, whose Gateaux derivative A'_x is B-symmetric and B-positive for some suitable linear operator B, was investigated by Lyashko, Nashed, Petryshyn, and others. In addition to Shalov [9], important applications of the variational principle to the solvability of linear PDE's of elliptic, hyperbolic, and parabolic type which are B-symmetric and B-positive for a suitable operator B were obtained in a series of papers in the 1970s by Filippov, Filippov-Skorokhodov, Didenko, and Tonti.

Before giving a more precise outline of the results contained in this monograph, it is important to point out that besides unifying many methods, the monograph contains also a number of new results, particularly the application of a general variational method developed in this book to linear PDE's of elliptic, hyperbolic, and parabolic type for which the classical variational principle is not applicable.

Chapter I is devoted to the theoretical study of the variational principles in the solvability of linear equations

$$(1) \qquad Au = f,$$

involving linear B-symmetric (B-sym.) and B-positive (B-p.) operators defined on some dense linear manifold $D(A) \subset H$, where H is a real Hilbert space with inner product $(\cdot, \cdot)$ and norm $\|\cdot\| = (\cdot, \cdot)^{1/2}$. We recall that $A : D(A) \to H$ is said to be B-sym. if there exists a linear operator B with $D(B) \supseteq D(A)$ and $\overline{B(D(A))} = H$ such that

$$(2) \qquad (Au, Bv) = (Bu, Av) \quad \forall u, v \in D(A);$$

A is said to be B-p. if

$$(3) \qquad (Au, Bu) \geq c\|u\|^2 \quad \forall u \in D(A)$$

and some constant $c > 0$. Among other results, the author unified the results of Shalov [10] and Petryshyn [8] concerning the Friedrichs type extension (see [4]) of the operator A in the case when A is, in addition, weakly closable w.r.t. B, i.e.

$$(4) \qquad u_n \to 0 \; in \; H \Longrightarrow (Au_n, Bv) \to 0 \; \forall v \in D(A)$$

or A is also B-positive definite (B-p.d.); i.e.

$$(5) \qquad (Au, Bu) \geq \beta\|Bu\|^2 \quad \forall u \in D(A)$$

and some constant $\beta > 0$. It is shown that $|(f, Bu)| \leq c_f(Au, Bu) \; \forall u \in D(A)$ iff A is B-p.d. Moreover, if A is B-p.d. and B-sym., then A has a unique Friedrichs' extension $A_0 \supseteq A$ which is continuously invertible and $D(A_0)$ consists of all elements in H_{AB} realizing the minimum of the functional

$$(6) \qquad D_f(u) = \|u\|^2_{AB} - 2(f, B_0u),$$

where B_0 is the extension of B to H_{AB}, f ranges through all of H, and H_{AB} is the completion of $D(A)$ in the metric $[u, v] = (Au, Bv)$, $\|u\|_{AB} = [u, u]^{1/2} \; \forall u, v \in D(A)$. Finally, the relation between the critical points of $D_f(u)$ and the weak solutions of (1) is investigated and a dual variational principle for (1) is constructed, which contains the results of Slobodyanskii [11] and which, among other things, could be used to provide an a posteriori estimate for the error $\|u_n - u_0\|$, where u_0 is a weak solution of (1) and u_n is its nth order approximation obtained by some (say, Galerkin) method. In the Comments to Chapter I the author provides an excellent survey of related recent results obtained by various researchers.

In Chapter II the author considers various classes of functionals that are not of the classical Euler-Lagrange type and studies the corresponding functional spaces which are determined by these new functionals, and which are more general than the Sobolev spaces.

In 1960 it was shown by Belatoni [1] that if the quasilinear PDE

$$(8)\quad Lu = au_{xx} + 2bu_{xy} + cu_{yy} = G(x, u, u_x, u_y), \quad (x, y) \in Q$$

(where a, b, c, and G are smooth functions on compact $\overline{Q} \subset R^2$) is of elliptic or hyperbolic type in Q, then there exists a "variational factor" $\mu = \mu(x, y, u, u_x, u_y) \neq 0$, $u \in \mathring{C}^2(Q)$, and a functional $F_0(u)$ in the class of Euler functionals

$$(9)\qquad F(u) = \int_Q F(x, y, u, u_x, u_y)dxdy$$

such that the condition $\delta F_0(u) = 0$ yields (8) or its equivalent

$$(8')\qquad \mu(Lu - G) = 0.$$

If the quasilinear equation (8) is of parabolic type on Q, then one *cannot* find an integration factor $\mu(x, y, u, u_x, u_y)$ and a functional F of type (9) so that the vanishing of its first variation would yield (8) or (8′). Earlier, a similar result was proved by Copson [2] for a more general class of linear second order PDE's and systems of PDE's. It is shown by Filippov that this negative aspect could be overcome if one uses the theory of B-sym. and B-p. operators studied in Chapter I. Indeed, if we want to solve the inverse problem of the calculus of variation for the (1), $Au = f$, where $A : D(A) \subset H \to H$ is B-sym. *and* B-p., then the functional

$$(10)\qquad D[u] = (Au, Bu) - 2(f, Bu), \qquad u \in D(A),$$

is such that

$$(11)\qquad \delta D[\overline{u}] = 0 \Leftrightarrow A\overline{u} = f, \qquad \overline{u} \in D(A).$$

It is shown that if, for example, (1) is a linear PDE of order 2, then the functional (10) constructed for concrete equations of elliptic, hyperbolic and parabolic type will have the following form for the case $n = 2$:

$$\begin{aligned}\Phi(u) = \int_Q \hat{\Phi}\Big[x, y, u(x, y), u_x, u_y, \int_a^x u(\xi, y)d\xi,\\ \int_b^y u(x, \eta)d\eta, \int_a^x d\xi \int_b^y u(\xi, \eta)d\eta,\\ \int_a^x u_y(\xi, y)d\xi, \int_b^y u_x(x, \eta)d\eta\Big] dxdy\end{aligned} \tag{12}$$

If $\hat{\Phi} = \hat{\Phi}(x, y, u, u_x, u_y)$, then such Euler functionals generate norms and functional Sobolev spaces. If, for example, $\hat{\Phi} = \hat{\Phi}(x, y, \int_a^x ud\xi, \int_b^y ud\eta)$, then the functionals (12) generate norms and functional spaces which are more general than Sobolev spaces and which require further study. This is particularly the case if $D[u]$ is of the form

$$D[u] = \int_Q F(x, u(x), L_1 u, \ldots, L_N u)dx \tag{13}$$

defined on some dense linear manifold in $L_2(Q)$, where $L_1, \ldots, L_N$ are suitable linear operators. The details of the study of these functionals, the corresponding functional spaces, and their relation to PDE's are considered by the author. The work of Nikolskii, Besov and Lasorkin appears to be particularly useful in this study.

Chapter III contains a survey of various approaches to the solvability of the inverse problems of variational calculus, which do not admit classical solutions. Sections 12–16 are devoted to the construction of "quasilinear" solutions of the inverse problems of the variational calculus for various classes of linear PDE's for which the classical variational principle fails, since the equations are not symmetric. Thus, given a linear PDE (1), $Au = f$, in which the operator A is "nonvariational" of elliptic, hyperbolic, or parabolic type, the author constructs a suitable functional space and the auxiliary operator B (in explicit form—this is the most difficult part of the problem in nontrivial applications) such that A becomes B-sym. and B-p. or B-p.d. This allows the abstract theory developed in Chapter I to be applicable to (1). As an example of the above approach, consider the following

Problem 1. Construct the quasiclassical solution of the inverse problem of variational calculus for the following boundary value problem for parabolic equation with the condition of periodicity in the time variable:

$$c(x, t)u_t - (k(x, t)u_x)_x = g(x, t), \qquad (x, t) \in R^2, \tag{14}$$

(15) $$u(x, 0) = u(x, T), \quad a \leq x \leq \gamma(0) = \gamma(T),$$

(16) $$ku_n = \psi(t), \qquad (x, t) \in \Gamma_0 = \{x, t : x = a, 0 \leq t \leq T\},$$

(17) $$u(x, t) = 0, \quad (x, t) \in \Gamma_\pi = \{x, t : a < x < \gamma(t), 0 < t < T\},$$

where the bounded domain $Q = \{x, t : a < x < \gamma(t), 0 < t < T\}$ is such that the boundary ∂Q is piecewise smooth.

To apply the scheme developed in Chapter I, let A be defined on $D(A) = \overset{\circ}{C}{}^{2,1}(\overline{Q}, \Gamma_\pi, T)$ by

(18) $$Au = \{cu_t - (ku_x)_x ; ku_n\}$$

and construct the auxiliary operator B in the form

(19)
$$Bv = \{Rv ; Rv\},$$
$$\text{where} \quad Rv = v(x, t) - \int_{\gamma(t)}^{x} 1/k(\theta, t)d\theta \int_0^\theta c(\xi, t)v_t(\xi, t)d\xi.$$

It follows that $(Au, Bv) = \int_Q (ku_x v_x + 1/k \int_a^x cu_t d\xi \int_a^x cv_t d\xi)dQ$, where $(\cdot, \cdot)$ is the inner product in $L_2(Q, \Gamma_0) = L_2(Q) \times L_2(\Gamma_0)$ for functions of the form $f(x, t) = \{g(x, t) ; \psi(t)\}$. It is shown that A is B-sym. and B-p. and thus the weak solution of (14)–(17) is found as the minimum of the functional

$$D[\overline{u}] = \|u\|^2_{\overset{\circ}{W}{}_2^A(Q, \Gamma_\pi, T)} - 2(f, Bu), \quad u \in \overset{\circ}{W}{}_2^A(Q, \Gamma_\pi T).$$

Problem 2. The author also constructs the auxiliary operator B for the hyperbolic equation

(20) $$(k(\xi, \eta)u_\eta)_\xi = g(\xi, \eta)$$

under certain boundary conditions, and shows that the operator A, defined by the left-hand side of (20), is B-sym. and B-p. The above problem contains, as a special case, the problems of Cauchy, Goursat, and Darboux.

Problem 3. The nonclassical elliptic BVP's as well as other problems are also treated by the author using the method of Chapter I.

Remark 1. The problem of constructing the operator B so that A is B-sym. and B-p.d. is quite difficult, however the author succeeds in constructing it for general classes of PD operators A.

In Chapter IV, using the results of Petryshyn [7], the author extends the results of Chapter I to nonlinear equations

$$(21) \qquad N(u) = f,$$

where $N : D(N) \subset H \to H$ is assumed to be **nonpotential** with $\overline{D(N)} = H$ and the Gateaux derivative N'_u is continuous $\forall u \in D(N)$. Again, the problem is to solve the inverse problem of variational calculus, i.e., find a functional $F(u)$ defined on $D(N)$ such that $F(u)$ has derivatives of lower order than N, $F(u)$ is bounded from below on $D(N)$, and $\delta F(\overline{u}) = 0$ iff $N(\overline{u}) = 0$. The author solves this problem under the assumption that N has a Gateaux derivative N'_0 for $u \in D(N)$ which is regular and there exists a linear closable operator B, with $D(B) \supset D(N)$, $\overline{B(D(N))} = H$, such that

$$(N'_u w, Bv) = (Bw, N'_u v) \; \forall u, v, w \in D(N)$$

and the following conditions hold:

(i) $(N'_0 u, Bu) \geq c\|u\|^2 \;\; \forall u \in D(N)$,
(ii) $(N'_u w, Bw) \geq c(N'_0 w, Bw) \;\; \forall u, w \in D(N)$,
(iii) $|(f, Bu)| \leq c_f (N'_0 u, Bu)^{1/2} \;\; \forall u \in D(N), f \in H.$

Among a number of results concerning the weak solvability of (21) in the space H_{AB}, where $A = N'_0$, it is shown (following the arguments in [7]) that under some additional conditions the operator N has a solvable extension $N_0 \supseteq N$ such that, for each $f \in H$, there exists a unique strong solution $u \in D(N_0) \subset H_{AB}$ of $N_0 u = f$ iff u_0 realizes the minimum of the functional $\Phi_0(u) = \int_0^1 (N'_0(tu), B_0 u)dt - (f, B_0 u)$.

To summarize, the monograph by Filippov on the variational principles for nonpotential operators is an important contribution to the theory and application of variational methods in the solvability of abstract and differential boundary value problems. It presents in a coherent manner a vast amount of technical material on the above subject, which thus far was scattered in periodical literature or technical reports. Of particular interest are the commentaries, included after each chapter, which contain a discussion of the subject from a historical perspective. Another positive aspect of the book is that a number of the articles cited in the extensive bibliography are not accessible in English. Hence, Filippov's book should be of great interest to the English speaking pure and applied mathematics community.

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